Symplectic Geometry

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This is a micro-course given at the CIMPA school "Harmonic Analysis and Mathematical Physics" held in Abidjan in 2025. The lectures were prepared last minute as I substituted for a lecturer who cancelled, and I am not an expert in the field. So the chance that there are still mistakes is high and I'd be grateful for any corrections and remarks!

Literature

1. Rolf Berndt, *Introduction to Symplectic Geometry*.
2. Ana Cannas Da Silva, *Lectures on Symplectic Geometry*.
3. Alan Weinstein, *Poisson Geometry*.
4. For Lie–Rinehart algebras and Lie algebroids, see the aticles of Johannes Huebschmann, e.g. *Lie–Rinehart algebras, Gerstenhaber algebras, and Batalin–Vilkovisky algebras*, but they are also mentioned in Jean-Louis Loday, Bruno Vallette, *Algebraic Operads*.
5. Dusa McDuff, Dietman Salamon, *Introduction to Sympectict Topoology*.
6. Vladimir Arnold, *Mathematical Methods of Classical Mechanics*.

Lecture 1 + Tutorial

Commutative algebras

$\mathbb{K}$ and A

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3. The multiplication in A is then a $\mathbb{K}$-bilinear map

$$A \otimes_{\mathbb{K}} A \to A, \quad \sum_i a_i \otimes_{\mathbb{K}} b_i \mapsto \sum_i a_i b_i,$$

so A becomes a **commutative $\mathbb{K}$-algebra**.

Examples from algebraic geometry

1. Example: The **tensor algebra** of a $\mathbb{K}$-module V is

$$\mathcal{T}_{\mathbb{K}} V := \mathbb{K} \oplus V \oplus (V \otimes_{\mathbb{K}} V) \oplus \dots .$$

   It is noncommutative, but the **symmetric algebra**

$$\mathcal{S}_{\mathbb{K}} V := \mathcal{T}_{\mathbb{K}} V / \langle \mathrm{u} \otimes_{\mathbb{K}} \mathrm{v} - \mathrm{v} \otimes_{\mathbb{K}} \mathrm{u} \mid \mathrm{u}, \mathrm{v} \in V \rangle$$

   is the **free commutative algebra on** V: expressing a commutative algebra A as $(\mathcal{S}_{\mathbb{K}} V)/I$ for an ideal $I \lhd \mathcal{S}_{\mathbb{K}} V$ with $V \cap I = 0$ means choosing a $\mathbb{K}$-module $V \subseteq A$ that **generates** A.

2. If $V \cong \mathbb{K}^d$ is a free $\mathbb{K}$-module with basis $x_1, \dots, x_d$, then $\mathcal{S}_{\mathbb{K}} V \cong \mathbb{K}[x_1, \dots, x_d]$ (polynomials in variables x_i with coeff.s in $\mathbb{K}$).

Examples from geometry

1. Smooth manialds M: $\mathbb{K} = \mathbb{R}$ field of real numbers, $A = C^\infty(M)$ ring of smooth $\mathbb{R}$-valued functions on M.
2. Affine varieties M: $\mathbb{K}$ any field, $M \subseteq \mathbb{K}^d$ given by polynomial equations, $A = \mathbb{K}[M]$ ring polynomial functions $M \to \mathbb{K}$.
3. Affine schemes: If M is the set M of irreducible representations (simple modules) p of any A, then the module is a field A/I and the residue class $[a] = a + I \in \mathbb{K}$ is the "value" of $a \in A$ in $p \in M$.
4. Alternatively, take M to be the set of algebra morphisms $A \to \mathbb{K}$, i.e. A-modules whose underlying $\mathbb{K}$-module is $\mathbb{K}$. Often this is the same set, see e.g. Hilbert's Nullstellensatz.

A mix

1. Let $\pi \colon E \to M$ is a vector bundle, $A = \mathrm{C}^\infty(M)$, and $V = \Gamma(E)$ be the A-module of smooth sections of E.
2. Then elements of $\mathcal{S}_A V*$ can be interpreted as functions on E that are smooth along the base and polynomial along the fibre,

$$\left[\sum a_{i_1 \dots i_n} \alpha_{i1} \otimes_A \cdots \otimes_A \alpha_{i_n}\right] \colon \mathrm{v} \mapsto \left(\sum a_{i_1 \dots i_n} \alpha_{i1}(\mathrm{v}) \cdots \alpha_{i_n}(\mathrm{v})\right)(\pi(\mathrm{v})).$$

Poisson algebras

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such that for all $a \in A$, the adjoint action

$$\{a,-\}\colon A \to A, \quad b \mapsto \{a,b\}$$

is a $\mathbb{K}$-linear **derivation**, that is, is an element of

$$\begin{aligned}&\mathrm{Der}_{\mathbb{K}}(A)\\ &:= \{\partial \in \mathrm{Hom}_{\mathbb{K}}(A,A) \mid \forall b,c \in A : \partial(bc) = b\partial(c) + \partial(b)c\}.\end{aligned}$$

Poisson algebra := commutative algebra with a Poisson bracket.

Recap: vector fields

1. Let M be a smooth manifold and $\Gamma(TM)$ be the **vector fields** on M. Recall this is a $C^\infty(M)$-module via

$$(a\mathrm{v})(p) := a(p)\mathrm{v}(p), \quad a \in C^\infty(M), \mathrm{v} \in \Gamma(TM), p \in M.$$

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2. Thm: The map

$$\Gamma(TM) \to \mathrm{Der}_{\mathbb{R}}(C^\infty(M)), \quad \mathrm{v} \mapsto \partial_{\mathrm{v}}, \quad \partial_{\mathrm{v}}(f) := df(\mathrm{v}).$$

is an isomorphism of $C^\infty(M)$-modules.

