von Neumann Algebras, Information and Quantum Field Theory

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Quantum Mechanics and Noncommutativity

Planck

↙ ↘

Schrödinger *Heisenberg*

↘ ↙

bon Neumann uniqueness

- Schrödinger:

$$i\hbar\frac{\partial}{\partial t}\psi(x,t) = H\psi(x,t)$$

Differential equations

- Heisenberg:

$$PQ - QP = i\hbar I$$

Linear operators on Hilbert space, **noncommutativity is essential**!

Operator Algebras

$\mathcal{H}$ = Hilbert space,

$B(\mathcal{H})$ = algebra of all bounded linear operators on $\mathcal{H}$.

Algebraic structure: linear structure, multiplication: $B(\mathcal{H})$ is a *algebra

Derived structures:

Order structure: $A \geq 0 \Leftrightarrow A = B^*B$: algebraic structure determines order structure

Metric structure:
$||A||^2 = \inf\{\lambda > 0 : A^*A \leq \lambda I\}$: algebraic structure determines metric structure

C^* property of the norm:
$||A^*A|| = ||A||^2$. $B(\mathcal{H})$ is a C^*-algebra

C^*-algebras = noncommutative topology

Gelfand-Naimark thm. $\exists$ contravariant functor F between category of *commutative* C^*-algebras and category of locally compact topological spaces:

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{F} & \mathrm{spec}(\mathfrak{A}) \\ || & & || \\ C(X) & \xleftarrow{F^{-1}} & X \end{array}$$

C^*-algebra = dual of a topological space

Every C^*-algebra is isomorphic to a norm closed *-subalgebras of $B(\mathcal{H})$.

Noncommutative geometry = *-subalgebras of C^*-algebras + structure (spectral triple), Connes NC geometry.

von Neumann algebras = noncommutative measure theory

$\mathcal{M} \subset B(\mathcal{H})$ is a von Neumann algebra if $\mathcal{M}$ is a $*$-algebra on $\mathcal{H}$ and is weakly closed. Equivalently (von Neumann density theorem)

$$\mathcal{M} = \mathcal{M}''$$

with $\mathcal{M}' = \{T \in B(\mathcal{H}) : TX = XT \quad \forall X \in \mathcal{M}\}$ the commutant.

$$\mathcal{M} \text{ abelian} \Leftrightarrow \mathcal{M} = L^\infty(X, \mu):$$

$(\mathcal{M} = \{M_f : g \in L^2 \mapsto fg \in L^2\})$

von Neumann algebra = dual of a measure space

Physics: *Observables* are selfadjoint elements X of $\mathcal{M}$, *states* are normalised positive linear functionals φ,

$\varphi(X)$ = expected value of the observable X in the state φ

Operator Algebras

Classical Commutative	Quantum Noncommutative
Manifold X $C^\infty(X)$	$*$-algebra A
Topological space X $C(X)$	C^*-algebra $\mathfrak{A}$
Measure space X $L^\infty(X, \mu)$	von Neumann algebra $\mathcal{A}$

Murray-von Neumann dimension

A *factor* is a von Neumann algebra with trivial center $Z(\mathcal{M}) = \mathcal{M} \cap \mathcal{M}'$.

Every von Neumann algebra is a direct integral of factors

With $\mathcal{M}$ a factor, $e, f \in \mathrm{Proj}(\mathcal{M})$

$$e \prec f \stackrel{def}{=} \exists v \in \mathcal{M} : \ v^* v = e, \ vv^* \leq f$$

This gives a comparison of projections (analog of Cantor-Bernstein thm) and there is a dimension function

$$d : \mathrm{Proj}(\mathcal{M}) \to [0, \infty]$$

$$e \prec f \Leftrightarrow d(e) \leq d(f)$$

$$e \sim f \Leftrightarrow d(e) = d(f)$$

Murray-von Neumann classification

At this point, von Neumann realised in half an hour (said by Murray) that the alternatives are

$$I : d(e) \in 0, 1, 2 \dots n \qquad (n \in \mathbb{N} \text{ or } n = \infty)$$
$$II : d(e) \in [0, 1] \qquad \textit{normalised}$$
$$III : d(e) \in \{0, 1\}$$

Factors can be of type I, II or III

- Type I:

$M = B(\mathcal{H}) \otimes 1$, Type I_n, $n = \dim\mathcal{H}$. Thus $\mathrm{Tr}(e) = 0, 1, 2, \dots \infty$, e projection
(∃ minimal projection)

- Type II: ∃ trace τ.

Type II_1, $\tau(e) \in [0, 1]$ or type II_∞, $\tau(e) \in [0, \infty]$.
$(II_\infty = II_1 \otimes I_\infty)$

- Type III: no trace.

(Every projection $e \neq 0$ is equivalent to 1)

Thermal equilibrium states

A primary role in thermodynamics is played by the equilibrium distribution.

Gibbs states

Finite quantum system: $\mathfrak{A}$ matrix algebra with Hamiltonian H and evolution $\tau_t = \mathrm{Ad}e^{itH}$. Equilibrium state φ at inverse temperature β is given by the Gibbs property

$$\varphi(X) = \frac{\mathrm{Tr}(e^{-\beta H}X)}{\mathrm{Tr}(e^{-\beta H})}$$

What are the equilibrium states at infinite volume where there is no trace, no inner Hamiltonian?

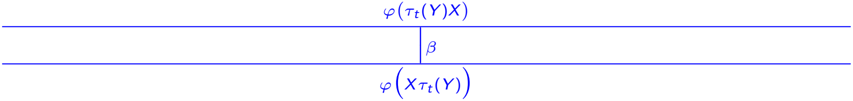
KMS states (Haag-Hugenoltz-Winnink 1967)

Infinite volume. $\mathfrak{A}$ a C^*-algebra, τ a one-par. automorphism group of $\mathfrak{A}$. A state φ of $\mathfrak{A}$ is KMS at inverse temperature $\beta > 0$ if for $X, Y \in \mathfrak{A}$ $\exists$ function F_{XY} s.t.

