Universal enveloping algebras of Lie-Rinehart algebras: connections as crossed products

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Broad motivation

Higher-spin gravity stands for a field theory including in its spectrum a metric (describing the "graviton") together with at least one symmetric tensor gauge field of rank strictly greater than two (describing a massless particle of "higher spin", i.e. higher than two).

Broad motivation

The geometries underlying the gauge theories of massless particles with "low" spins are well known, however the geometry underlying higher-spin gravity remains elusive.

Spin	Theory	Geometry	Field	Gauge syms
1	Yang-Mills	Principal bundles	Connection	Vertical autom.
2	Einstein	Riemannian	Metric	Diffeomorphisms
$1, 2, 3, \ldots$	HS gravity	?	Sym. tensors	HS symmetries

Broad motivation

Nevertheless, several general lessons (or recipes) are known about higher-spin generalisations of these symmetries, among which :

- Extension from low-spin to higher-spin symmetry algebras:

$$\text{Finite-dim. Lie algebra } \mathfrak{g} \longrightarrow \text{Universal enveloping algebra } U(\mathfrak{g})$$

- Extension from spacetime symmetries to higher-spin symmetries:

$$\text{Vector fields} \longrightarrow \text{Differential operators}$$

These two recipes can be unified as follows:

$$\text{Projective Lie-Rinehart algebra } \mathfrak{L} \text{ over } A$$
$$\downarrow$$
$$\text{Universal enveloping algebra } \mathcal{U}_A(\mathfrak{L})$$

since the above two examples correspond respectively to

- $A = \mathbb{R}$ and $\mathfrak{L} = \mathfrak{g}$: $\mathcal{U}_{\mathbb{R}}(\mathfrak{g}) = U(\mathfrak{g})$
- $A = C^\infty(M)$ and $\mathfrak{L} = \mathfrak{X}(M)$: $\mathcal{U}_{C^\infty(M)}\big(\mathfrak{X}(M)\big) = \mathcal{D}(M)$

Mathematical problem under scrutiny

Question: Does there exist a generalisation for Lie-Rinehart algebras (at least in the projective case, with the geometric example of Lie algebroids in mind) of the two following classical results for Lie algebras?

- **semidirect sum (Lie) $\Leftrightarrow$ smash product (Associative)**
  Let $\mathfrak{g} = \mathfrak{i} \ni \mathfrak{h}$ be a Lie algebra which is a semidirect sum of an ideal $\mathfrak{i} \subseteq \mathfrak{g}$ and a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. Then, as an associative algebra,

$$U(\mathfrak{i} \ni \mathfrak{h}) \simeq U(\mathfrak{i}) \# U(\mathfrak{h}).$$

- **split extension (Lie) "$\Leftrightarrow$" crossed product (Associative)**
  Let $\mathfrak{g}$ be a Lie algebra extension of $\mathfrak{h}$ by $\mathfrak{i}$, *i.e.*, a short exact sequence of Lie algebras

$$0 \to \mathfrak{i} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{h} \to 0.$$

  Then, as an associative algebra,

$$U(\mathfrak{g}) \simeq U(\mathfrak{i}) * U(\mathfrak{h}).$$

  This isomorphism defines/relies on a $\mathbb{K}$-linear splitting $\mathfrak{h} \hookrightarrow \mathfrak{g}$ of the short exact sequence.

Quick reminder of smash product

Let H be a bialgebra with counit denoted ϵ.

Definition (left H-module algebra)

An algebra A which is an H-module via an action $H \otimes A \to A : h \otimes a \mapsto h \triangleright a$ of H on A measuring A to A, i.e.

$$h \triangleright (a_1 a_2) = (h_{(1)} \triangleright a_1)(h_{(2)} \triangleright a_2) \quad \text{and} \quad h \triangleright 1_A = \epsilon(h) 1_A \,,$$

is called an H-module algebra.

Definition (smash product)

Let A be an H-module algebra. The smash product $A\#H$ is the tensor product $A \otimes H$ endowed with a structure of associative algebra via the product

$$(a \otimes h)(a' \otimes h') = (a\, h_{(1)} \triangleright a') \otimes (h_{(2)} h') \,.$$

Quick reminder of crossed product

Remark: The crossed product is just a more sophisticated version ("twisted" by a "Hopf cocycle") of the smash product when $\triangleright$ is only a weak action of H on A:

$$(a \otimes h)(a' \otimes h') = (a\, h_{(1)} \triangleright a')\, \sigma(h_{(2)}, h'_{(1)}) \otimes (h_{(3)} h'_{(2)})\,,$$

where $\sigma \in \mathrm{Hom}(H \otimes H, A)$ must obey some conditions in order for this product to be associative and unital.