Hamiltonian mechanics

1. Definition: A **Poisson manifold** is a smooth manifold M together with a Poisson bracket on $C^\infty(M)$.
2. Any $a \in C^\infty(M)$ then yields a dynamical system:

$$\frac{db}{dt} = \{a, b\}, \quad b \in C^\infty(M).$$

3. Interpretation: M is the set of states of some dynamical system, a is its Hamiltonian that governs its dynamics, $b(p)$ is the value of an observable b in $p \in M$.
4. For any Poisson algebra $(A, \{-,-\})$ I call $\{a,-\} \in \mathrm{Der}_{\mathbb{K}}(A)$ the **Hamiltonian vector field** associated to $a \in A$.

The Poisson bivector (field)

1. Remark: Since $\{-,-\}$ is a derivation in each entry, a Poisson bracket on $C^\infty(M)$ is always of the form
$$\{a,b\} = \sum_i (\mathrm{d}a(\mathrm{u}_i)\mathrm{d}b(\mathrm{v}_i) - \mathrm{d}b(\mathrm{u}_i)\mathrm{d}a(\mathrm{v}_i))$$
for a unique bivector (field) called the **Poisson bivector**,
$$\Pi = \sum_i \mathrm{u}_i \wedge \mathrm{v}_i = [\sum_i \mathrm{u}_i \otimes \mathrm{v}_i] \in \Lambda^2_{C^\infty(M)}\Gamma(\mathrm{T}M).$$
2. Here $V = \Gamma(\mathrm{T}M)$ is viewed as module over $A = C^\infty(M)$ and $\Lambda_A V = \mathcal{T}_A V / \langle \mathrm{v} \otimes_A \mathrm{v} \mid \mathrm{v} \in V\rangle$ is the **exterior algebra** of V.
3. Remark: If you take any bivector and define $\{-,-\}$ as above, this is a Poisson bracket iff $[\Pi,\Pi] = 0$ (Schouten-Nijnhuis bracket).

Examples

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2. Exercise 2: If $\mathfrak{g}$ is a Lie algebra over $\mathbb{K}$, then $A := \mathcal{S}_{\mathbb{K}}\mathfrak{g}$ carries a unique Poisson bracket for which

$$\{\iota(\mathrm{u}), \iota(\mathrm{v})\} = \iota([\mathrm{u}, \mathrm{v}]), \quad \mathrm{u}, \mathrm{v} \in \mathfrak{g}.$$

Here $\iota\colon V \to \mathcal{S}V$ is the canonical embedding.

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3. Exercise 3: If $\mathbb{K} = \mathbb{R}$, we may view $M = \mathfrak{g}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g}, \mathbb{R})$ as a smooth manifold and $\mathcal{S}_{\mathbb{K}}\mathfrak{g}$ as a subalgebra of $C^\infty(M)$ consisting of polynomials in the coordinates $\mathrm{v} \in \mathfrak{g}$ on $\mathfrak{g}^*$. Show that the Poisson bracket on $\mathcal{S}_{\mathbb{K}}\mathfrak{g}$ extends uniquely to a Poisson bracket on $C^\infty(\mathfrak{g}^*)$.

Filtered algebras

1. Definition: A **filtered algebra** is an algebra U with a sequence of $\mathbb{K}$-submodules

$$0 = F_{-1}U \subseteq F_0U \subseteq F_1U \subseteq \dots$$

such that $(F_iU)(F_jU) \subseteq F_{i+j}U$.

2. Definition: The **associated graded algebra** is

$$\mathrm{gr}(U) := \bigoplus_{i \geqslant 0} \mathrm{gr}\, U_i, \quad \mathrm{gr}\, U_i := F_iU/F_{i-1}U$$

with (graded) product given by

$$\mathrm{gr}(U)_i \otimes_{\mathbb{K}} \mathrm{gr}(U)_j \to \mathrm{gr}(U)_{i+j}, \quad [a] \otimes_{\mathbb{K}} [b] \mapsto [ab].$$

Example and exercise

1. Definition: U is **almost commutative** if $\mathrm{gr}(U)$ is commutative.
2. Exercise 4: In this case, $\mathrm{gr}(U)$ is a Poisson algebra via

$$\mathrm{gr}(U)_i \otimes_{\mathbb{K}} \mathrm{gr}(U)_j \to \mathrm{gr}(U)_{i+j-1}, \quad [a] \otimes_{\mathbb{K}} [b] \mapsto \{[a],[b]\} := [ab-ba].$$

3. Example: The Poincaré-Birkhoff-Witt theoem says that the **universal enveloping algbera**

$$U(\mathfrak{g}) := \mathcal{T}_{\mathbb{K}}\mathfrak{g} / \langle \mathrm{u} \otimes_{\mathbb{K}} \mathrm{v} - \mathrm{v} \otimes_{\mathbb{K}} \mathrm{u} - [\mathrm{u},\mathrm{v}] \mid \mathrm{u},\mathrm{v} \in \mathfrak{g} \rangle$$

of a Lie algebra $\mathfrak{g}$ that is projective as a $\mathbb{K}$-module (this is automatic if $\mathbb{K}$ is a field) is almost commutative, with $\mathrm{gr}(U(\mathfrak{g})) \cong \mathcal{S}_{\mathbb{K}}\mathfrak{g}$. This yields a sophisticated solution to Exercise 2.

Lie–Rinehart algebras

Lie–Rinehart algebras

1. Definition: A **Lie–Rinehart algebra** over A (aka $(\mathbb{K}, A)$-algebra) is a Lie algbera V over $\mathbb{K}$ equipped with an A-module structure and an A-linear Lie algebra morphism (called the **anchor**)

$$\partial\colon V \to \mathrm{Der}_{\mathbb{K}}(A)$$

such that

$$[\mathrm{v}, a\mathrm{w}] = a[\mathrm{v}, \mathrm{w}] + \partial_{\mathrm{v}}(a)\mathrm{w}, \quad a \in A, \mathrm{v}, \mathrm{w} \in V.$$

2. Definition: When $A = \mathrm{C}^{\infty}(M)$, $V = \Gamma(E)$ for some vector bundle E, then E is called a **Lie algebroid** over M.