(a) $F_{XY}(t) = \varphi\big(X\tau_t(Y)\big)$

(b) $F_{XY}(t + i\beta) = \varphi\big(\tau_t(Y)X\big)$

F_{XY} bounded analytic on $S_\beta = \{0 < \Im z < \beta\}$



KMS states generalise Gibbs states, equilibrium condition for infinite systems

Tomita-Takesaki modular theory

$\mathcal{M}$ be a von Neumann algebra on $\mathcal{H}$, $\varphi = (\Omega, \cdot\Omega)$ normal faithful state on $\mathcal{M}$. Embed $\mathcal{M}$ into $\mathcal{H}$

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow[\text{isometric}]{X \mapsto X^*} & \mathcal{M} \\ {\scriptstyle X \to X\Omega}\big\downarrow & & \big\downarrow{\scriptstyle X \to X\Omega} \\ \mathcal{H} & \xrightarrow[\text{non isometric}]{S_0 : X\Omega \mapsto X^*\Omega} & \mathcal{H} \end{array}$$

$\bar{S}_0 = J\Delta^{1/2}$ polar decomposition

$$t \in \mathbb{R} \mapsto \sigma_t^\varphi \in \mathrm{Aut}(\mathcal{M})$$

$$\sigma_t^\varphi(X) = \Delta^{it} X \Delta^{-it}$$

intrinisic dynamics associated with φ (*modular automorphisms*);

$$J\mathcal{M}J = \mathcal{M}'$$

(*modular conjugation*)

Modular group

By a remarkable historical accident, at the 1967 Baton Rouge conference, Tomita distributed his preprint and Haag announced the KMS condition.

Takesaki participated in Baton Rouge conference and soon later he completed the Tomita's theory and characterised the modular group by the KMS condition.

• σ^φ is a purely noncommutative object (trivial in the commutative case)

• σ^φ is a thermal equilibrium evolution If $\varphi(X) = \mathrm{Tr}(\rho X)$ (type I case) then $\sigma_t^\varphi(X) = \rho^{it} X \rho^{-it}$

• *The Connes Radon-Nikodym cocycle relates the modular groups of different states*

$$u_t = (D\psi : D\varphi)_t \in \mathcal{M} \ , \quad \sigma_t^\psi = u_t \sigma_t^\varphi(\cdot) u_t^*$$

the modular group does not depend on the state up to inners

Connes classification, '73

$\mathcal{M}$ a factor. Set

$$S(\mathcal{M}) = \bigcap_{\varphi} \mathrm{sp}(\Delta_\varphi) \setminus \{0\}$$

intersecion over all normal states.

$S(\mathcal{M}) \setminus \{0\}$ is a closed, *multiplicative subgroup* of $\mathbb{R}^+$. Therefore one of the following holds for a type *III* factor

- $S(\mathcal{M}) = \{0, 1\}$, factors of type III_0
- $S(\mathcal{M}) = \{\lambda^n : n \in \mathbb{Z}\}^-$, for some $\lambda > 0$, factors of type III_λ
- $S(\mathcal{M}) = \mathbb{R}^+$, factors of type III_1

for a type *II* factor $S(\mathcal{M}) \subset \{0, 1\}$ and one can distinguish this case e.g. by considering weights.

Crossed product. Takesaki duality, '73

$\mathcal{M}$ von Neumann algebra on $\mathcal{H}$, α an action of an abelian group G on $\mathrm{Aut}(\mathcal{M})$ (here, $G = \mathbb{R}$ for simplicity). The crossed product on $\mathcal{H} \otimes L^2(G)$

$$\hat{\mathcal{M}} \equiv \mathcal{M} \rtimes_\alpha G = \{\mathcal{M} \otimes 1, U(g) \otimes \lambda(g)\}''$$

U implements α, λ reg. representation of G. Then

$$\hat{\hat{\mathcal{M}}} = \hat{\mathcal{M}} \rtimes_{\hat{\alpha}} \hat{G} \simeq \mathcal{M} \otimes B(\mathcal{H}), U \otimes \lambda$$

(crossed product by the dual action $\hat{\alpha}$)

$$\mathcal{M} \rtimes_{\sigma^\varphi} \mathbb{R} \text{ type } III \implies \hat{\mathcal{M}} \text{ type } II_\infty$$

(crossed product by the modular group)

The modular Hamiltonian

The generator of the modular operator unitary group Δ_φ^{it} is called the modular Hamiltonian $\log \Delta_\varphi$

One may consider the the relative modular operator $\Delta_{\Phi,\Omega}$, and the more general

modular Hamiltonian $\log \Delta_{\Phi,\Omega}$

The modular Hamiltonian is the generator of an intrinsic evolution where positivity of the energy is replaced by the KMS condition

The study of the modular Hamiltonian is getting a hot topic in Theoretical Physics

Quantum Field Theory. Local nets of von Neumann algebras, '64

Haag-Kastler nets. $\mathcal{K}$ set of double cones on Minkowski spacetime M.

Local net $\mathcal{A}$ on M: map $O \in \mathcal{K} \mapsto \mathcal{A}(O) \subset B(\mathcal{H})$ s.t.

- *Isotony* $O_1 \subset O_2 \implies \mathcal{A}(O_1) \subset \mathcal{A}(O_2)$
- *Locality* O_1, O_2 spacelike $\implies [\mathcal{A}(O_1), \mathcal{A}(O_2)] = \{0\}$
- *Poincaré covariance*
- *Positive energy and vacuum vector* Ω.

Reeh-Schlieder theorem

The vacuum vector Ω is cyclic and separating for $\mathcal{A}(O)$ if both O and O' have non-empty interior

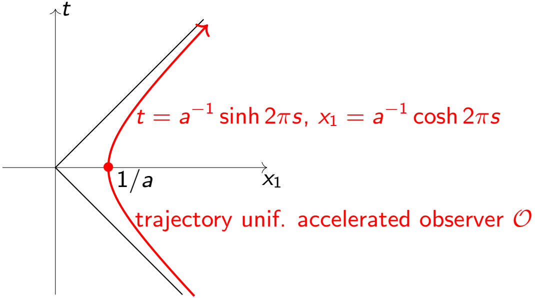
Given such a spacetime region O we may consider the vacuum modular group of O and the

the vacuum modular Hamiltonian of O : $\log \Delta_O$

What is the meaning of $\log \Delta_O$?