Geometric motivation

Generalisation: It is well-known that a flat (or curved) connection on a transitive Lie algebroid is equivalent to a decomposition of the Lie-Rinehart algebra of its global sections as a ("curved") semidirect sum.

Question: Does the universal enveloping algebra of the latter semidirect sum of Lie-Rinehart algebras factorise as a smash (or crossed) product?

Geometric motivation

Example: Consider a principal H-bundle of total space P and base space B. An invariant Ehresmann connection on P is equivalent to a splitting of the Atiyah sequence

$$0 \to \frac{VP}{H} \hookrightarrow \frac{TP}{H} \twoheadrightarrow TB \to 0\,,$$

i.e. in terms of Lie-Rinehart algebras of global sections,

$$\mathfrak{X}(P)^H \simeq \mathfrak{V}(P)^H \niplus \mathfrak{X}(B)\,.$$

Question: Does a flat (or curved) invariant Ehresmann connection on a principal bundle provide a factorisation of the associative algebra $\mathcal{D}(P)^H$ of invariant differential operators on the total space as a smash (or crossed) product of the associative algebras $\mathcal{V}(P)^H$ and $\mathcal{D}(B)$ respectively spanned by invariant differential operators tangential to the fibre and by differential operators on the base manifold?

$$\mathcal{D}(P)^H \overset{?}{\simeq} \mathcal{V}(P)^H \# \mathcal{D}(B)$$

Main result

Theorem (XB-NK-PS)

Let A be a commutative algebra. If

$$0 \to \mathfrak{i} \overset{\iota}{\hookrightarrow} \mathfrak{g} \overset{\pi}{\twoheadrightarrow} \mathfrak{h} \to 0$$

is a split short exact sequence of Lie-Rinehart algebras over A, which are projective as left A-modules, then we have an isomorphism of A-rings and right $\mathcal{U}_A(\mathfrak{h})$-comodule algebras:

$$\mathcal{U}_A(\mathfrak{g}) \simeq U_A(\mathfrak{i}) \#_\sigma \mathcal{U}_A(\mathfrak{h}),$$

where σ is a suitable $U_A(\mathfrak{i})$-valued Hopf 2-cocycle and $\#_\sigma$ denotes the corresponding crossed product.

Main result

More precisely, the technical answer to the initial question is:

- If $\mathfrak{g} \simeq \mathfrak{i} \ni \mathfrak{h}$ is a semi-direct sum of the A-Lie algebra $\mathfrak{i}$ and of the Lie-Rinehart algebra $\mathfrak{h}$ over A, then we have an isomorphism of A-rings and right $\mathcal{U}_A(\mathfrak{h})$-comodule algebras:

$$\mathcal{U}_A(\mathfrak{i} \ni \mathfrak{h}) \simeq U_A(\mathfrak{i}) \# \mathcal{U}_A(\mathfrak{h}).$$

- If $\mathfrak{g} \simeq \mathfrak{i} \ni_\tau \mathfrak{h}$ is a curved semi-direct sum of the A-Lie algebra $\mathfrak{i}$ and of the Lie-Rinehart algebra $\mathfrak{h}$ over A, then we have an isomorphism of A-rings and right $\mathcal{U}_A(\mathfrak{h})$-comodule algebras

$$\mathcal{U}_A(\mathfrak{i} \ni_\tau \mathfrak{h}) \simeq U_A(\mathfrak{i}) \#_\sigma \mathcal{U}_A(\mathfrak{h}),$$

  where τ is a Lie cocycle and σ is a Hopf cocycle.

Sample of some definitions

Let A be a commutative algebra over a field $\mathbb{K}$.

Definition (Lie-Rinehart algebra)

A Lie-Rinehart algebra over A is a Lie algebra $\mathfrak{g}$ endowed with a (left) A-module structure $A \otimes \mathfrak{g} \to \mathfrak{g}$, $a \otimes X \mapsto a \cdot X$, and with a Lie algebra morphism $\rho : \mathfrak{g} \to \mathfrak{Der}_{\mathbb{K}}(A)$ such that

$$\rho(a \cdot X) = a \cdot \rho(X) \qquad \text{and} \qquad [X, a \cdot Y] = a \cdot [X, Y] + \Big(\rho(X)(a)\Big) \cdot Y$$

for all $a \in A$ and $X, Y \in \mathfrak{g}$. The Lie algebra morphism ρ with the above property is called the *anchor* of the Lie-Rinehart algebra. If the anchor is trivial, then the Lie-Rinehart algebra $\mathfrak{g}$ is called an A-Lie algebra.