Examples

1. A Lie algebra over A is the same as a Lie–Rinehart algbera whose anchor is trivial, meaning that $\partial_{\mathrm{v}} = 0$ for all $\mathrm{v} \in V$.
2. Example: The anchor $\Gamma(\mathrm{T}M) \cong \mathrm{Der}_{\mathbb{R}}(\mathrm{C}^\infty(M))$ turns $\Gamma(\mathrm{T}M)$ into a Lie–Rinehart algebra over $\mathrm{C}^\infty(M)$, so $\mathrm{T}M$ is a Lie algebroid.

Poisson algebras as Lie–Rinehart algebras

1. Definition: If A is a Poisson algebra, define

$$\mathrm{d}a \in \mathrm{Der}_{\mathbb{K}}(A)^* = \mathrm{Hom}_A(\mathrm{Der}_{\mathbb{K}}(A), A), \quad \partial \mapsto \mathrm{d}a(\partial) := \partial(a)$$

and

$$V := \mathrm{span}_A\{\mathrm{d}a \mid a \in A\} \subseteq \mathrm{Der}_{\mathbb{K}}(A)^*.$$

2. Exercise 5: V carries a unique Lie–Rinehart algbera structure with

$$[\mathrm{d}a, \mathrm{d}b] = \mathrm{d}\{a, b\}, \quad a, b \in A.$$

3. In particular: The cotangent bundle T^*M of a Poisson manifold is canonicaly a Lie algebroid.

A theorem of Frobenius

1. A d-**dimensional geometric distribution** on a manifold M is a smooth choice of d-dimensional subspaces $E_p \subseteq T_pM$, $p \in M$.
2. Question: Is there a submanifold $N \subseteq M$ with $E_p = T_pN$?
3. Thm: This holds if and only if for all vector fields u, v with $\mathrm{u}(p), \mathrm{v}(p) \in E_p$ for all $p \in M$, we have $[\mathrm{u}, \mathrm{v}](p) \in E_p$.
4. Upshot: A regular foliation (integrable geometric distribution) is the same as a sub Lie algebroid of TM.

Alternating forms

1. Definition: Fix a Lie–Rinehart algebra V and abbreviate

$$\Omega_A(V) := (\Lambda_A V)^* = \mathrm{Hom}_A(\Lambda_A V, A).$$

2. This is a graded A-module, and we identify $\omega \in \Omega_A^p(V) = (\Lambda_A^p V)^*$ with an **alternating p-form**, that is, a multilinear map

$$\omega\colon V^{\otimes_A p} = V \otimes_A \cdots \otimes_A V \to A,$$

for which $\omega(\mathrm{v}_1, \ldots, v_p) = 0$ if $\mathrm{v}_i = \mathrm{v}_{i+1}$ for some i.

Cartan–Chevalley-Eilenberg–De Rham–Rinehart

1. Definition: We define $\mathrm{d}\colon \Omega^p_A(V) \to \Omega^{p+1}_A(V)$ by

$$
\begin{aligned}
&\mathrm{d}\omega(\mathrm{v}_1, \dots, \mathrm{v}_{p+1}) \\
&= \sum_{j=1}^{p+1} (-1)^{j+1} \partial_{\mathrm{v}_j}(\omega(\mathrm{v}_1, \dots, \hat{\mathrm{v}}_j, \dots \mathrm{v}_{p+1})) + \\
&\sum_{1 \leqslant i < j \leqslant p+1} (-1)^{i+j} \omega([\mathrm{v}_i, \mathrm{v}_j], \mathrm{v}_1, \dots, \hat{\mathrm{v}}_i, \dots, \hat{\mathrm{v}}_j, \dots, \mathrm{v}_{p+1}),
\end{aligned}
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where $\mathrm{v}_1, \dots, \mathrm{v}_{p+1} \in V$.

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\end{aligned}
$$

where $\mathrm{v}_1, \dots, \mathrm{v}_{p+1} \in V$.

2. Thm/Definition: We have $\mathsf{d} \circ \mathsf{d} = 0$; the **cohomology** of V is

$$H(V, A) := \ker \mathsf{d} / \mathrm{im}\, \mathsf{d}.$$

Example

1. $p = 0$: $a \in A, \mathrm{d}a \in \Omega^1(V) = V^*$, $\mathrm{d}a(\mathrm{v}) = \partial_{\mathrm{v}}(a)$.
2. $p = 2$: For a bilinear form $\omega\colon V \otimes_A V \to A$, we obtain

$$\begin{aligned}\mathrm{d}\omega(\mathrm{u},\mathrm{v},\mathrm{w}) = \partial_{\mathrm{u}}(\omega(\mathrm{v},\mathrm{w})) - \partial_{\mathrm{v}}(\omega(\mathrm{u},\mathrm{w})) + \partial_{\mathrm{w}}(\omega(\mathrm{u},\mathrm{v}))\\ - \omega([\mathrm{u},\mathrm{v}],\mathrm{w}) + \omega([\mathrm{u},\mathrm{w}],\mathrm{v}) - \omega([\mathrm{v},\mathrm{w}],\mathrm{u}).\end{aligned}$$

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3. For $A = \mathrm{C}^\infty(M)$, $V = \Gamma(\mathrm{T}M)$, the cohomology $H(V, A)$ is the **De Rham cohomology** of the manifold M.
4. For $\partial = 0$, $H(V, A)$ is the **Lie algebra cohomology** of V.
5. For $V = \Omega^1(M)$, M a Poisson manifold, it is the **Poisson cohomology** of M.

