# $L_\infty$ extensions for the Poisson algebra of a symplectic manifold

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# References


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# The Problem

**Theorem (JV)**

Let $(M,\omega)$ be symplectic. Then the *universal central extension* of the Fréchet Lie algebra $\mathfrak{X}_{Ham}(M,\omega)$ of Hamiltonian vector fields is given by

$$\frac{\Omega^1(M)}{\delta\Omega^2(M)}$$

where $\delta = \pm *_\omega d *_\omega$ is the Poisson homology differential.

And when we see a Lie algebra on cohomology ... we can not avoid looking for an $L_\infty$ algebra on the complex.

**Question**

Is there an according $L_\infty$-algebra on $(\Omega^{\geq 1}(M),\delta)$, and if yes how does it look ?

$$\Omega^{\geq 1}(M)$$

$$\downarrow$$

$$\frac{\Omega^{(}M)}{\delta\Omega^{2}(M)}$$

$$\downarrow$$

$$C^{\infty}(M)$$

Image credit : Anna Marklova, instagram : @sophiehardy5

$L_\infty$-algebras

Example : The Observable algebra of a multisymplectic manifold

Operators in (symplectic) Poisson homology

The $L_\infty$-algebra of a symplectic manifold

# What is an $L_\infty$-algebra ?

**Definition : $L_\infty$-algebra**

$L_i, l_i$, with $\{L_i\} = L_\bullet$ graded vector space and $l_i : \Lambda^i L_\bullet \to L_\bullet$, $i \geq 1$ of degrees $2-i$, such that

$$0 = \sum_{\substack{i+j=n+1 \\ \sigma \in ush(i,n-i)}} \pm l_j(l_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)} \dots, x_{\sigma(n)})$$

$\pm = (-1)^{i(j+1)} sgn(\sigma) \epsilon(\sigma; x)$

- generalization of differential graded Lie algebras
- $Q$-manifolds over a point
- graded coderivations squaring to zero
- Lie algebras up to homotopy

# The transfer theorem

**Transfer Theorem (LV)**

Let $L_\bullet$ be an $L_\infty$-algebra and $f : L_\bullet \to R_\bullet$ a quasi-isomorphism of complexes. Then there is an $L_\infty$-algebra structure on $R_\bullet$ and an $L_\infty$-morphism $F : L_\bullet \to R_\bullet$ refining $f$.

**Theorem (BFLS)**

Let $L_\bullet = (\ldots \to L_1 \to L_0 \to \mathcal{F})$ be a resolution of the Lie algebra $\mathcal{F}$.

- A skew-symmetric $l_2 : L_0 \times L_0 \to L_0$ covering the Lie bracket on $\mathcal{F}$ can be extended to an $L_\infty$-algebra structure on $L_\bullet$.
- If $l_2$ is zero on boundaries, then the structure can be chosen such that $l_i$ are non-zero only on $L_0$.

This still can work when the complexes are not trival cohomologically.

# The multisymplectic $L_\infty$-algebra

**Theorem (Rogers)**

Let $(M, \omega)$ be multisymplectic of degree $k+1$. The following forms an $L_\infty$-algebra :

- $L_0 = \{\alpha \mid \exists X_\alpha \in \mathfrak{X}(M) : d\alpha = -\iota_{X_\alpha}\omega\} \subset \Omega^{k-1}(M)$
- $L_{-i} = \Omega^{k+1-i}(M)$ for $i > 1$.
- $l_1 = d$
- For $i > 1$, $l_i : \bigwedge^i L_0 \to L_{2-i}$,

  $l_i(\alpha_1, ..., \alpha_i) = -(-1)^{\frac{i(i+1)}{2}} \omega(X_{\alpha_1}, ..., X_{\alpha_i}, \cdot, \cdots, \cdot)$

This algebra has a very specific type :

- The complex $L$ is degree-bounded in between 0 and $-k+1$.
- Except for $l_1$, only bracket starting in $\Lambda^i L_0$ are non-trivial.
- The equations boil down to $l_1 \circ l_{i+1} = l_i \circ \partial_{l_2}$.