Bisognano-Wichmann theorem '75

Rindler spacetime (wedge $x_1 > |t|$), vacuum modular group

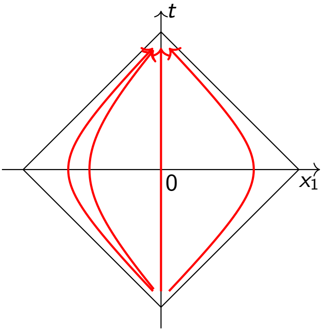


a : uniform acceleration of O
s/a : proper time of O
$\beta = 2\pi/a$: inverse KMS temperature of O

Hawking-Unruh effect!

Double cone, conformal case, Hislop, L. '80

For a bounded region O (double cone, causal envelop of a space ball B), in the conformal case the modular group is given by the geometric transformation



local modular trajectories

$(u, v) \mapsto \big((Z(u, s), Z(v, s)\big)$

$Z(z, s) = \frac{(1+z)+e^{-s}(1-z)}{(1+z)-e^{-s}(1-z)}$

$u = x_0 + r, \quad v = x_0 - r, \quad r = |\mathbf{x}| \equiv \sqrt{x_1^2 + \cdots + x_d^2}$

massive case: open problem

Doplicher-Haag-Roberts theory, '71

$\mathfrak{A} = \overline{\cup_O \mathcal{A}(O)}$ (spacetime dimension > 2)

$$\pi \text{ DHR rep. of } \mathfrak{A} \overset{\text{def}}{\iff} \pi\restriction\mathfrak{A}(O') \simeq \iota\restriction\mathfrak{A}(O'), \quad O \in \mathcal{K}$$

Given $O \in \mathcal{K}$, $\exists\rho \in \mathrm{End}(\mathfrak{A})$, $\rho \simeq \pi$

$$\rho\restriction\mathfrak{A}(O') = \mathrm{id}$$

ρ is a localized endomorphism of $\mathfrak{A}$. DHR endom. form a tensor C^*-category.

Statistics. Choose $\rho_1 \sim \rho$ localized in $O_1 \subset O'$: $\rho_1 = u\rho(\cdot)u^*$.
$\rho\rho_1 = \rho_1\rho$ gives $\epsilon = u^*\rho(u) \in \rho^2(\mathfrak{A})'$
$\epsilon_i \equiv \rho^{i-1}(\epsilon)$, $i \in \mathbb{N}$,

$$\begin{cases} \epsilon_i^2 = 1, \\ \epsilon_i\epsilon_j = \epsilon_j\epsilon_i \quad \text{if } |i-j| \geq 2, \\ \epsilon_i\epsilon_{i+1}\epsilon_i = \epsilon_{i+1}\epsilon_i\epsilon_{i+1} \end{cases}$$

unitary representation of $\mathbb{S}_\infty$, the statistics of ρ.

ρ irreducible: the statistics is classified by statistics parameter

$$\lambda_\rho = 0, \pm 1, \pm \frac{1}{2}, \pm \frac{1}{3}, \ldots$$

$\lambda_\rho = \kappa_\rho / d_{\mathrm{DHR}}(\rho)$ with $d_{\mathrm{DHR}}(\rho) > 0$ and $\kappa_\rho = \pm 1$.
$d_{\mathrm{DHR}}(\rho)$ is the *statistical dimension* of ρ;

$$d_{\mathrm{DHR}}(\rho) \in \mathbb{N} \cup \infty$$

κ_ρ is the *univalence* of ρ.

Jones index '83

Factors (von Neumann algebras with trivial center) are "very infinite-dimensional" objects. For an inclusion of factors $\mathcal{N} \subset \mathcal{M}$ the Jones index $[\mathcal{M} : \mathcal{N}]$ measure the relative size of $\mathcal{N}$ in $\mathcal{M}$. Surprisingly, the index values are quantised:

$$[\mathcal{M} : \mathcal{N}] = 4\cos^2\left(\frac{\pi}{n}\right) \ , \quad n = 3, 4, \ldots \quad \text{or} \quad [\mathcal{M} : \mathcal{N}] \geq 4$$

Jones index appears in many places in math and in physics.

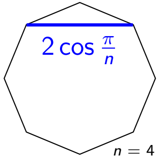


Figure: Jones index values

Index and statistics

Natural connection between the Jones index and DHR, L. '89

$$\text{DHR statistical dimension} = \sqrt{\text{Jones index of } \rho}$$

$[\mathcal{M} : \rho(\mathcal{M})] = d^2_{DHR}(\rho)$. Here $\mathcal{M} = \mathcal{A}(O)$ and ρ is a DHR endomorphism localised in O

Low dimensional statistics The statistical dimension is quantised. The statistics is a representation of the Artin braid group (first analysis: Fredenhagen, Rehren, Schrör, - L., '89). Jones, HOMPLY and Kauffman polynomials appear.

Analog of the Kac-Wakimoto formula. L. '97

The root of our work relies in this formula for the incremental free energy of a black hole (cf. the Kac-Wakimoto formula, Kawahigashi, Xu, L.)

H_ρ be the Hamiltonian for a uniformly accelerated observer in the Minkowski spacetime with acceleration $a > 0$ in representation ρ (localised in the wedge for H_ρ)

$$(\Omega, e^{-tH_\rho}\Omega)\big|_{t=\beta} = d(\rho)$$

with Ω the vacuum vector and $\beta = \frac{2\pi}{a}$ the inverse Hawking-Unruh temperature. $d(\rho)^2$ is Jones' index.

The left hand side is a generalised partition formula, so $\log d(\rho)$ has an entropy meaning in accordance with Pimsner-Popa work.

The proof of formula is based on a tensor categorical and spacetime symmetries analysis.