The universal enveloping algebra

1. Rinehart has generalised the construction of $U(\mathfrak{g}$ to universal enveloping algebras $U(V, A)$ of Lie–Rinehart algebras.
2. He also has generalised the Poincaré-Birkhoff-Witt theorem, which gives for A-projective V a Poisson structure on $\mathcal{S}_A V$. We then have
$$H(V, A) \cong \mathrm{Ext}_{U(V,A)}(A, A).$$
3. Example: For $A = \mathrm{C}^\infty(M)$, $V = \Gamma(\mathrm{T}M)$, $U(V, A)$ is the algbera of differential operators on M. $\mathcal{S}_A V$ is the algebra of functions on T^*M that are smooth along M and polynomial along the cotangent spaces. These are the **symbols** of differential operators. The Poisson bracket will be recovered in the next section from the canonical symplectic form on T^*M.

Symplectic forms

Symplectic forms

1. Definition: A **symplectic form** on a Lie–Rinehart algebra V is an alternating 2-form ω on V for which

$$-^\flat\colon V \to V^*, \quad \mathrm{v} \mapsto \mathrm{v}^\flat := \omega(\mathrm{v}, -)$$

is an isomorphism of A-modules and which is **closed**,

$$\mathrm{d}\omega = 0.$$

2. Thus ω defines a class $[\omega] \in H^2(V, A)$.

Symplectomorphisms

1. Definition: A **symplectomorphism** is an A-linear map $J\colon V \to W$ between symplectic Lie–Rinehart algebras $(V,\omega),(W,\eta)$ such that

$$\eta(J\mathrm{u}, J\mathrm{v}) = \omega(\mathrm{u},\mathrm{v}) \quad \forall \mathrm{u},\mathrm{v} \in V.$$

2. Definition: When $V = W$, $\omega = \eta$, and J is invertible, we speak of an **automorphism** of (V,ω). We denote the group of all these by

$$Sp(V,\omega) \subseteq GL(V).$$

Examples

1. Example: If $A = \mathbb{K}$ is a field, $[-,-] = \partial = 0$, then V is simply a $\mathbb{K}$-vector space with a non-degenerate alternating bilinear form ω on it. We then call (V, ω) a **symplectic vector space**.

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2. Example: If $A = \mathrm{C}^\infty(M)$, $V = \Gamma(\mathrm{T}M)$, we call (M, ω) a **symplectic manifold**. In particular, we have:
3. Thm: $M = T^*N$ is canonically symplectic.

Sketch of proof

1. Recall: Local coordinates $q^1, \dots, q^n$ on N induce local coordinates $q^1, \dots, q^n, p_1, \dots, p_n$ on M, with $T_q^*N \ni \alpha = \alpha_i \mathrm{d}q^i =: p_i(\alpha)\mathrm{d}q^i$.
2. The form $\omega := \mathrm{d}p_i \wedge \mathrm{d}q^i$ is independent of the chosen coordinates.
3. $\mathrm{d}\omega = 0$ is obvious from the local represetation.
4. Non-degeneracy follows because in the local coordinates $s^1, \dots, s^{2n} = p_i, q^j$, the form ω is represented by the matrix

$$\omega_{ij} := \omega\left(\frac{\partial}{\partial s^i}, \frac{\partial}{\partial s^j}\right)$$

which is non-singular. $\square$

5. Remark: Darboux's theorem will show that locally, every symplectic manifold looks like this.

The Poisson bracket

1. Recall: $a \in A$ defines $\mathrm{d}a \in V^*$ by $\mathrm{d}a(\mathrm{w}) := \partial_{\mathrm{w}}(a)$. If ω is a symplectic form, there is now a unique $\mathrm{v}_a \in V$ wtth $\mathrm{v}_a^\flat = \mathrm{d}a$.
2. Thm: $\{a, b\} := \omega(\mathrm{v}_a, \mathrm{v}_b)$ is a Poisson bracket on A.
3. By construction, we have $\partial_{\mathrm{v}_a} = \{a, -\}$, so v_a is/represents the Hamiltonian vector field associated to $a \in A$, and

$$\{a, b\} = \partial_{\mathrm{v}_a}(b) = \mathrm{d}b(\mathrm{v}_a).$$

4. Warning: Some authors write $-\mathrm{v}_a$ for v_a!
5. So: Every symplectic manifold is canonically a Poisson manifold.
6. Exercise 6: $[\mathrm{v}_a, \mathrm{v}_b] = \mathrm{v}_{\{a,b\}}$. When checking this, you automatically generalise this to arbitrary Poisson algebras: the Hamiltonian vector fields form a quotient Lie algebra of A with repsect to $\{-, -\}$.

The converse

1. Conversely, if M is a Poisson manifold, let $\sim$ be the smallest equivalence relation on M such that $p \sim q$ if they lie on an integral curve of a Hamiltonian vector field.
2. Thm: The equivalence class $[p] \subseteq M$ of $p \in M$ is a submanifold and carries a unique symplectic form for which the restriction $C^\infty(M) \to C^\infty([p])$ is a map of Poisson algebras.
3. Upshot: Every Poisson manifold has a (usually singular!) foliation whose leaves are symplectic manifolds.
4. We will not prove this in general, but show that for $M = \mathfrak{g}^*$, these leaves are the orbits of the coadjoint action.

Exercise 7: Maxwell

1. Consider $M = \mathbb{R}^6 \cong T^*N$, $N = \mathbb{R}^3$ with coordinates $q^1, q^2, q^3, p_1, p_2, p_3$. Fix $B_i \in C^\infty(N)$, $i = 1, 2, 3$, in the variables q^1, q^2, q^3 und put

$$\omega := \omega_0 - \omega_B, \quad \omega_0 = \sum_{i=1}^{3} \mathrm{d}q^i \wedge \mathrm{d}p_i,$$

$$\omega_B := B_1 \mathrm{d}q^2 \wedge \mathrm{d}q^3 + B_2 \mathrm{d}q^3 \wedge \mathrm{d}q^1 + B_3 \mathrm{d}q^1 \wedge \mathrm{d}q^2$$

