Pfaffian Morita equivalence and geometric structures

Noncommutative Geometry and Higher Structures, Scalea

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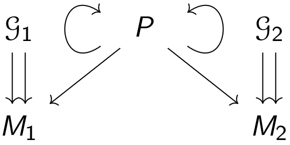
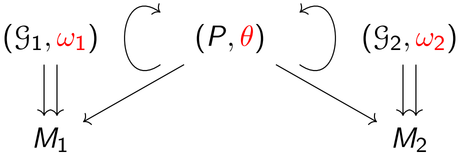
Morita equivalence between Lie groupoids = principal bibundle

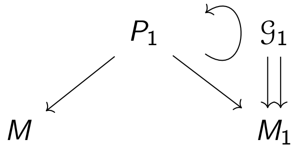


Geometric Morita equivalence between geometric Lie groupoids = principal geometric bibundle?

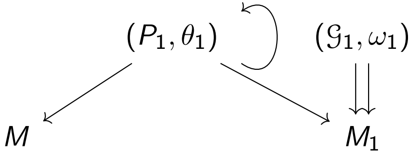


Motivating example (Xu, '91): symplectic Morita equivalence between symplectic groupoids

Morita equivalence preserves (among many other objects and properties!) the category of principal bundles



Geometric Morita equivalence preserves the category of geometric principal bundles?



Motivating example: Hamiltonian principal bundles

Definition (Salazar, '12)

A **Pfaffian groupoid** consists of a Lie groupoid $\mathcal{G} \rightrightarrows M$ together with a representation E and a **multiplicative** form $\omega \in \Omega^1(\mathcal{G}, t^*E)$ such that

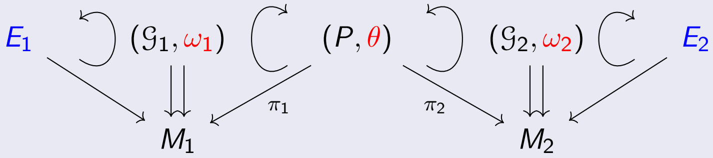
- $T\mathcal{G} = \ker(ds) + \ker(\omega)$
- $\mathcal{S} := \ker(\omega) \cap \ker(ds) \subseteq T\mathcal{G}$ involutive distribution
- $\ker(\omega) \cap \ker(ds) = \ker(\omega) \cap \ker(dt)$

The **symbol bundle** $\mathcal{S}_{|M} \subseteq A$ satisfies $0 \subseteq \mathcal{S}_{|M} \subseteq \ker(\rho)$ (Lie ideal).

Examples

- (G, ω_{MC}), for $E = \mathfrak{g}$ ($\mathcal{S} = 0$)
- $(G, 0)$, for $E = V$ any representation ($\mathcal{S} = \mathfrak{g}$)
- $(\mathcal{G} := J^1\Gamma, \omega)$, for $\Gamma \subseteq \mathrm{Diff}_{loc}(M)$ Lie pseudogroup, $E = TM$ and ω restriction of the Cartan form of the jet bundles
- $J^\infty\Gamma$ (more subtleties and technicalities)

Definition (**Pfaffian Morita equivalence**)



θ multiplicative ("$\mathrm{act}_1^*\theta = \mathrm{pr}_1^*\omega_1 + \mathrm{pr}_2^*\theta$", "$\mathrm{act}_2^*\theta = \mathrm{pr}_1^*\theta + \mathrm{pr}_2^*\omega_2$")

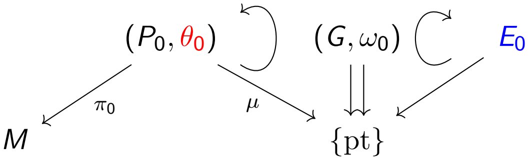
$\ker(\theta) \cap \ker(d\pi_1) = \ker(\theta) \cap \ker(d\pi_2)$ involutive

$\pi_1^* E_1 \cong \pi_2^* E_2$ as $(\mathcal{G}_1 \ltimes P)$- and $(P \rtimes \mathcal{G}_2)$-representations

Theorem (Accornero, C.)

Pfaffian Morita equivalence preserves the category of **principal Pfaffian bundles**.

We "simplify" principal Pfaffian bundles for transitive Lie groupoids (via the ME to their isotropy)



Theorem (Accornero, C.)