Classification of local conformal nets (Kawahigashi, L. '04)

Classification of local conformal nets on the circle, $c = 1 - \frac{6}{m(m+1)}$

m	Labels for Z
n	(A_{n-1}, A_n)
$4n+1$	(A_{4n}, D_{2n+2})
$4n+2$	(D_{2n+2}, A_{4n+2})
11	(A_{10}, E_6)
12	(E_6, A_{12})
29	(A_{28}, E_8)
30	(E_8, A_{30})

Local conformal nets with central charge $c < 1$ are classified by pair of Dynkin diagrams $A - D_{2n} - E_{6,8}$ s.t. the difference of Coxeter numbers is 1.

Entropy of finite systems

$X = \{x_1, \dots x_n\}$ a set of events. If x_i occurs with probability p_i, its information is $-\log p_i$

$$\text{Shannon entropy}: S(P) = -\sum p_i \log p_i \ .$$

If $Q = \{q_1, \dots q_n\}$ other probability distribution (state)

$$\text{Relative entropy}: S(P\|Q) = \sum p_i(\log p_i - \log q_i)$$

mean value in the state P of the difference between the information carried by the state P and the state Q.

Noncommutative entropy: $\varphi = -\mathrm{Tr}(\rho_\varphi \,\cdot)$ state on a matrix algebra

$$\text{von Neumann entropy}: S(\varphi) = \ -\mathrm{Tr}(\rho_\varphi \log \rho_\varphi)$$

Umegaki's relative entropy

$$S(\psi\|\varphi) =: \mathrm{Tr}\big(\rho_\psi(\log \rho_\psi - \log \rho_\varphi)\big)$$

Araki's relative entropy, '76

Araki's relative entropy between two faithful normal states φ and ψ on $\mathcal{M}$ is

$$S(\psi\|\varphi) \equiv -(\Psi, \log \Delta_{\Phi,\Psi}\, \Psi)$$

Φ, Ψ are cyclic vector representatives of φ, ψ

$$S(\psi\|\varphi) \geq 0$$

positivity of the relative entropy

In QFT the von Neumann algebra $\mathcal{M} = \mathcal{A}(O)$ is a factor of type III

$$\mathcal{M} \text{ not of type I} \implies \text{von Neumann entropy } = \infty$$

Relative entropy is defined.

CP maps, quantum channels

$\mathcal{N}, \mathcal{M}$ vN algebras. A linerar map $\alpha : \mathcal{N} \to \mathcal{M}$ is completely positive is

$$\alpha \otimes \mathrm{id}_n : \mathcal{N} \otimes \mathrm{Mat}_n(\mathbb{C}) \to \mathcal{M} \otimes \mathrm{Mat}_n(\mathbb{C})$$

is positive $\forall n$. We always assume α to be unital and normal.
ω faithful normal state of $\mathcal{M}$ Set

$$\mathrm{H}_\omega(\alpha) \equiv \sup_{(\omega_i)} \sum_i S(\omega|\omega_i) - S(\omega \cdot \alpha|\omega_i \cdot \alpha)$$

supremum over all ω_i with $\sum_i \omega_i = \omega$.
The conditional entropy $H(\alpha)$ of α is defined by

$$\mathrm{H}(\alpha) = \inf_\omega \mathrm{H}_\omega(\alpha)$$

infimum over all "full" states ω for α. Clearly $\mathrm{H}(\alpha) \geq 0$ because $\mathrm{H}_\omega(\alpha) \geq 0$

α is a quantum channel if its conditional entropy $\mathrm{H}(\alpha)$ is finite.

Generalisation of Stinespring dilation

Let $\alpha : \mathcal{N} \to \mathcal{M}$ be a normal, completely positive unital map between the vN algebras $\mathcal{N}$, $\mathcal{M}$. A pair (ρ, v) $\rho : \mathcal{N} \to \mathcal{M}$ a homomorphism, $v \in \mathcal{M}$ an isometry s.t.

$$\alpha(n) = v^* \rho(n) v \ , \quad n \in \mathcal{N} \ .$$

(ρ, v) is *minimal* if the left support of $\rho(\mathcal{N}) v \mathcal{H}$ is qual to 1.

Thm Let $\alpha : \mathcal{N} \to \mathcal{M}$ be a normal, CP unital map with $\mathcal{N}$, $\mathcal{M}$ properly infinite. There exists a minimal dilation pair (ρ, v) for α. If (ρ_1, v_1) is another minimal pair, $\exists!$ unitary $u \in \mathcal{M}$ such that

$$u\rho(n) = \rho_1(n) u \, , \quad v_1 = uv, \qquad n \in \mathcal{N}$$

We have

$$H(\alpha) = \log Ind(\alpha) \qquad \text{(minimal index)}$$

Landauer's bound for infinite systems. L. 18

Landauer's principle: *any logically irreversible manipulation of information, such as the erasure of a bit or the merging of two computation paths, must be accompanied by a corresponding entropy increase in non-information bearing degrees of freedom of the information processing apparatus or its environment* (cf. C. Bennet)

Let $\alpha : \mathcal{N} \to \mathcal{M}$ be a quantum channel the free energy is

$$F_\alpha \equiv \left\langle K_{\mathcal{H}}(\varphi_1|\varphi_2) + \beta^{-1} log \Delta_{\mathcal{H}}(\varphi_1|\varphi_2)\right\rangle = \beta^{-1} \log d \geq 0$$

with $K_{\mathcal{H}}$ the modular Hamiltonian determied by the tensor categorocal structure. If α is irreversible, then

$$F_\alpha \geq \frac{1}{2} kT \log 2$$

The *original lower bound* for the incremental free energy is

$$F_\alpha \geq kT \log 2$$

it remains true for finite-dimensional systems

Bekenstein's bound

Bekenstein argued in the early 80s that an upper limit is to exist for the entropy/energy ratio of a bounded system

$$S \leq 2\pi R E \,;$$

here S, E are the entropy and the energy contained in the system of effective radius R.

In 2008, Casini gave a precise formulation of this inequality by subtracting the vacuum entropy and energy contribution to eliminate divergencies and reducing the above bound to the positivity of the relative entropy

$$S(\Phi\|\Omega) = \langle K \rangle_{\Phi} - \delta S \geq 0 \,,$$

where δS is the vacuum-subtracted von Neumann entropy of Φ, and $\langle K \rangle_{\Phi}$ is the modular energy, namely the expectation value on Φ of (half of) the Tomita-Takesaki modular Hamiltonian.