   Show: ω is symplectic if $\mathrm{div} B = 0$.
2. Find a formula for the Poisson bracket $\{f, g\}$, $f, g \in C^\infty(M)$.
3. Let $V \in C^\infty(N)$ be another function and put

1. $E_i := -\frac{\partial V}{\partial q^i}, \quad H := \frac{1}{2}\sum_{i=1}^3 p_i^2 + V(q^1, q^2, q^3)$.
2. Consider $\mathrm{v} = (p_1, p_2, p_3), E, B$ as vectors in $\mathbb{R}^3$ and show:

$$\{H, q^i\} = p_i, \quad \{H, p_i\} = (E + \mathrm{v} \times B)_i,$$

   where $\times$ is the vector product in $\mathbb{R}^3$.
3. Let $\gamma \colon \mathbb{R} \to M$ be a curve and $\alpha \colon \mathbb{R} \to N$ be its projection onto the first three variables. Show: γ is an integral curve of $\{H, -\}$ if and only if the acceleration $\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2}$ is given by the Lorentz force (just google, $F = ma$ wtth $m = 1$).

Lecture 2

Darboux's theorem

For symplectic vector spaces

1. Thm: If (V,ω) is a finite-dimensional symplectic vector space, there is a basis $\mathrm{e}_i, \mathrm{v}_i$, $i = 1, \ldots, n$, with
$$\omega(\mathrm{e}_i, \mathrm{v}_j) = \delta_{ij}, \quad \omega(\mathrm{e}_i, \mathrm{e}_j) = \omega(\mathrm{v}_i, \mathrm{v}_j) = 0.$$
In particular, $\dim_{\mathbb{R}} V = 2n$.
2. Proof: This is a variant of Gram–Schmid: take any $\mathrm{e}_1 \neq 09$ and v_1 mit $\omega(\mathrm{e}_1, \mathrm{v}_1) = 1$. Put $V_1 := \mathrm{span}_{\mathbb{K}}\{\mathrm{e}_1, \mathrm{v}_1\}$.
3. Consider $\pi \colon V \to V, \quad \mathrm{u} \mapsto \omega(\mathrm{u}, \mathrm{v}_1)\mathrm{e}_1 + \omega(\mathrm{e}_1, \mathrm{u})\mathrm{v}_1$. We have $\pi \circ \pi = \pi$ and $\pi|_{V_1} = \mathrm{id}$.

The induction step

1. Put $W := \ker \pi$. Then $V = V_1 \oplus W$ and $\omega|_W$ is symplectic: if $\mathrm{e}_2 \in W$ and $\mathrm{v} \in V$ with $\omega(\mathrm{e}_2, \mathrm{v}) = 1$, put

$$\mathrm{v}_2 := \mathrm{v} - \pi(\mathrm{v}) \in W.$$

Then we have by the definition of W

$$\omega(\mathrm{e}_2, \mathrm{v}_1) = \omega(\mathrm{e}_2, \mathrm{e}_1) = 0 \Rightarrow \omega(\mathrm{e}_2, \mathrm{v}_2) = \omega(\mathrm{e}_2, \mathrm{v}) = 1,$$

so $\omega|_W$ is non-degenerate.

2. Now continue by induction on the dimension. $\square$

Moser's trick

1. Moser stability: If M is a compact manifold and ω_t is a smooth curve of symplectic forms wit constant cohomology class $[\omega_t] = [\omega_0]$, then there exists $\varphi \in \mathrm{Diff}(M)$ with $\varphi^*\omega_1 = \omega_0$.
2. Proof: This is not entirely trivial (see [McDuff–Salamon]) , but $\frac{\partial[\omega]}{\partial t} = 0$ shows there are $\alpha_t \in \Omega^1(M)$ with
$$\frac{\partial \omega_t}{\partial t} = \mathrm{d}\alpha_t.$$
3. Let v_t be a time-dependent vector field and φ_t its flow, so
$$\frac{\partial \varphi_t}{\partial t} = \mathrm{v}_t \circ \varphi_t.$$

Proof (Moser)

1. Then $\mathrm{d}\omega_t = 0$ gives
$$\frac{\partial}{\partial t}\varphi_t^*\omega_t = \varphi_t^*\Big(\frac{\partial \omega_t}{\partial t} + i_{\mathrm{v}_t}\mathrm{d}\omega_t + \mathrm{d}(i_{\mathrm{v}_t}\omega_t)\Big) = \mathrm{d}\varphi_t^*(\alpha_t + i_{\mathrm{v}_t}\omega_t).$$
2. If we apply this with $\mathrm{v}_t := -\alpha_t^\sharp$, the unique vector field for which
$$\alpha_t = -\omega_t(\mathrm{v}_t, -) = -i_{\mathrm{v}_t}\omega_t,$$
we obtain
$$\frac{\partial}{\partial t}\varphi_t^*\omega_t = 0.$$
3. M compact $\Rightarrow$ there is φ_t, $\varphi_t^*\omega_t = \varphi_0^*\omega_0 = \omega_0$. Take $\varphi := \varphi_1$. $\square$

Application: Darboux, global version

1. Thm: If (M, ω) is a symplectic manifold and $p \in M$, there exists a chart $\kappa \colon U \to \mathbb{R}^{2n}$ with $(\kappa^{-1})^*\omega = \omega_0 := \omega_{Darboux}$.
2. Proof: Start with any chart, put $q := \kappa(p)$. Wlog (Darboux for vector spaces, compose κ with $\varphi \in Sp(2n)$ if needed) the forms $\omega_1 := (\kappa^{-1})^*\omega$ and ω_0 agree in q.
3. Put $\omega_t := t\omega_0 + (1-t)\omega_1$. In q these are independent of t, so in some ball B they are symplectic.
4. Since $\omega_1 - \omega_0$ is closed, Poincaré's lemma $(H^2(B) = 0)$ shows $\exists \alpha \in \Omega^1(B) : \omega_1|_B - \omega_0|_B = \mathrm{d}\alpha$. Now Moser. $\square$

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5. Corollary: $\omega^n = \omega \wedge \cdots \wedge \omega$ is a volume form, so symplectic manifolds are canonically oriented.

