BV_∞ Quantization of (-1) shifted Derived Poisson manifolds

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Khudaverdian–Voronov : Homotopy Poisson algebras (P_∞)
Cattaneo–Felder : Homotopy Schouten algebras

(+1) shifted derived Poisson algebra ←···→ Homotopy Schouten algebras in the context of $\mathbb{Z}$-grading

Definition : A (+1) shifted derived Poisson algebra is a $\mathbb{Z}$-graded comm. alg. A with deg +1 brackets

$$\lambda_n : A^{\otimes n} \longrightarrow A, \quad n \geq 1$$

1) $L_\infty[1]$ - algebra

2) for all $n \geq 1$, $a \longmapsto \lambda_n(a_1, \ldots, a_{n-1}, a)$

(graded) derivation

Remark $H^0(A, \lambda_1)$ - Schouten algebra.

Definition A (-1) shifted derived Poisson manifold

= $\mathbb{Z}$-graded manifold M s.t.

$C^\infty(M)$ is a $(+1)$-shifted Poisson algebra

Equivalent Def (J. Pridham)

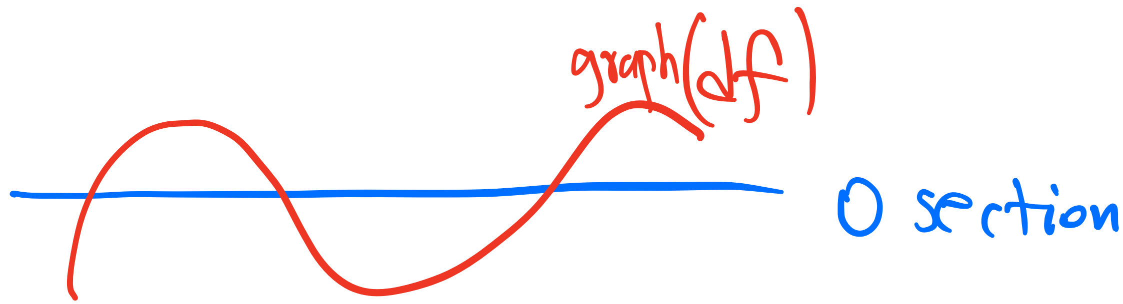
= A dg manifold (M, Q) equipped with $\pi = \sum_{n=2} \pi_n$, $\pi_n \in \Gamma(S^n T_M)$ s.t.

$$\{Q, \pi\} + \frac{1}{2}\{\pi, \pi\} = 0$$

$\pi \in Pol(T_M^\vee)$, $\{\cdot\,;\cdot\}$ standard Poisson bracket on $T_M^\vee$

Example: (Derived intersection of coisotropic submanifolds)

$T^\vee_M$



Lagrangian submanifolds

$\left(T^\vee_M[-1],\ (df,\ \Pi_2)\right)$ ...... (-1) shifted derived Poisson

- $C^\infty(T_M^\vee[-1]) \cong \Gamma(\Lambda^{-\bullet} T_M)$
- $i_{df} : \Gamma(\Lambda^{-\bullet} T_M) \longrightarrow \Gamma(\Lambda^{-(\bullet+1)} T_M)$

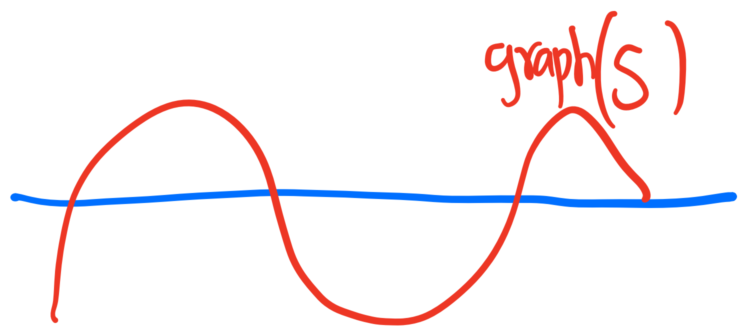
  contraction.
- $\pi_2 \in S^2(T_{T_M^\vee[-1]})$ defining the Schouten bracket on $\Gamma(\Lambda^{-\bullet} T_M)$

In general, A - Lie algebroid

$s \in \Gamma(A^\vee)$ cocycle

$(A^\vee[-1], i_s, \pi_2)$ - shifted derived Poisson

$A^\vee$ — Lie Poisson



coistropic

0 section

derived intersection of two coisotropic

Example

$\mathfrak{L}$ - L_∞- algebroid.

$\mathfrak{L}^\vee[-1]$ (-1) shifted derived Poisson.

$$C^\infty(\mathfrak{L}^\vee[-1]) \simeq \Gamma(\hat{S}^\bullet(\mathfrak{L}[1]))$$

$\Gamma(\mathfrak{L}[1])$ — $L_\infty[1]$-algebra

Using Leibniz rule & multi-anchor maps

$$\tilde{\lambda}_n : \Gamma(S^{k_1}(\mathfrak{L}[1])) \times \cdots \times \Gamma(S^{k_n}(\mathfrak{L}[1]))$$

$$\longrightarrow \Gamma(S^{k_1+k_2+\cdots+k_n+(1-n)}(\mathfrak{L}[1]))$$

BV_∞ - quantization

$\mathrm{Dens}^{\frac{1}{2}}_{M}$ = Berezinian half-density line bundle over M

$DO^{\leq n}(\mathrm{Dens}^{\frac{1}{2}}_{M})$ - diff operators on $\Gamma(M, \mathrm{Dens}^{\frac{1}{2}}_{M})$ of order $\leq n$