Bekenstein-type bound. L. 25

I will provide a universal, rigorous, model independent bound on the relative entropy/true energy ratio in a given region B, which is derived from first principles

B a space region of width $2R > 0$; so, B lies within two parallel planes with distance $2R$; then

$$S(\Phi\|\Omega)_B \leq 2\pi R(\Phi, P\Phi)\,,$$

P is the usual Hamiltonian, $P \geq 0$.

Φ localized in B i.e. $\langle\cdot\rangle_\Phi = \langle\cdot\rangle_\Omega$ on $\mathcal{A}(B')$, with B' complem. of B.

Simplified proof, sketch

I sketch a simpler but less general proof:

$$\begin{aligned}
S(\Phi\|\Omega)_B &\leq S(\Phi\|\Omega)_{I_R} && \text{(monot. rel. entr.)}\\
&\leq \frac{1}{2}\big(S(\Phi\|\Omega)_{W'_{-R}} + S(\Phi\|\Omega)_{W_R}\big) && \text{(monot. rel. entr.)}\\
&= \frac{1}{2}\big((\Phi, \log\Delta_{\Omega,\Phi;W'_{-R}}\Phi) + (\Phi, \log\Delta_{\Omega,\Phi;W_R}\Phi)\big) && \text{(definition)}\\
&= \frac{1}{2}\big((\Phi, \log\Delta_{\Omega;W'_{-R}}\Phi) + (\Phi, \log\Delta_{\Omega;W_R}\Phi)\big) && \text{(state is localised)}\\
&= \pi\big((\Phi, K_R\Phi) - (\Phi, K_{-R}\Phi)\big) && \text{(BW thm)}\\
&= 2\pi R(\Phi, P\Phi) && \text{(Lie alg. relation)}
\end{aligned}$$

The rigorous proof works in exponential form, with complex analysis methods

More general states (S. Hollands and R. L.)

Approximately localised states

If $\langle\cdot\rangle_\Phi \leq (1+\varepsilon)\langle\cdot\rangle_\Omega$ on $\mathcal{A}(B')$, with $\varepsilon \geq 0$, then

$$S(\Phi\|\Omega)_B \leq 2\pi R\,(\Phi, P\Phi) + \varepsilon\,.$$

Charged states

$\Phi \in \mathcal{H}_\rho$ a state localised in B in *charged representation* ρ:

$$S(\Phi\|\Omega)_B \leq 2\pi R\,(\Phi, P_\rho\Phi) + \log d(\rho)$$

with $d(\rho) =$ DHR *statistical dimension* of ρ.

I showed long ago $d(\rho) = \sqrt{\textit{Jones index of } \rho}$ *and* $\log d(\rho)$ *is the entropy of the charge* ρ

Putting together

$\Phi \in \mathcal{H}_\rho$ charged state, $\langle\cdot\rangle_\Phi \leq (1+\varepsilon)\langle\cdot\rangle_\Omega$ on $\mathcal{A}(B')$, then

$$S(\Phi\|\Omega)_B \leq 2\pi R\,(\Phi, P_\rho\Phi) + \log d(\rho) + \varepsilon\,.$$

The entropy balance and ant formulas; null translations

$\mathcal{A}$ wedge dual net of vN algebras, $W(x)$ right wedge with vertex $x = \langle x_0, x_1 \rangle \in \mathbb{R}^2$, U the transl. unitaries,

$$S(x) = S(\Phi \| \Omega)_{W(x)}\,, \qquad \bar{S}(x) = S(\Phi \| \Omega)_{W(x)'}\,.$$

rel. entr. between vect. states Φ, Ω on $\mathcal{A}(W(x))$ and $\mathcal{A}(W(x)')$.
Entropy balance formula

$$S(a,a) - S(b,b) = \bar{S}(a,a) - \bar{S}(b,b) + 2\pi(b-a)(\Phi, P_+\Phi)\,,$$

$a, b \in \mathbb{R}$, with P_+ null-translation generator.

Ant formula $\partial_a S(a,a)$ exists a.e. and

$$-\partial_a S(a,a) = 2\pi \inf_{u' \in \mathcal{A}(a,a)'} (u'\Phi, Pu'\Phi)\,,$$

infimum over isometries $u' \in \mathcal{A}(a,a)'$.

The entropy balance and ant formulas; spatial translations

The ant formula was considered Wall and proved in Ceylan and Faulkner; a different proof was given in Hollands and L.

So far proved for a dense subspace of vectors

The entropy balance formula for *spatial translations* is

$$S(0,a) - S(0,b) = \bar{S}(0,a) - \bar{S}(0,b) + 2\pi(b-a)(\Phi, P\Phi)\,,$$

with P is the *usual, positive* Hamiltonian, namely the generator of the time-translation unitary group.

The ant formula for space translation s(Hollands, Morsella, work in progress)

QNEC

In QFT, the stress-energy tensor is locally non-positive. Bousso, Fisher, Liechenauer, and Wall proposed the QNEC: For null direction deformations

$$S''(\lambda) \geq 0$$

$S(\lambda)$ vacuum relative entropy of every state in the deformed region

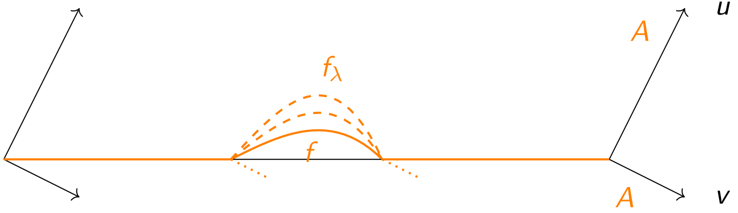


Figure: The function f is the boundary of the deformed region on the null horizon. The entire deformed region is its causal envelop A.