Sample of some definitions

Definition (Connection of a Lie-Rinehart algebra)

A connection of a Lie-Rinehart algebra $\mathfrak{h}$ over A with anchor ρ on a left A-module N is a map

$$\nabla : \mathfrak{h} \to \mathfrak{gl}_{\mathbb{K}}(N) : X \mapsto \nabla_X$$

such that

$$\nabla_{a\cdot X} n = a \cdot \nabla_X n \qquad \text{and} \qquad \nabla_X (a \cdot n) = \rho(X)(a) \cdot n + a \cdot \nabla_X n$$

for all $X \in \mathfrak{h}$, $n \in N$, $a \in A$. If the above map ∇ is a Lie algebra morphism, then it is called a *flat connection* (or *representation*, or *action*) of $\mathfrak{h}$ on N.

Sample of some definitions

Proposition (Curved semidirect sum)

Let $\mathfrak{h}$ be a Lie-Rinehart algebra over A and $\mathfrak{i}$ be an A-Lie algebra, with a connection $\nabla : \mathfrak{h} \to \mathfrak{der}_{\mathbb{K}}(\mathfrak{i})$ of $\mathfrak{h}$ on $\mathfrak{i}$ such that its curvature obeys

$$[\nabla_X, \nabla_Y] - \nabla_{[X,Y]} = ad_{\tau(X,Y)}$$

where $\tau \in \mathrm{Hom}_A(\wedge^2_A \mathfrak{h}, \mathfrak{i})$ is a *Lie cocycle* in the sense that

$$\sum_{\text{cyclic}\{X,Y,Z\}} \nabla_X \tau(Y,Z) + \tau(X,[Y,Z]) = 0\,.$$

Then the A-module $\mathfrak{i} \oplus \mathfrak{h}$ can be made into a Lie-Rinehart algebra.

Sample of some definitions

Definition (Curved semidirect sum)

This Lie-Rinehart algebra with anchor

$$\rho_\tau : \mathfrak{i} \oplus \mathfrak{h} \to \mathfrak{Der}_{\mathbb{K}}(A) : n \oplus X \mapsto \rho_{\mathfrak{h}}(X)\,,$$

and Lie bracket

$$\big[n \oplus X, m \oplus Y\big]_\tau := \Big([n,m]_{\mathfrak{i}} + \nabla_X m - \nabla_Y n + \tau(X,Y)\Big) \oplus [X,Y]_{\mathfrak{h}}\,,$$

$$\text{for any } n,m \in \mathfrak{i} \text{ and } X,Y \in \mathfrak{h},$$

is denoted here $\mathfrak{i} \ni_\tau \mathfrak{h}$ and called *curved semidirect sum*.

Sample of some definitions

Proposition (Connection as curved semidirect sum)

Conversely, any A-linear splitting $s:\mathfrak{h}\hookrightarrow\mathfrak{g}$ of a Lie-Rinehart extension

$$0\to\mathfrak{i}\overset{\iota}{\hookrightarrow}\mathfrak{g}\overset{\pi}{\twoheadrightarrow}\mathfrak{h}\to 0$$

defines a decomposition $\mathfrak{g}=\mathfrak{i}\ni_\tau\mathfrak{h}$ of the extension of $\mathfrak{h}$ by $\mathfrak{i}$ as a curved semidirect sum.
In particular, the connection ∇ of $\mathfrak{h}$ on $\mathfrak{i}$ is defined via the adjoint action

$$\nabla=ad\circ s\;:\;\mathfrak{h}\to\mathfrak{Der}_{\mathbb{K}}(\mathfrak{i})\;:\;X\mapsto\nabla_X=ad_{s(X)}$$

and its curvature defines the Lie cocycle τ

$$[s(X),s(Y)]-s\big([X,Y]\big)=\tau(X,Y)$$

Sample of some definitions

Definition (Universal enveloping algebra of a Lie-Rinehart algebra)