The form $\omega_0 = \omega_{\theta_0, G}$ is fully determined by θ_0 and G...
... and the new principal Pfaffian G-bundle $\pi_0 : (P_0, \theta_0) \to M$ satisfies

- θ_0 *is pointwise surjective*
- θ_0 *is G-equivariant*
- $\ker(\theta_0) \subseteq \ker(d\pi_0)$
- $\ker(\theta_0) \subseteq TP_0$ *is involutive*
- $(\theta_0)_p(v_p^\dagger)$ *is independent from* $p \in P_0$

Definition

A **Cartan bundle** is a principal G-bundle $\pi : P \to M$ together with a representation $V \in \mathrm{Rep}(G)$ and a form $\theta \in \Omega^1(P, V)$ such that

- θ is pointwise surjective
- θ is G-equivariant
- $0 \subseteq \ker(\theta) \subseteq \ker(d\pi)$
- $\ker(\theta) \subseteq TP$ is involutive
- $\theta_p(v_p^\dagger)$ is independent from $p \in P$

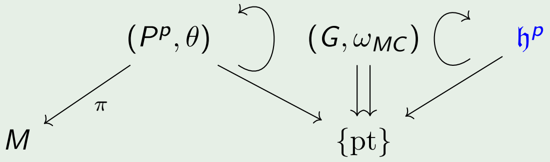
$$\dim(M) \leq \dim(V) \leq \dim(P)$$

$\ker(\theta)$ foliation of rank $\dim(P) - \dim(V)$ given by the orbits of the infinitesimal action by the Lie ideal $\mathfrak{k} := \{\alpha \in \mathfrak{g} \mid \alpha^\dagger \in \Gamma(\ker(\theta))\}$

Theorem (C., '21)

Cartan bundles $\Leftrightarrow$ transitive Pfaffian groupoids

Example (Cartan geometries)



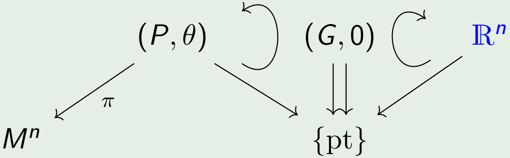
θ **Cartan connection** modelled on $\mathfrak{h} \supseteq \mathfrak{g}$

- $\ker(\theta) = 0$
- $V = \mathfrak{h}$
- $\theta(v^\dagger) = v$
- $\mathfrak{k} = 0$ (defines the foliation by points on P)

Theorem (C., '21)

Fixing a model, Cartan geometries $\Leftrightarrow$ transitive Pfaffian groupoids with zero symbol $\mathcal{S} = 0$

Example (G-structures)



θ **tautological form** of $Fr(M)$, $G \subseteq GL(n, \mathbb{R})$

- $\ker(\theta) = \ker(d\pi)$
- $V = \mathbb{R}^n$
- $\theta(v^\dagger) = 0$
- $\mathfrak{k} = \mathfrak{g}$ (defines the vertical foliation on P)

Theorem (Salazar, '12)

G-structures $\Leftrightarrow$ transitive Pfaffian groupoids with maximal symbol $\mathcal{S} = \ker(ds) \cap \ker(dt)$

"Intermediate" example ($0 \subsetneq \ker(\theta) \subsetneq \ker(d\pi)$, $0 \subsetneq \mathfrak{k} \subsetneq \mathfrak{g}$)

Higher order structures $P^k \subseteq Fr^k(M)$, with $G^k \subseteq GL^k(n, \mathbb{R})$ and $\theta^k \in \Omega^1(P^k, \mathbb{R}^n \oplus \mathfrak{h}^{k-1})$

Integrability for Cartan bundles (in progress)

- Integrability for Pfaffian $J^1\Gamma$-bundles is not Morita invariant!
- $J^1\Gamma_{sympl-fol}$ and $J^1\Gamma_{cont}$ are Pfaffian Morita equivalent but integrate to codim 1 symplectic foliations and contact structures
- New notion of integrability involving the choice of a Lie bracket on V and a Lie algebra extension $\mathfrak{a} = \mathfrak{k} \oplus V$ of $\mathfrak{k}$ by V
- For V abelian algebra or Heisenberg algebra, recover "ordinary" integrability and twisted integrability (Albert, Molino, '84)
- For $\mathfrak{k} = 0$, recover flatness of a Cartan geometry (the curvature $\Theta := d\theta + \frac{1}{2}[\theta, \theta] \in \Omega^2(P, \mathfrak{h})$ vanishes)

Thank you for your attention