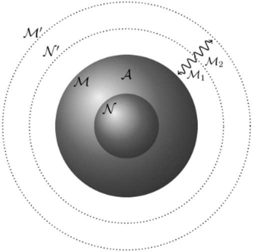
$S''(\lambda) \geq 0$ unexpected. Ceyhan and Faulkner in 2020 by half-sided modular inclusions. Hollands, L. recent new proof

Recovery information protocol. Van Der Heijden, Verlinde 25 - Palumbo 26

Jones tower, my canonical endomorphism

$$\Gamma : a \in \mathcal{M} \to \Gamma a \Gamma^* \in \mathcal{N}$$

$\Gamma = J_{\mathcal{N}} J_{\mathcal{M}}$



$$\langle \Psi | e_{\mathcal{M}_1} \Gamma(A) e_{\mathcal{M}_1} | \Psi \rangle = [\mathcal{M} : \mathcal{N}]^{-1} \,\mathrm{Tr}_{\mathcal{A}}(A) = [\mathcal{M} : \mathcal{N}]^{-1} \langle \Psi | A | \Psi \rangle.$$

Standard subspaces and modular theory

$\mathcal{H}$ complex Hilbert space and $H \subset \mathcal{H}$ a *closed, real linear subspace*. *Symplectic complement*:

$$H' = \{\xi \in \mathcal{H} : \Im(\xi, \eta) = 0 \ \forall \eta \in H\}$$

H is *cyclic* if $\overline{H + iH} = \mathcal{H}$ and *separating* if $H \cap iH = \{0\}$.

H is a standard subspace if it is $\mathcal{H}$ *cyclic* if $\overline{H + iH} = \mathcal{H}$ and *separating* $H \cap iH = \{0\}$

H standard subspace $\rightarrow$ anti-linear operator S_H

$$S_H : \xi + i\eta \rightarrow \xi - i\eta, \ \xi, \eta \in H$$

$S_H^2 = 1|_{D(S_H)}$, $D(S_H) = H + iH$. S_H is closed, densely defined, $S_H^* = S_{H'}$

Modular theory for standard subspaces

Conversely, S densely defined, closed, anti-linear involution on $\mathcal{H} \to H_S = \{\xi \in D(S) : S\xi = \xi\}$ is a standard subspace:

$$H \leftrightarrow S \textit{ is a bijection}$$

Set $S_H = J_H \Delta_H^{1/2}$, polar decomposition. Then J_H is an anti-unitary involution, $\Delta_H > 0$ is non-singular called the modular conjugation and the modular operator they satisfy $J_H \Delta_H J_H = \Delta_H^{-1}$ and

$$H \leftrightarrow (J, \Delta) \textit{ is a bijection.}$$

Main relations:

$$\Delta_H^{it} H = H, \quad J_H H = H'$$

Every closed, real linear H is

$$\text{standard} \oplus (0 \subset \mathcal{H}) \oplus (\mathcal{H} \subset \mathcal{H})$$

Examples

Example 1: $\mathcal{M}$ von Neumann algebra on $\mathcal{H}$, Ω cyclic separating vector

$$H = \overline{\mathcal{M}_{\rm s.a.}\Omega} \text{ is a standard subspace of } \mathcal{H}$$

$$\Delta_H = \Delta_{\mathcal{M}}, \quad J_H = J_{\mathcal{M}}$$

Example 2: $\mathcal{H}$ (one-particle) Hilbert space, $H \subset \mathcal{H}$ real Hilbert space (of vectors localized in a region O)

$$\Gamma(\Delta_H) = \Delta_{\mathcal{A}(H)} \quad \Gamma(J_H) = J_{\mathcal{A}(H)}$$

$\mathcal{A}(H)$ von Neumann algebra on the Fock space $e^{\mathcal{H}}$

$$\mathcal{A}(H) = \{V(\xi) : \xi \in H\}''$$

$V(\xi)$ Weyl unitary

Passivity

$\log \Delta_H$ is characterised by complete passivity, following Pusz and Woronowicz in the von Neumann algebra case

$\mathcal{H}$ a complex Hilbert space, $H \subset \mathcal{H}$ a standard subspace and A a selfadjoint linear operator on $\mathcal{H}$ such that $e^{isA}H = H$, $s \in \mathbb{R}$.

A is passive with respect to H if

$$-(\xi, A\xi) \geq 0\,, \quad \xi \in D(A) \cap H\,.$$

A is completely passive w.r.t. H if the generator of $e^{itA} \otimes e^{itA} \cdots \otimes e^{itA}$ is passive with respect to the n-fold tensor product $H \otimes H \otimes \cdots \otimes H$, all $n \in \mathbb{N}$.

A is completely passive with respect to H iff $\log \Delta_H = \lambda A$ *for some* $\lambda \geq 0$.

positivity of energy $\leftrightsquigarrow$ *comp. passivity of modular Hamiltonian* (equivalence in principle)

Cutting projection

Let H be a factorial standard sunbspace, i.e. $H \cap H' = \{0\}$. The cutting projection P_H is the symplectic projection

$$P_H : H + H' \to H\,, \quad h + h' \mapsto h$$

We have

$$P_H = (1 + S_H)(1 - \Delta_H)^{-1}$$

(P_H is the closure of the right-hand side).

In free QFT, P_H is geometric (Figliolini, Guido)

Entropy operator

Let H a be standard (not necessarily factorial) subspace.

The entropy operator $\mathcal{E}_H$ is defined (factorial case) by

$$\mathcal{E}_H = A(\Delta_H) + J_H B(\Delta_H)\,,$$

(closure of the right-hand side), $A(\lambda) \equiv -(1-\lambda)^{-1}\log\lambda$,
$B(\lambda) \equiv \lambda^{1/2}a(\lambda)\log\lambda$

In the factorial case

$$\mathcal{E}_H = iP_H i \log\Delta_H$$

The entropy operator $\mathcal{E}_H$ is a real linear on $\mathcal{H}$, positive, selfadjoint w.r.t. the real part of the scalar product. Its its expectation values correspond to entropy quantities

If H is any closed, real linear subspace, $\mathcal{E}_H$ is defined on the standard component, and zero on the othogonal.

