Formal exponential maps and Atiyah class of differential graded manifolds

Mathieu Stiénon

PennState

NonCommutative Geometry and Higher Structures
Scalea, Italy — 2022/06/08

1. Kapranov: $L_\infty[1]$ algebra hiding behind Atiyah class of Kähler mfd

2. Atiyah class of a dg manifold

3. Formal exponential maps

4. Formal exponential map on a dg mfd and Kapranov's $L_\infty[1]$ algebra

Atiyah class of holomorphic vector bundle

- $E \to X$: holomorphic vector bundle E over complex manifold X
- (smooth) connection $\nabla^{1,0} : \Gamma(E) \to \Omega^{1,0}(E)$ of type $(1,0)$:

$$\nabla^{1,0}(f \cdot s) = \partial(f) \cdot s + f \cdot \nabla^{1,0}(s), \quad \forall s \in \Gamma(E), f \in C^\infty(X;\mathbb{C})$$

- Then $\nabla^{1,0}\bar{\partial} + \bar{\partial}\nabla^{1,0} : \Gamma(E) \to \Omega^{1,1}(E)$ can be regarded as a Dolbeault 1-cocycle

$$\mathcal{R} \in \Omega^{1,1}(\mathrm{End}(E)).$$

Definition (Atiyah, 1957): The Atiyah class α_E of E is the cohomology class

$$\alpha_E = [\mathcal{R}] \in H^1\big(X; \Omega^1_X \otimes \mathrm{End}(E)\big)$$

The Atiyah class α_E captures the obstruction to the existence of a holomorphic connection on E.

Kapranov's theorem (1999)

Theorem (Kapranov): Let X be a Kähler manifold.
The Dolbeault complex $\Omega^{0,\bullet}(T_X^{1,0})$ admits an $L_\infty[1]$ algebra structure $(q_k)_{k\geq 1}$ where q_k is the wedge product

$$\Omega^{0,j_1}(T_X^{1,0}) \odot \cdots \odot \Omega^{0,j_k}(T_X^{1,0}) \to \Omega^{0,j_1+\cdots+j_k}(S^k(T_X^{1,0}))$$

composed with

$$R_k : \Omega^{0,\bullet}(S^k(T_X^{1,0})) \to \Omega^{0,\bullet+1}(T_X^{1,0})$$

where $\odot$ is the graded (w.r.t to $j_1, j_2, \cdots$) symmetric tensor product and

- $R_1 = \bar\partial$,
- $R_2 = \mathcal{R} = \mathcal{R}^{\nabla^{1,0}}$ is an Atiyah cocycle,
- $R_{n+1} = d^{\nabla^{1,0}}(R_n)$ for $n \geq 2$

Differential graded manifolds

Let M be a smooth manifold with structure sheaf $\mathcal{O}_M$.

Definition: A $\mathbb{Z}$-graded manifold $\mathcal{M}$ with body M is a sheaf $\mathcal{R}$ of $\mathbb{Z}$-graded commutative $\mathcal{O}_M$-algebras over M such that $\mathcal{R}(U) \cong \mathcal{O}_M(U)\hat{\otimes}\hat{S}(V^\vee)$ for all sufficiently small open subsets U of M — here $\hat{S}(V^\vee)$ denotes the algebra of formal power series on some fixed $\mathbb{Z}$-graded vector space V. $C^\infty(\mathcal{M}) := \mathcal{R}(M)$

Example: Given a $\mathbb{Z}$-graded vector bundle $E \to M$, we get a $\mathbb{Z}$-graded manifold:

$$\mathcal{R}(U) = \Gamma\big(U; \hat{S}(E^\vee)\big).$$

Definition: A dg manifold is a $\mathbb{Z}$-graded manifold $\mathcal{M}$ endowed with a vector field $Q \in \mathfrak{X}(\mathcal{M})$ of degree $+1$ such that $[Q, Q] = 2\ Q \circ Q = 0$.

We say that Q is a *homological* vector field on the graded manifold $\mathcal{M}$.