The (co)adjoint adctions

Notation

1. If M is a smooth manifold and $p \in M$, then a tangent vector $\mathrm{v} \in \mathrm{T}_p M$ is an equivalence class $[\gamma]$ of a smooth curve $\gamma\colon (-1,1) \to M$, $\gamma(0) = p$ through p. Two curves γ, β are equivalent if for all $f \in C^\infty(M)$, we have

$$\frac{\mathrm{d}(f \circ \gamma)}{\mathrm{d}t}\Big|_{t=0} = \frac{\mathrm{d}(f \circ \beta)}{\mathrm{d}t}\Big|_{t=0} =: \mathrm{d}f_p(\mathrm{v}) =: (\partial_{\mathrm{v}}(f))(p).$$

2. The derivative of a smooth map $\varphi\colon M \to N$ in $p \in M$ is the map

$$(\varphi_*)_p\colon \mathrm{T}_p M \to \mathrm{T}_{\varphi(p)} N, \quad [\gamma] \mapsto [\varphi \circ \gamma].$$

The adjoint action of G on G

1. Let G be a Lie group and $\mathfrak{g} = \mathrm{T}_e G$ be its Lie algbera (over $\mathbb{K} = \mathbb{R}$).
2. Definition: The **adjoint action** of G on G is given by

$$Ad_g \colon G \to G, \quad h \mapsto ghg^{-1}.$$

3. This defines a smooth action of G on itself,

$$Ad \colon G \to \mathrm{Diff}(G), \quad g \mapsto Ad_g, \quad Ad_e = \mathrm{id}_G, \quad Ad_f \circ Ad_g = Ad_{fg}.$$

The adjoint action of G on $\mathfrak{g}$

1. Definition: Taking the derivative of $Ad_g : G \to G$ in $e \in G$ (unit element in G) we obtain the **adjoint action** of G on $\mathfrak{g}$,

$$\mathfrak{Ad}_g := ((Ad_g)_*)_e : \mathfrak{g} = \mathrm{T}_e G \to \mathfrak{g}, \quad \mathrm{w} = [\gamma] \mapsto [Ad_g \circ \gamma] = [g \gamma g^{-1}].$$

Here $\gamma : (-1, 1) \to G$ is a curve through $\gamma(0) = e$.

2. By the chain rule $((\varphi \circ \psi)_*)_p = (\varphi_*)_{\psi(p)} \circ (\psi_*)_p$, this is a linear representation of G,

$$\mathfrak{Ad} : G \to GL(\mathfrak{g}).$$

The adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$

1. We initially define the **adjoint action** of $\mathfrak{g}$ on $\mathfrak{g}$ without any reference to Ad or $\mathfrak{Ad}$ by

$$\mathfrak{ad}_{\mathrm{v}} \colon \mathfrak{g} \to \mathfrak{g}, \quad \mathrm{w} \mapsto [\mathrm{v}, \mathrm{w}].$$

2. By the Jacobi identity, this is a representation of $\mathfrak{g}$ as a Lie algebra,

$$\mathfrak{ad} \colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), \quad \mathrm{v} \mapsto \mathfrak{ad}_{\mathrm{v}}, \quad \mathfrak{ad}_{\mathrm{v}} \circ \mathfrak{ad}_{\mathrm{w}} - \mathfrak{ad}_{\mathrm{w}} \circ \mathfrak{ad}_{\mathrm{v}} = \mathfrak{ad}_{[\mathrm{v},\mathrm{w}]}.$$

3. Advantage: Makes sense for abstract Lie algebras, we don't need G.

Coadjoint actions

1. Definition: The **coadjoint actions** of G respecitvely $\mathfrak{g}$ on $\mathfrak{g}^*$ are simply the dual representation,

$$(\mathfrak{Coad}^*_g)(\alpha)(\mathrm{v}) = \alpha(\mathfrak{Ad}_{g^{-1}}(\mathrm{v})), \quad g \in G, \alpha \in \mathfrak{g}^*, \mathrm{v} \in \mathfrak{g}.$$

respectively

$$(\mathfrak{coad}(\mathrm{v})(\alpha))(\mathrm{w}) := -\alpha([\mathrm{v}, \mathrm{w}]), \quad \mathrm{v}, \mathrm{w} \in \mathfrak{g}, \alpha \in \mathfrak{g}^*.$$

More notation

1. The Lie bracket on $\mathfrak{g}$ is obtained by identifying $\mathfrak{g}$ with the **left-invariant vector fields** on G (those ones with $(L_g)_* \mathrm{v} = \mathrm{v}$ for all left shifts $L_g \colon G \to G, h \mapsto gh$). The isomorphism with $\mathrm{T}_e G$ is given by evaluation of a vector field in $e \in G$.
2. The Lie bracket on $\mathfrak{g}$ is then just the Lie bracket of vector fields. In other words,

$$[\mathrm{v}, \mathrm{w}] = \mathcal{L}_{\mathrm{v}}(\mathrm{w})$$

is the **Lie derivative** of w along v.