Entropy of a vector relative to a real linear subspace

Let $\mathcal{H}$ be a complex Hilbert space and $H \subset \mathcal{H}$ a closed, real linear subspace

The entropy of a vector $h \in \mathcal{H}$ with respect to $H \subset \mathcal{H}$ is defined by

$$S(h\|H) = \Re(h, \mathcal{E}_H h)\,, \quad h \in \mathcal{H}\,.$$

real quadratic form sense, namely $S(h\|H) = ||\mathcal{E}_H^{1/2} h||^2$.

Properties of the entropy of a vector

Some of the main properties of the entropy of a vector are:

- $S(h\|H) \geq 0$ or $S(h\|H) = +\infty$ positivity
- If $K \subset H$, then $S(h\|K) \leq S(h\|H)$ monotonicity
- If $h_n \to h$, then $S(h\|H) \leq \liminf_n S(h_n|H)$ lower semicontinuity
- If $H_n \subset H$ is an increasing sequence with $\overline{\bigcup_n H_n} = H$, then $S(h\|H_n) \nearrow S(h\|H)$ monotone continuity
- If $h \in D(\log \Delta_H)$ then $S(h\|H) < \infty$ finiteness on smooth vectors
- $S(h\|H) = S(k\|H)$ if $k - h \in H'$ locality
- $S(h\|H) = S(V(h)\Omega\|\Omega)_{R(H)}$ entropy of coherent stae on the Fock space $e^{\mathcal{H}}$

Borchers theorem (real subspace analog)

H standard subspace, T a one-parameter group with positive generator s.t. $T(s)H \subset H, \;\; s > 0.$

Then:

$$\begin{cases} \Delta^{it} T(s) \Delta^{-it} = T(e^{-2\pi t} s) \\ JT(s)J = T(-s), \;\; t, s \in \mathbb{R} \end{cases}$$

i.e. Δ^{it} and $T(s)$ give a rep. of the "$ax + b$" group!

Comments. The converse construction

Note 1: If T has no non-zero fixed vector, H is unique up to multiplicity

Note 2: If we start with a unitary, positive energy U of the "$ax + b$" group, and translations have no non-zero fixed points, we may define $H \equiv \{\xi \in \mathcal{H} : S\xi = \xi\}$ where $S \equiv J\Delta^{1/2}$ and $\log \Delta \equiv -2\pi\cdot$(generator of dilations).

Standard pair $(T, H) \leftrightarrow$ rep of the "$ax + b$" group

Analog of Wiesbrock, Araki-Zsido thm, standard subspace analog

$K \subset H$ standard subspaces. Suppose that

$$\Delta_H^{-is} K \subset K \;\; s \geq 0, \quad \text{(hsm condition)}$$

Then

- $\Delta_{\mathcal{M}}^{is}, \Delta_{\mathcal{N}}^{-it}$ generate a representation of the "ax + b" group
- the translation generator $T = \frac{1}{2\pi}(\log \Delta_{\mathcal{N}} - \log \Delta_{\mathcal{M}})^{-}$ is positive
- $e^{iTt} H \subset H$, $t \geq 0$, and $K = e^{iT} H$

Abstract result

Let $H \subset \mathcal{H}$ be a standard subspace and $T(t) = e^{iAt}$ a one-parameter unitary group on $\mathcal{H}$ such that

- $A \geq 0$
- $T(t)H \subset H$, $t \geq 0$

Define $H_\lambda = T(\lambda)H\,,\ \lambda \in \mathbb{R}$, translated subspaces. Then the entropy function

$$\lambda \mapsto S(\lambda) = S(\psi \| H_\lambda) \text{ is convex for all } \psi$$

and finite for a dense set of vectors. If $S(\lambda_0) < \infty$, then

(i) $S(\lambda)$ is finite and C^1 on $[\lambda_0, \infty)$;

(ii) $S'(\lambda)$ is absolutely continuous in $[\lambda_0, \infty)$ with almost everywhere non-negative derivative $S''(\lambda) \geq 0$.

Weyl algebra and Gaussian states

If H is a real linear space with a non-degenerate symplectic form β, The Weyl C^*-algebra $C^*(H)$ linearly generated by the (unitaries) $V(h)$, $h \in H$, that satisfy the commutation relations (CCR)

$$V(h+k) = e^{i\beta(h,k)} V(h)V(k) \,, \quad h, k \in H \,,$$

$V(h)^* = V(-h)$. A state φ_α on $C^*(H)$ is called Gaussian, if

$$\varphi_\alpha\big(V(h)\big) = e^{-\frac{1}{2}\alpha(h,h)} \,,$$

with α a real bilinear form α on H, compatible with β.

H standard linear subspace of $\mathcal{H} \to$ von Neumann $\mathcal{A}(H) = C^*(H)''$ algebra on Fock space $e^{\mathcal{H}}$, i.e. GNS of $\omega = \varphi_\alpha$ with $\alpha = \Re(\cdot,\cdot)$

Entropy of coherent sectors

Given $\xi \in \mathcal{H}$ consider coherent state φ_ξ on Weyl von Neumann algebra $\mathcal{A}(H)$ on the Bose Fock space $e^{\mathcal{H}}$:

The vacuum relative entropy of φ_ξ on $\mathcal{A}(H)$ is given by

$$S(\varphi_\xi \| \varphi_0) = S(\xi \| H)$$

Araki's relative entropy

Entropy of vector

Ω vacuum vector, $\varphi_\xi = (V(\xi)\Omega, \cdot V(\xi)\Omega)$, $V(\xi)$ Weyl unitary

First and second quantisation

First quantisation: map

$$O \subset \mathbb{R}^d \mapsto H(O) \text{ real linear space of } \mathcal{H}$$

local, covariant, etc.