Examples of dg manifolds

Example: If $\mathfrak{g}$ is a Lie algebra, then $\mathcal{M} = \mathfrak{g}[1]$ is a dg-manifold.

- its algebra of functions: $C^\infty(\mathfrak{g}[1]) \cong \Lambda^\bullet \mathfrak{g}^\vee$
- its homological vector field: $Q = d_{\mathrm{CE}}$ (Chevalley–Eilenberg)

$C^\infty(\mathfrak{g}[1]) = \bigoplus_k (\Lambda^k \mathfrak{g}^\vee)[-k]$

Example: If M is a smooth manifold, then $\mathcal{M} = T_M[1]$ is a dg manifold.

- its algebra of functions: $C^\infty(T_M[1]) \cong \Omega^\bullet(M)$
- its homological vector field: $Q = d_{\mathrm{dR}}$ (de Rham)

$C^\infty(T_M[1]) = \bigoplus_k \Omega^k(M)[-k]$

Example: If X is a complex mfd, then $\mathcal{M} = T_X^{0,1}[1]$ is a dg manifold.

- its algebra of functions: $C^\infty(T_X^{0,1}[1]) \cong \Omega^{0,\bullet}(X)$
- its homological vector field: $Q = \bar{\partial}$ (Dolbeault operator)

$C^\infty(T_X^{0,1}[1]) = \bigoplus_k \Omega^{0,k}(X)[-k]$

Affine connection on a graded manifold

Definition: An affine connection on a graded mfd $\mathcal{M}$ is a $\mathbb{k}$-linear map

$$\nabla : \mathfrak{X}(\mathcal{M}) \otimes \mathfrak{X}(\mathcal{M}) \to \mathfrak{X}(\mathcal{M})$$

of degree 0 satisfying

$$\nabla_{fX} Y = f \nabla_X Y,$$
$$\nabla_X (fY) = X(f) Y + (-1)^{|X||f|} f \nabla_X Y,$$

for all homogeneous $f \in C^\infty(\mathcal{M})$ and $X, Y \in \mathfrak{X}(\mathcal{M})$.

Atiyah class of a dg manifold

- Choose a torsion-free affine connection

$$\Gamma(T_{\mathcal{M}}) \times \Gamma(T_{\mathcal{M}}) \xrightarrow{\nabla} \Gamma(T_{\mathcal{M}})$$

  on the graded manifold $\mathcal{M}$.
- Consider the section $\mathrm{At}^{\nabla}_{(\mathcal{M},Q)}$ of $\mathrm{Hom}\left(S^2(T_{\mathcal{M}}), T_{\mathcal{M}}\right)$ of degree $+1$ defined by

$$\mathrm{At}^{\nabla}_{(\mathcal{M},Q)}(X,Y) = \mathcal{L}_Q(\nabla_X Y) - \nabla_{\mathcal{L}_Q X} Y - (-1)^{|X|} \nabla_X(\mathcal{L}_Q Y)$$

  for all homogeneous $X, Y \in \mathfrak{X}(\mathcal{M})$.
- Since $\mathcal{L}_Q \circ \mathcal{L}_Q = 0$, $\mathrm{At}^{\nabla}_{(\mathcal{M},Q)} = \mathcal{L}_Q \nabla$ is a 1-cocycle of the cochain complex

$$\Big(\Gamma\big(\mathrm{Hom}(S^2(T_{\mathcal{M}}), T_{\mathcal{M}})\big), \mathcal{L}_Q\Big).$$

- Fact: Its cohomology class is independent of the connection ∇.

Definition: The Atiyah class of the dg manifold $(\mathcal{M}, Q)$

$$\alpha_{(\mathcal{M},Q)} := [\mathrm{At}^{\nabla}_{(\mathcal{M},Q)}] \in H^1\big(\Gamma(\mathrm{Hom}(S^2(T_{\mathcal{M}}), T_{\mathcal{M}})), \mathcal{L}_Q\big)$$

is the obstruction to existence of a connection on $\mathcal{M}$ compatible with the homological vector field Q.