The universal enveloping algebra of a Lie-Rinehart algebra $\mathfrak{h}$ over A is a $\mathbb{K}$-algebra $\mathcal{U}_A(\mathfrak{h})$ endowed with a morphism $\iota_A\colon A \to \mathcal{U}_A(\mathfrak{h})$ of $\mathbb{K}$-algebras and a morphism $\iota_{\mathfrak{h}}\colon \mathfrak{h} \to \mathcal{U}_A(\mathfrak{h})$ of Lie algebras over $\mathbb{K}$ such that

$$\iota_{\mathfrak{h}}(aX) = \iota_A(a)\,\iota_{\mathfrak{h}}(X) \quad \text{and} \quad \iota_{\mathfrak{h}}(X)\,\iota_A(a) - \iota_A(a)\,\iota_{\mathfrak{h}}(X) = \iota_A\big(\rho(X)(a)\big)$$

for all $a \in A$ and $X \in \mathfrak{h}$, which is universal with respect to this property.

This means that if $\mathcal{U}$ is another $\mathbb{K}$-algebra endowed with a morphism $j_A\colon A \to \mathcal{U}$ of $\mathbb{K}$-algebras and a morphism $j_{\mathfrak{h}}\colon \mathfrak{h} \to \mathcal{U}$ of $\mathbb{K}$-Lie algebras such that

$$j_{\mathfrak{h}}(aX) = j_A(a)\,j_{\mathfrak{h}}(X) \quad \text{and} \quad j_{\mathfrak{h}}(X)\,j_A(a) - j_A(a)\,j_{\mathfrak{h}}(X) = j_A\big(\rho(X)(a)\big),$$

then there exists a unique $\mathbb{K}$-algebra morphism $j\colon \mathcal{U}_A(\mathfrak{h}) \to \mathcal{U}$ such that $j \circ \iota_A = j_A$ and $j \circ \iota_{\mathfrak{h}} = j_{\mathfrak{h}}$.

Sample of some definitions

Etc...

Conclusion

Algebra (generalisation)	Lie algebra (Lie-Rinehart algebra)	Hopf algebra (left Hopf algebroid)
Extension	$0 \to \mathfrak{i} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{h} \to 0$	$0 \to U(\mathfrak{i}) \hookrightarrow U(\mathfrak{g})$ & $U(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{h}) \to 0$
Kernel of projection	Lie ideal $\mathfrak{i} \subseteq \mathfrak{g}$	Hopf kernel $U(\mathfrak{i}) \subseteq U(\mathfrak{g})$
Adjoint module	$\mathfrak{g}$-module Lie algebra $\mathfrak{i}$	$U(\mathfrak{g})$-module Hopf algebra $U(\mathfrak{i})$
Algebra splitting (aka flat connection)	$\mathfrak{g} \leftarrow \mathfrak{h}$ Lie algebra map	$U(\mathfrak{g}) \leftarrow U(\mathfrak{h})$ Asso. algebra map
Restriction of adjoint action	$\mathfrak{h}$-module Lie algebra $\mathfrak{i}$	$U(\mathfrak{h})$-module Hopf algebra $U(\mathfrak{i})$
Decomposition/ Factorisation	$\mathfrak{g} = \mathfrak{i} \ni \mathfrak{h}$ semidirect sum	$U(\mathfrak{i}) = U(\mathfrak{i}) \# U(\mathfrak{h})$ smash product

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Adjoint module	$\mathfrak{g}$-module Lie algebra $\mathfrak{i}$	$U(\mathfrak{g})$-module Hopf algebra $U(\mathfrak{i})$
Splitting (aka curved connection)	$\mathfrak{g} \leftarrow \mathfrak{h}$ Linear map	$U(\mathfrak{g}) \leftarrow U(\mathfrak{h})$ Coalgebra map
Decomposition/ Factorisation	$\mathfrak{g} = \mathfrak{i} \ni_\tau \mathfrak{h}$ curved semidirect sum	$U(\mathfrak{i}) = U(\mathfrak{i})\#_\sigma U(\mathfrak{h})$ crossed product
2-cocycle	Lie cocycle τ (“curvature”)	Hopf cocycle σ (“twisting”)

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Decomposition/ Factorisation	$\mathfrak{g} = \mathfrak{i} \ni \mathfrak{h}$ semidirect sum	$U(\mathfrak{g}) = U(\mathfrak{i}) \# U(\mathfrak{h})$ smash product
Trivial 2-cocycle	$\tau = 0$ (zero curvature)	$\sigma = \epsilon \circ m$ (Trivial twisting)