Explicitly

1. Thm: $\mathfrak{ad}_{\mathrm{v}}(\mathrm{w})$ is the differntial of $\mathfrak{Ad}\colon G \to GL(\mathfrak{g})$, meaning that
$$\mathfrak{ad}_{\mathrm{v}}(\mathrm{w}) = \frac{\partial^2}{\partial s \partial t} \exp(s\mathrm{v}) \exp(t\mathrm{w}) \exp(-s\mathrm{v})|_{s,t=0} = [\mathrm{v},\mathrm{w}].$$
2. Proof: $\varphi_{\mathrm{v},t}(g) = g \exp_{t\mathrm{v}}$ (product in G) is the flow of v starting in $g \in G$. Now stare for $g = e$ on (see [Arnold] for the Lie derivative)
$$\begin{aligned}[\mathrm{v},\mathrm{w}](g) = (\mathcal{L}_{\mathrm{v}}\mathrm{w})(g) &= \frac{\mathrm{d}}{\mathrm{d}s} (\varphi_{\mathrm{v},-s})_*(\mathrm{w}_{\varphi_{\mathrm{v},s}(g)})|_{s=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}s} (\varphi_{\mathrm{v},-s})_*(\varphi_{\mathrm{v},s}(g)\mathrm{w}_e)|_{s=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\frac{\mathrm{d}}{\mathrm{d}t} (\varphi_{\mathrm{v},-s})(g \exp(s\mathrm{v}) \exp(t\mathrm{w}))|_{s,t=0}. \qquad \square\end{aligned}$$

The Kostant–Kirillov aka Lie–Poisson form

Goal

1. We will now show that the orbit

$$M := \mathfrak{Coad}_G(\alpha) \subseteq \mathfrak{g}^*$$

of any $\alpha \in \mathfrak{g}^*$ under the coadjoint action of G is canonically a symplectic manifold; as we will see, these are the symplectic leaves of $\mathfrak{g}^*$ with respect to the Poisson bracket defined by $[-,-]$.

2. We follow Homework 17 in [Da Silva] where this is broken nicely into several steips.

Step 1

1. The adjoint representation of G on $\mathfrak{g}$ associates to $\mathrm{v} \in \mathfrak{g}$ an "action vector field" $\hat{\mathrm{v}}$ on the manifold $\mathfrak{g}$ whose value in $\mathrm{w} \in \mathfrak{g}$ is

$$\hat{v}(\mathrm{w}) = [\gamma] \in \mathsf{T}_{\mathrm{w}}\mathfrak{g}, \quad \gamma\colon (-1,1) \to \mathfrak{g}, \quad t \mapsto \mathfrak{Ad}_{\exp t\mathrm{v}}(\mathrm{w}).$$

2. The manifold $\mathfrak{g}$ is a vector space, which yields an isomorphism $\mathsf{T}_{\mathrm{w}}\mathfrak{g} \cong \mathfrak{g}$, and if we insert our computation of $\mathfrak{ad}$ as the differential of $g \mapsto \mathfrak{Ad}_g$, we conclude that under this identification, we have

$$\hat{v}(\mathrm{w}) = \frac{\mathsf{d}}{\mathsf{d}t}\mathfrak{Ad}_{\exp(t\mathrm{v})}(\mathrm{w}) \mid_{t=0} = \mathfrak{ad}_{\mathrm{v}}(\mathrm{w}) = [\mathrm{v}, \mathrm{w}].$$

Step 2

1. Now we do the same computation for the coadjoint representation: for $\mathrm{v} \in \mathfrak{g}$ let $\check{\mathrm{v}}$ be the vector field on the manifold $\mathfrak{g}^*$ whose value in the point $\alpha \in \mathfrak{g}^*$ is

$$\check{\mathrm{v}}(\alpha) = [\gamma] \in \mathrm{T}_\alpha \mathfrak{g}^*, \quad \gamma \colon (-1, 1) \to \mathfrak{g}^*, \quad t \mapsto \mathfrak{Coad}_{\exp(t\mathrm{v})}(\alpha).$$

2. Now $\mathfrak{Coad}_{\exp(t\mathrm{v})}(\alpha) \in \mathfrak{g}^*$ is given by

$$(\mathfrak{Coad}_{\exp(t\mathrm{v})}(\alpha))(\mathrm{w}) = \alpha(\mathfrak{Ad}_{\exp(t\mathrm{v})^{-1}}(\mathrm{w})) = \alpha(\mathfrak{Ad}_{\exp(-t\mathrm{v})}(\mathrm{w})).$$

3. Thus taking $\frac{\mathrm{d}}{\mathrm{d}t}$ in $t = 0$ yields under the identification $\mathrm{T}_\alpha \mathfrak{g}^* \cong \mathfrak{g}^*$

$$(\check{\mathrm{v}}(\alpha))(\mathrm{w}) = \alpha(-[\mathrm{v}, \mathrm{w}]) = \mathfrak{coad}_{\mathrm{v}}(\alpha)(\mathrm{w}) = \alpha([\mathrm{w}, \mathrm{v}]).$$

Step 3

1. The expression
$$\omega_\alpha(\mathrm{v},\mathrm{w}) := \alpha([\mathrm{w},\mathrm{v}])$$
is clearly an alternating 2-form on $\mathfrak{g}$.
2. By Step 2, we have
$$\omega_\alpha(\mathrm{v},-) = 0 \Leftrightarrow \check{\mathrm{v}}(\alpha) = 0.$$
3. In other words, the radical of ω_α is the Lie algebra $\mathfrak{h} \subseteq \mathfrak{g}$ of the isotropy subgroup (the stabiliser) $H \subseteq G$ of $\alpha \in \mathfrak{g}^*$,
$$H = \{g \in G \mid \mathfrak{Coad}_g(\alpha) = \alpha\},$$

$$\mathfrak{h} = \{\mathrm{v} \in \mathfrak{g} \mid \mathfrak{coad}_{\mathrm{v}}(\alpha) = \check{v}(\alpha) = 0\}.$$

Step 4

1. The **orbit-stabiliser theorem** yields a diffeomorphism

$$G/H \to M, \quad gH \mapsto \mathfrak{Coad}_g(\alpha),$$

and its derivative defines an isomorphism

$$\mathfrak{g}/\mathfrak{h} \to \mathrm{T}_\alpha M, \quad \mathrm{v} + \mathfrak{h} \mapsto \check{\mathrm{v}}(\alpha) = \mathfrak{coad}_{\mathrm{v}}(\alpha),$$

where at the end, we identify $\mathrm{T}_\alpha M \subseteq \mathrm{T}_\alpha \mathfrak{g}^* \cong \mathfrak{g}^*$.