A connection ∇ on a dg manifold $(\mathcal{M}, Q)$ is said to be *compatible* with the homological vector field if

$$\mathcal{L}_Q(\nabla_X Y) = \nabla_{\mathcal{L}_Q X} Y + (-1)^{|X|} \nabla_X(\mathcal{L}_Q Y)$$

for all homogeneous $X, Y \in \mathfrak{X}(\mathcal{M})$.

Example: dg manifold $(\mathbb{R}^{m|n}, Q)$

- $(x_1, \cdots, x_m; x_{m+1} \cdots x_{m+n})$ are coordinate functions on $\mathbb{R}^{m|n}$
- $Q = \sum_k Q_k(x) \frac{\partial}{\partial x_k}$
- trivial connection $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = 0$
- $\alpha_{\mathbb{R}^{m|n}} \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right) = (-1)^{|x_i|+|x_j|} \sum_k \frac{\partial^2 Q_k}{\partial x_i \partial x_j} \frac{\partial}{\partial x_k}$

Example: $\mathfrak{g}$ is a finite-dimensional Lie algebra

- $(\mathcal{M}, Q) = (\mathfrak{g}[1], d_{\mathrm{CE}})$ is corresponding dg manifold
- $T_{\mathcal{M}} \cong \mathfrak{g}[1] \times \mathfrak{g}[1]$ implies

$$H^1\big(\Gamma(S^2(T_{\mathcal{M}}^{\vee}) \otimes T_{\mathcal{M}}), \mathcal{L}_Q\big) \cong H^0_{\mathrm{CE}}(\mathfrak{g}; \Lambda^2 \mathfrak{g}^{\vee} \otimes \mathfrak{g}) \cong (\Lambda^2 \mathfrak{g}^{\vee} \otimes \mathfrak{g})^{\mathfrak{g}}$$

- $\alpha_{\mathfrak{g}[1]} \in (\Lambda^2 \mathfrak{g}^{\vee} \otimes \mathfrak{g})^{\mathfrak{g}}$ is precisely the Lie bracket of $\mathfrak{g}$

1 Kapranov: $L_\infty[1]$ algebra hiding behind Atiyah class of Kähler mfd

2 Atiyah class of a dg manifold

3 Formal exponential maps
- ordinary manifolds
- graded manifolds
- differential graded manifolds

4 Formal exponential map on a dg mfd and Kapranov's $L_\infty[1]$ algebra

Exponential maps arise naturally in relation with linearization problems:

1. Lie theory
2. smooth manifolds

PBW isomorphism in Lie theory

- $\mathfrak{g}$, a finite dimensional Lie algebra
- $\exp : \mathfrak{g} \to G$
- $\exp$ is a local diffeomorphism from nbhd of 0 to nbhd of 1
- induced map on distributions $(\exp)_* : \mathcal{D}'(0) \xrightarrow{\cong} \mathcal{D}'(1)$
- canonical identifications: $\mathcal{D}'(0) \cong S\mathfrak{g}$ and $\mathcal{D}'(1) \cong U\mathfrak{g}$
- $(\exp)_* : S\mathfrak{g} \xrightarrow{\cong} U\mathfrak{g}$ is the symmetrization map

$$X_1 \odot \cdots \odot X_n \longmapsto \frac{1}{n!} \sum_{\sigma \in S_n} X_{\sigma(1)} \cdots X_{\sigma(n)}$$

- Poincaré–Birkhoff–Witt isomorphism
- $S\mathfrak{g} \xrightarrow{\mathrm{pbw}:=(\exp)_*} U\mathfrak{g}$ is an isomorphism of **co**algebras but not a morphism of *algebras*.