Filtration

$$DO^{0} \subseteq DO^{\leq 1} \subset DO^{\leq 2} \subset \cdots$$

$$[DO^{\leq n}, DO^{\leq m}] \subseteq DO^{\leq n+m-1}$$

$$Gr^n DO := \frac{DO^{\leq n}}{DO^{\leq n-1}} \xrightarrow{\sim} \Gamma(S^n T_M)$$

Principal map

$$\sigma_n : DO^{\leq n} \xrightarrow{pr} Gr^n DO \xrightarrow{\sim} \Gamma(S^n T_M)$$

Definition (Kravchenko)

BV_∞ - operator:

- $\Delta = \sum_{n \geq 1} \hbar^n \Delta_n$, $\Delta_n \in DO^{\leq n}$
- degree $+1$
- $\Delta^2 = 0$

Given a BV_∞-operator Δ

$$\lambda_n(f_1, \ldots, f_n)s = [\,[[\Delta_n, f_1], f_2]; \cdots f_n]\,s$$

$$\Rightarrow \left(C^\infty(M), (\lambda_n) \right)$$ — deg +1 multi-Poisson brackets

$$Q + \Pi = \sigma_\hbar(\Delta)\Big|_{\hbar=1} := \sum \hbar^n \sigma_n(\Delta_n)\Big|_{\hbar=1} \in \Gamma(S T_M)$$

(−1)-shifted derived Poisson tensor

Example Assume $\Delta = \hbar^2 \Delta_2$ — only one term.

Δ_2 — deg +1, order 2 diff operator on $\mathrm{Dens}^{\frac{1}{2}}_M$

$\Delta_2 \circ \Delta_2 = 0$

$$\lambda_2(f, g)\, s = \Delta_2(fgs) \pm g\Delta_2(fs) \pm f\Delta_2(gs) \pm fg\Delta_2(s)$$

$s \in \Gamma(M, \mathrm{Dens}^{\frac{1}{2}} M)$

Assume
- $\rho \in \Gamma(M, \mathrm{Dens}^{\frac{1}{2}} M)$ nowhere vanishing
- $\Delta_2(\sqrt{\rho}) = 0$

$$\Delta_\rho(f) := \frac{1}{\sqrt{\rho}}\,\Delta_2\left(f\sqrt{\rho}\right), \quad f \in C^\infty(M)$$

Then

- $\Delta_\rho^2 = 0$, Δ_ρ — deg $+1$
- $\{f, g\} = \Delta_\rho(fg) - \Delta_\rho(f)g + (-1)^{|f|} f \Delta_\rho(g)$ — Schouten bracket
- Δ_ρ — BV operator

Quantization Problem

Given a (-1) shifted derived Poisson

$$(\mathcal{M}, Q, \Pi)$$

Find a self-adjoint BV_∞-operator

Δ s.t.

$$\sigma_\hbar(\Delta)\Big|_{\hbar=1} = Q + \Pi \in$$

$$\mathfrak{X}(\mathcal{M}) \oplus \Gamma(S^2 T_{\mathcal{M}}) \oplus \Gamma(S^3 T_{\mathcal{M}}) \oplus \cdots$$

Theorem A. A (-1) shifted derived Poisson manifold $(\mathcal{M}, Q, \pi)$ quantizable if $H^2(\mathcal{M}, Q+\pi) = 0$

$H(\mathcal{M}, Q+\pi)$ = Poisson cohomology:

$$\Gamma(\hat{S}\, T_{\mathcal{M}}) \longrightarrow \Gamma(\hat{S}\, T_{\mathcal{M}})$$

$$X \longmapsto \{Q+\pi, X\}$$

Theorem B For any L_∞-algebroid $\mathcal{L}$, its corresponding linear (-1)-shifted derived Poisson manifold $\mathcal{L}^\vee[-1]$ admits a canonical quantization.

Theorem C For A - Lie algebroid
$s \in \Gamma(A^\vee)$ 1-cocycle
$(A^\vee[-1], \iota_s, \pi_2)$ - admits a canonical quantization

$$\Delta = \hbar \iota_s + \hbar^2 \Phi \circ d_{CE}^{ELW} \circ \Phi^{-1} :$$

$$\Gamma\left(\Lambda^{-\bullet} A \otimes \left(\Lambda^{top} A^\vee \otimes \Lambda^{top} T_M^\vee\right)^{\frac{1}{2}}\right)$$

$$\longrightarrow \Gamma\left(\Lambda^{-(\bullet+1)} A \otimes \left(\Lambda^{top} A^\vee \otimes \Lambda^{top} T_M^\vee\right)^{\frac{1}{2}}\right)$$

$$\Phi : \Gamma\left(\wedge^k A^\vee \otimes \left(\wedge^{top} A \otimes \wedge^{top} T_M^\vee\right)^{\frac{1}{2}}\right)$$

$$\xrightarrow{\sim} \Gamma\left(\wedge^{top-k} A \otimes \left(\wedge^{top} A^\vee \otimes \wedge^{top} T_M^\vee\right)^{\frac{1}{2}}\right)$$

canonical isomorphism.

Example $\left(T_M^\vee[-1],\ i_{df},\ \pi_2\right)$

$$\Delta = \hbar\, i_{df} + \hbar^2\, \Phi \circ d_{DR} \circ \Phi^{-1}$$

$$\Gamma\left(\wedge^{-\bullet} T_M \otimes \left(\wedge^{top} T_M^\vee\right)\right)$$

$$\longrightarrow \Gamma\left(\wedge^{-(\bullet+1)} T_M \otimes \left(\wedge^{top} T_M^\vee\right)\right)$$

$$\Phi : \quad \Omega^k(M) \xrightarrow{\sim} \Gamma\left(\wedge^{top-k} T_M \otimes \wedge^{top} T_M^\vee\right)$$

(Koszul – 1985)

GRAZIE !!!