2. So Step 3 shows that ω_α descends to a non-degenerate 2-form on the tangent space $\mathrm{T}_\alpha M$, $M = \mathfrak{Coad}_G(\alpha)$.

Step 5

1. For each $\beta \in M$, we have constructed a non-degenerate 2-form ω_β on $\mathrm{T}_\beta M$, and this clearly depends smoothly on β. So we have constructed a non-degenerate 2-form

$$\omega \in \Omega^2(M), \quad M = \mathfrak{Coad}_G(\alpha) \cong G/H.$$

   Note that ω does not depend on the chosen base point α, it is canonically defined on M.
2. Our final aim is to show that $\mathrm{d}\omega = 0$. So we show that $\mathrm{d}\omega = 0$ in α.

Step 5 continued

1. Above, we considered the isomorphism

$$\mathfrak{g}/\mathfrak{h} \to \mathrm{T}_\alpha M \subseteq \mathrm{T}_\alpha \mathfrak{g}^* \cong \mathfrak{g}^*, \quad \mathrm{v} + \mathfrak{h} \mapsto \check{\mathrm{v}}(\alpha) = \alpha([-, \mathrm{v}]).$$

2. If $\beta = \mathfrak{Coad}_g(\alpha) \in M$ is another point, then the derivative of

$$\mathfrak{Coad}_g \colon M \to M$$

in α maps $\mathrm{T}_\alpha M \cong \mathfrak{g}/\mathfrak{h}$ to $\mathrm{T}_\beta M \cong \mathfrak{g}/\mathfrak{l}$, where $\mathfrak{l}$ is the Lie algebra of tie stabiliser of β, which is $L = gHg^{-1} = Ad_g(H)$.

3. As $\mathfrak{Coad}_g$ is a linear map on $\mathfrak{g}^*$, the derivative is also $\mathfrak{Coad}_g$. Let us compute the corresponding map $\mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{l}$.

Step 5 continued

1. The original derivative $\mathfrak{Coad}_g : \mathrm{T}_\alpha M \to \mathrm{T}_\beta M$ is given by

$$\begin{aligned} & \alpha([-,\mathrm{v}]) = \check{\mathrm{v}}(\alpha) \\ \mapsto\ & \mathfrak{Coad}_g(\check{\mathrm{v}}(\alpha)) = \check{\mathrm{v}}(\alpha)(\mathfrak{Ad}_{g^{-1}}(-)) = \alpha([\mathfrak{Ad}_{g^{-1}}(-),\mathrm{v}]) \\ =\ & \alpha([\mathfrak{Ad}_{g^{-1}}(-),\mathfrak{Ad}_{g^{-1}}(\mathfrak{Ad}_g(\mathrm{v}))]) \\ =\ & \alpha(\mathfrak{Ad}_{g^{-1}}[-,\mathfrak{Ad}_g(\mathrm{v})]) \\ =\ & \mathfrak{Coad}_g(\alpha)([-,\mathfrak{Ad}_g(\mathrm{v})]) = \beta([-,\mathfrak{Ad}_g(\mathrm{v})]). \end{aligned}$$

So the derivative corresponds simply to

$$\mathfrak{Ad}_g : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{l}, \quad \mathrm{v} + \mathfrak{h} \mapsto \mathfrak{Ad}_g(\mathrm{v}) + \mathfrak{l}.$$

Step 5...

1. We can now assign to a given $\mathrm{v} \in \mathfrak{g}$ a "left-invariant" vector field $\bar{\mathrm{v}}$ on M which $\underbrace{\text{in } \beta}$ $= \mathfrak{Coad}_g(\alpha) \in M$ takes the value $\alpha(\mathfrak{Ad}_{g^{-1}}(-), \mathrm{v})$, and this is $\mathfrak{Ad}_g(\mathrm{v})(\beta)$.
2. In a neighbourhood U of α, these left-invariant vector fields span the $C^\infty(U)$-module of all vector fields. As $d\omega$ is multilinear, it thus suffices to test $d\omega = 0$ only on such left-invariant vector fields.

Step 5 continued

1. We have

$$\mathrm{d}\omega(\bar{\mathrm{u}},\bar{\mathrm{v}},\bar{\mathrm{w}}) = \partial_{\bar{u}}(\omega(\bar{v},\bar{w})) - \partial_{\bar{\mathrm{v}}}(\omega(\bar{u},\bar{w})) + \partial_{\bar{w}}(\omega(\bar{\mathrm{u}},\bar{v})) \\ - \omega([\bar{u},\bar{v}],\bar{\mathrm{w}}) + \omega([\bar{u},\bar{w}],\bar{v}) - \omega([\bar{v},\bar{w}],\bar{u}).$$

2. We show that the first three terms are individually zero.
3. Indeed, if $\beta = \mathfrak{Coad}_g(\alpha) \in U$, then

$$\omega(\bar{\mathrm{v}},\bar{\mathrm{w}})(\beta) = \mathfrak{Coad}_g(\alpha)([\mathfrak{Ad}_g(\mathrm{w}),\mathfrak{Ad}_g(\mathrm{v})]) = \omega(\bar{\mathrm{v}},\bar{\mathrm{w}})(\alpha).$$

So this is a constant function whose partial derivatives vanish.

The end

1. The final three terms can be directly evaluated in α, giving

$$-\alpha([\mathrm{w}, [\mathrm{u}, \mathrm{v}]]) + \alpha([\mathrm{v}, [\mathrm{u}, \mathrm{w}]]) - \alpha([\mathrm{u}, [\mathrm{v}, \mathrm{w}]]),$$

and this is indeed 0 by the Jacobi identity.

Kostant–Souriau

1. Thm: If $H^1(\mathfrak{g}, \mathbb{R}) = H^2(\mathfrak{g}, \mathbb{R}) = 0$, then there is (up to coverings) a bijective correspondence between coadjoint orbits and transitive symplectic G-manifolds.