Geodesic exponential map and PBW isomorphism

- Choose an affine connection ∇ on smooth manifold M.
- $\exp : T_M \to M \times M$ (bundle map)
  defined by $\exp(X_m) = (m, \gamma(1))$ where γ is the smooth path in M satisfying $\dot{\gamma}(0) = X_m$ and $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$
- $\Gamma(S(T_M))$ seen as space of differential operators on T_M, all derivatives in the direction of the fibers, evaluated along the zero section of T_M
- $\mathcal{D}(M)$ seen as space of differential operators on $M \times M$, all derivatives in the direction of the fibers, evaluated along the diagonal section $M \to M \times M$
- map induced by $\exp$ on fiberwise differential operators:
  $\mathrm{pbw} := (\exp)_* : \Gamma(S(T_M)) \xrightarrow{\cong} \mathcal{D}(M)$ is an isomorphism of left modules over $C^\infty(M)$ called Poincaré–Birkhoff–Witt isomorphism

pbw as infinite jet of exp

The Taylor series of the composition

$$T_m M \xrightarrow{\exp} \{m\} \times M \xrightarrow{f} \mathbb{R}$$

at the point $0_m \in T_m M$ is

$$\sum_{J \in \mathbb{N}_0^n} \frac{1}{J!} \big(\operatorname{pbw}(\partial_x^J) f \big)(m) \cdot y^J \quad \in \hat{S}(T_m^\vee M),$$

where

- $(x_i)_{i \in \{1,\dots,n\}}$ are local coordinates on M
- $(y_j)_{j \in \{1,\dots,n\}}$ induced local frame of $T_M^\vee$ regarded as fiberwise linear functions on T_M

Hence pbw is the fiberwise infinite jet of the bundle map $\exp : T_M \to M \times M$ along the zero section of $T_M \to M$.

Algebraic characterization of pbw

Theorem (Laurent-Gengoux, S, Xu, 2014): This map

$$\Gamma(ST_M) \xrightarrow{\mathrm{pbw}} \mathcal{D}(M)$$

is the unique isomorphism of left $C^\infty(M)$-modules satisfying

$$\begin{aligned} \mathrm{pbw}(f) &= f, \quad \forall f \in C^\infty(M); \\ \mathrm{pbw}(X) &= X, \quad \forall X \in \mathfrak{X}(M); \\ \mathrm{pbw}(X^{n+1}) &= X \cdot \mathrm{pbw}(X^n) - \mathrm{pbw}(\nabla_X X^n), \quad \forall n \in \mathbb{N}. \end{aligned}$$

Equivalently, for all $n \in \mathbb{N}$ and $X_0, \dots, X_n \in \mathfrak{X}(M)$, we have

$$\mathrm{pbw}(X_0 \odot \cdots \odot X_n) = \tfrac{1}{n+1} \sum_{k=0}^{n} \Big\{ X_k \cdot \mathrm{pbw}(X^{\{k\}}) - \mathrm{pbw}\left(\nabla_{X_k}(X^{\{k\}})\right) \Big\}$$

where $X^{\{k\}} = X_0 \odot \cdots \odot X_{k-1} \odot X_{k+1} \odot \cdots \odot X_n$.

Example:

- $M = G$ (Lie group)
- Let $X^L \in \mathfrak{X}(G)$ denote the left invariant vector field associated with a vector $X \in \mathfrak{g}$ in the Lie algebra.
- Consider the torsion-free connection ∇ defined by

$$\nabla_{X^L} Y^L = \frac{1}{2}[X, Y]^L \qquad \forall X, Y \in \mathfrak{g}.$$

- The associated formal exponential map is

$$\operatorname{pbw}(X_1 \odot \cdots \odot X_n) = \frac{1}{n!} \sum_{\sigma \in S_n} X^L_{\sigma(1)} \cdots X^L_{\sigma(n)}.$$

Both $\Gamma(S(T_M))$ and $\mathcal{D}(M)$ are left coalgebras over $C^\infty(M)$.