# Operators for the Poisson homology of a symplectic manifold

Let $M, \omega$ be symplectic of dimension $2n$, then there are the following operators on the de Rham complex

$$L := \omega\wedge, \qquad \Lambda := \iota_\pi, \qquad H := (n - deg(\cdot))\cdot$$

Then we have, for $\delta = [\Lambda, d]$

$$\begin{array}{rrr} [\Lambda, L] = H & [H, \Lambda] = 2\Lambda & [H, L] = -2L \\ [L, d] = 0 & [\Lambda, d] = \delta & [H, d] = -d \\ [\Lambda, \delta] = 0 & [L, \delta] = d & [H, \delta] = \delta \end{array}$$

Universal central extension formulas :

$$p : \frac{\Omega^1(M)}{\delta\Omega^2(M)} \to \mathfrak{X}_{Ham}(M), p(\alpha) = X_{\delta\alpha}$$

$$[\alpha, \beta] := (\delta\alpha d\delta\beta)$$

# First steps to the solution

- BFLS implies that it suffices to find maps $\tilde{l}_k : \Lambda^k C^\infty(M) \to \Omega^{k-1}(M)$ such that $\tilde{l}_k \circ \partial_{\{\cdot,\cdot\}} = \delta \tilde{l}_{k+1}$.
- Very naive approach : $m_k : \bigotimes^k C^\infty(M) \to \Omega^{k-1}(M)$, given by $m_k(f_1, ..., f_k) = f_1 df_2 \wedge ... \wedge df_k$
- Naive approach : $Alt(m_k) = \Lambda^k C^\infty(M) \to \Omega^{k-1}(M)$ $Alt(m_k)(f_1, ..., f_k) = \frac{1}{k} \sum_{i=1}^k (-1)^{i+1} f_i df_1 \wedge ... \widehat{df_i} ... \wedge df_k$

Problem :

$$Alt(m_k) \circ \partial_{\{\cdot,\cdot\}} = (-\delta + \frac{1}{k} d\Lambda) Alt(m_{k+1})$$

# Repairing the lower degrees

$$m_2 \circ \partial_{\{\cdot,\cdot\}} = (-\delta + \frac{1}{2} d\Lambda) Alt(m_3)$$
$$= (-\delta - \frac{1}{2}\delta L\Lambda) Alt(m_3) = \delta \circ (-id - \frac{1}{2} L\Lambda) Alt(m_3).$$

So we can set

$$\tilde{l}_3 = (-id - \frac{1}{2} L\Lambda) Alt(m_3).$$

We go on calculating

$$\tilde{l}_3 \circ \partial_{CE} = (-id - \frac{1}{2} L\Lambda) \circ Alt(m_3) \circ \partial_{CE}$$
$$= ... = (\delta + \frac{1}{2}\delta L\Lambda - \frac{1}{6}\delta L\Lambda) Alt(m_4) = \delta(id + \frac{1}{3} L\Lambda) Alt(m_4)$$

Hence, we can set

$$\tilde{l}_4 = (id + \frac{1}{3} L\Lambda) Alt(m_4)$$

# Repairing the lower degrees (2)

- Less naive approach : $\tilde{l}_k = \pm(id + \frac{1}{k-1}L\Lambda)Alt(m_k)$

However,

$$\tilde{l}_4 \circ \partial_{CE} = \delta(-id - \frac{1}{4}L\Lambda - \frac{1}{24}L^2\Lambda^2)Alt(m_5)$$

$$\tilde{l}_5 \circ \partial_{CE} = \delta(id + \frac{1}{5}L\Lambda + \frac{1}{40}L^2\Lambda^2)Alt(m_6)$$

But

$$\tilde{l}_7 = -(id + \frac{1}{6}L\Lambda + \frac{1}{60}L^2\Lambda^2 + \frac{1}{something}L^3\Lambda^3)Alt(m_7)$$

# The solution

We set

$$\tilde{l}_k = (-1)^k \left( \sum_{l \geq 0} a_k^l L^l \Lambda^l \right) Alt(m_k)$$

with

$$a_k^l = \frac{(k-l-1)!}{(k-1)!l!}.$$

**Theorem (JRV)**

The brackets $l_k = \tilde{l}_k \circ \delta_* : \Lambda^k \Omega^1(M) \to \Omega^{k-1}(M)$ together with $\delta$ define an $L_\infty$-algebra, $L_\bullet$, such that the natural map $L_\bullet \to \frac{\Omega^1(M)}{\delta \Omega^2(M)}$ is an $L_\infty$-homomorphism.

Thank you a lot for your attention !