Comultiplication by deconcatenation in both $\Gamma(S(T_M))$ and $\mathcal{D}(M)$:

$$\begin{aligned}\Delta(X_1\cdots X_n) = 1\otimes(X_1\cdots X_n) \\ + \sum_{\substack{p+q=n \\ p,q\in\mathbb{N}}}\sum_{\sigma\in\mathfrak{S}_p^q}(X_{\sigma(1)}\cdots X_{\sigma(p)})\otimes(X_{\sigma(p+1)}\cdots X_{\sigma(n)}) \\ + (X_1\cdots X_n)\otimes 1\end{aligned}$$

for all $X_1,\dots,X_n\in\mathfrak{X}(\mathcal{M})$.

Proposition: $\mathrm{pbw}:\Gamma(S(T_M))\to\mathcal{D}(M)$ is an isomorphism of coalgebras over $C^\infty(M)$.

- $\mathrm{pbw}^{-1} : \mathcal{D}(M) \to \Gamma(S(T_M))$ takes a differential operator to its *complete symbol*
- both $\Gamma(S(T_M))$ and $\mathcal{D}(M)$ are bi-algebroids
- pbw preserves comultiplication but does not respect multiplication
- Unlike $\exp$, the formal exponential map pbw can be evaluated (recursively) without resorting to points of M and geodesic curves of ∇.

Formal exponential maps on graded manifolds

WHAT ABOUT REPLACING THE SMOOTH MANIFOLD M BY A DIFFERENTIAL GRADED MANIFOLD $\mathcal{M}$?

Definition (Liao, Mehta, S, Xu): Let $\mathcal{M}$ be a graded manifold. The formal exponential map associated to an affine connection ∇ on $\mathcal{M}$ is the morphism of left $C^\infty(\mathcal{M})$-modules

$$\mathrm{pbw} : \Gamma(S(T_{\mathcal{M}})) \to \mathcal{D}(\mathcal{M}),$$

inductively defined by the relations

$$\mathrm{pbw}(f) = f \qquad \forall f \in C^\infty(\mathcal{M}),$$
$$\mathrm{pbw}(X) = X \qquad \forall X \in \Gamma(T_{\mathcal{M}}),$$

and, for all $n \in \mathbb{N}$ and homogeneous $X_0, \dots, X_n \in \Gamma(T_{\mathcal{M}})$,

$$\mathrm{pbw}(X_0 \odot \cdots \odot X_n) = \tfrac{1}{n+1} \sum_{k=0}^{n} \epsilon_k \left\{ X_k \cdot \mathrm{pbw}(X^{\{k\}}) - \mathrm{pbw}\left(\nabla_{X_k} X^{\{k\}}\right) \right\}.$$

- $\epsilon_k = (-1)^{|X_k|(|X_0|+\cdots+|X_{k-1}|)}$
- $X^{\{k\}} = X_0 \odot \cdots \odot X_{k-1} \odot X_{k+1} \odot \cdots \odot X_n$

Proposition (Liao, S): The formal exponential map

$$\mathrm{pbw} : \Gamma(S^{\leqslant k}(T_{\mathcal{M}})) \to \mathcal{D}^{\leqslant k}(\mathcal{M})$$

is a well defined isomorphism of filtered coalgebras over $C^\infty(\mathcal{M})$.

Formal exponential map on diff'l graded manifolds

Given a dg manifold $(\mathcal{M}, Q)$,
we get two induced differential graded coalgebras:

- $(\Gamma(S(T_{\mathcal{M}})), L_Q)$
- $(\mathcal{D}(\mathcal{M}), \mathcal{L}_Q := [\![Q, -]\!])$

Question: When is

$$(\Gamma(S(T_{\mathcal{M}})), L_Q) \xrightarrow{\text{pbw}} (\mathcal{D}(\mathcal{M}), \mathcal{L}_Q)$$

an isomorphism of differential graded coalgebras?

Theorem (Seol, S, Xu): The Atiyah class $\alpha_{(\mathcal{M},Q)}$ vanishes if and only if there exists a torsion-free affine connection ∇ on $\mathcal{M}$ such that

$$\mathrm{pbw} : (\Gamma(S(T_{\mathcal{M}})), L_Q) \to (\mathcal{D}(\mathcal{M}), \mathcal{L}_Q)$$

is an isomorphism of differential graded coalgebras over $C^\infty(\mathcal{M})$.

In general, the failure of pbw to preserve the dg structure is measured by

$$(\mathrm{pbw})^{-1} \circ \mathcal{L}_Q \circ \mathrm{pbw} - L_Q = \sum_{k=0}^{\infty} R_k$$

where $R_k \in \Gamma(\mathrm{Hom}(S^k(T_{\mathcal{M}}), T_{\mathcal{M}}))$ are sections of degree $+1$.

$R_0 = 0, \quad R_1 = 0, \quad R_2 = -\mathrm{At}^{\nabla}$

Theorem (Seol, S, Xu):

1. The R_k for $k \geq 2$, together with L_Q induce an $L_\infty[1]$ algebra on the space of vector fields $\mathfrak{X}(\mathcal{M})$.
2. The R_k for $k \geq 2$ are completely determined by the Atiyah cocycle At^{∇}, the curvature R^{∇}, and their exterior derivatives.
   In particular, if the curvature vanishes (i.e. $R^{\nabla} = 0$), then

$$R_2 = -\mathrm{At}^{\nabla}, \quad R_{n+1} = \frac{1}{n+1} d^{\nabla} R_n \quad \text{for } n \geq 2$$

Theorem (Seol, S, Xu):

- Given a Kähler manifold X,
- $(\mathcal{M}, Q) = (T_X^{0,1}[1], \bar{\partial})$ is a dg manifold
- and $\mathfrak{X}(\mathcal{M}) = \mathfrak{X}(T_X^{0,1}[1])$ admits an $L_\infty[1]$ algebra structure.
- There is an $L_\infty[1]$ quasi-isomorphism

$$(\mathfrak{X}(T_X^{0,1}[1]), \{R_i\}) \xrightarrow{L_\infty[1] \text{ quasi iso.}} (\Omega^{0,\bullet}(T_X^{1,0}), \{\lambda_i\})$$

Moreover, our $L_\infty[1]$ algebra structure on $\mathfrak{X}(T_X^{0,1}[1])$ can be transferred to Kapranov's $L_\infty[1]$ algebra structure on $\Omega^{0,\bullet}(T_X^{1,0})$.

Thank You

[1] Zhuo Chen, Maosong Xiang, and Ping Xu. "Atiyah and Todd classes arising from integrable distributions." In: *J. Geom. Phys.* 136 (2019), pp. 52–67. DOI: 10.1016/j.geomphys.2018.10.011.

[2] Camille Laurent-Gengoux, Mathieu Stiénon, and Ping Xu. "Poincaré-Birkhoff-Witt isomorphisms and Kapranov dg-manifolds." In: *Adv. Math.* 387 (2021), Paper No. 107792, 62 pp. DOI: 10.1016/j.aim.2021.107792.

[3] Hsuan-Yi Liao and Mathieu Stiénon. "Formal exponential map for graded manifolds." In: *Int. Math. Res. Not. IMRN* 3 (2019), pp. 700–730. DOI: 10.1093/imrn/rnx130.

[4] Rajan A. Mehta, Mathieu Stiénon, and Ping Xu. "The Atiyah class of a dg-vector bundle." In: *C. R. Math. Acad. Sci. Paris* 353.4 (2015), pp. 357–362. DOI: 10.1016/j.crma.2015.01.019.

[5] Seokbong Seol, Mathieu Stiénon, and Ping Xu. "Dg Manifolds, Formal Exponential Maps and Homotopy Lie Algebras." In: *Comm. Math. Phys.* 391.1 (2022), pp. 33–76. DOI: 10.1007/s00220-021-04265-x.

[6] Mathieu Stiénon and Ping Xu. "Atiyah classes and Kontsevich-Duflo type theorem for DG manifolds." In: *Homotopy algebras, deformation theory and quantization*. Vol. 123. Banach Center Publ. Polish Acad. Sci. Inst. Math.,