Second quantisation: map

$$O \subset \mathbb{R}^d \mapsto \mathcal{A}(O) \text{ v.N. algebra on } e^{\mathcal{H}}$$

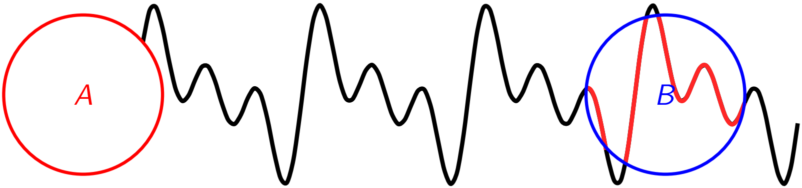
$$\mathcal{A}(O) = \mathcal{A}(H(O)) \equiv \{V(h) : h \in O\}'' , \quad V \text{ Weyl unitary on } e^{\mathcal{H}} .$$

In our case $H(O)$ is generated by the waves with Cauchy data in B (O double cone with time-zero basis B)

Our analysis in this talk will be done in first quantisation, then we may apply the second quantisation functor.

The information carried by a classical wave

Suppose that Alice encodes and sends information by an undulatory signal, what information can Bob get by the wave packet in a given region at later time?



Bob has access only the portion of the wave that is in his lab at a given time. We are interested in the local information or information density of the wave packet.

Wave packets

By a Klein-Gordon wave (or wave packet), we mean a real solution of the wave equation

$$(\Box + m^2)\Phi = 0 \ ,$$

with compactly supported, smooth Cauchy data $\Phi|_{x^0=0}$, $\Phi'|_{x^0=0}$.

The waves form a real linear space, which is dense in the complex one-particle Hilbert space. The closure of real linear subspace of waves with Cauchy data supported in a ball B or half-space M are elements gives $H(B)$ or $H(M)$. We set $H(O)$ and $H(W)$, with O, W the causal completion of B, W (double cone, wedge)

Entropy of a wave (F. Ciolli, G. Ruzzi, R. L.)

Let Φ be a Klein-Gordon wave and $W_{\lambda_0,\lambda_1} = W + (\lambda_0, \lambda_1, \dots)$.

The entropy $S_\Phi(\lambda_0, \lambda_1) = S(\Phi \| H(W_{\lambda_0,\lambda_1})$ is

$$S_\Phi(\lambda_0, \lambda_1) = 2\pi \int_{x^0=\lambda_0,\, x^1 \geq \lambda_1} (x^1 - \lambda_1)\, T_{00}(x)\, dx$$

Setting in particular $\lambda = \lambda_1 = \lambda_2$,

$$S''_\Phi(\lambda) = 2\pi \int_{x^0=\lambda,\, x^1=\lambda} \langle v, Tv \rangle dx \geq 0 \ ,$$

where v is the light-like vector $v = (1, 1, 0 \dots, 0)$.

Here T is the stress-energy tensor; the energy density is
$T_{00} = \frac{1}{2}\big(\Phi'^2 + |\nabla\Phi|^2 + m^2\Phi^2\big)$

Entropy of localised states: $U(1)$-current model

One-dimensional case.

$U(1)$-current j: ℓ real function in $S(\mathbb{R})$, $L(x) \equiv \int_{-x}^{\infty} \ell(t)dt$.

$$S(\lambda) \equiv S(L \| H(\lambda, \infty)) = \pi \int_{\lambda}^{+\infty} (x - \lambda)\ell^2(x)\mathrm{d}x \ ,$$

$S(\lambda)$ vacuum relative entropy of excited state by $j \mapsto j + \ell$ (Buchholz-Mack-Todorov sector with charge $q = \int \ell$)

$$S'(\lambda) = -\pi \int_{\lambda}^{+\infty} \ell^2(x)\mathrm{d}x \leq 0 \ ,$$

$$S''(\lambda) = \pi \ell^2(\lambda) \geq 0$$

positivity of S''

The local entropy of a massless wave (G. Morsella, R. L.)

The modular Hamiltonian $\log\Delta_B$ associated with the unit ball B in the free scalar, massless QFT is (on Cauchy data)

$$-2\pi A = \log\Delta_B\,.$$

$$\log\Delta_B = 2\pi\imath_0 \begin{bmatrix} 0 & \frac{1}{2}(1-r^2) \\ \frac{1}{2}(1-r^2)\nabla^2 - r\partial_r - D & 0 \end{bmatrix}$$

with L_0 the higher dimensional Legendre operator

$$L_D = \frac{1}{2}(1-r^2)\nabla^2 - r\partial_r - D$$

Massless case

Bose case (G. Morsella, R. L.) The entropy of a massless wave Φ in the unit ball B is

$$S(\Phi \| B) = 2\pi \int_B \frac{1-r^2}{2} \langle T^{(0)}_{00} \rangle_\Phi(x) dx + \pi D \int_B \Phi^2 dx$$

Fermi case (F. La Piana, G. Morsella) With S_Φ the entropy of Φ in the unit ball cantered at 0, we have

$$S_\Phi = \frac{1}{8\pi^2} \int_{\mathbb{R}^3} (1-r^2) \langle T_{00}(t, \mathbf{x}) \rangle_\Phi d\mathbf{x}$$

provided Φ is supported in B

Local entropy density of a massive wave packet

- Describe the local, massive modular Hamiltonian: old problem.

$$\log \Delta_B = -\pi \imath_m \begin{bmatrix} 0 & M_m \\ L_m & 0 \end{bmatrix}$$

Bostelmann, Cadamuro, Mintz: computer numerical analysis

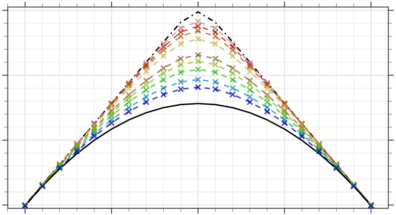


Figure: M_m as m varies

- Get rigorous bound on the local entropy in the massive case

New strategy: use the new notion of entropy operator and compare with the half-space entropy

Nets of standard subspaces

$\mathcal{H}$ complex Hilbert space, $\mathcal{O}$ the family of double cones of the Minkowski spacetime $\mathbb{R}^{d+1}$.
A local Poincaré covariant *net of real linear subspaces* is a map

$$O \in \mathcal{O} \mapsto H(O) \subset \mathcal{H}\,,$$

with $H(O)$ real linear, closed subspace of $\mathcal{H}$, s.t.

- $O_1 \subset O_2 \implies H(O_1) \subset H(O_2)$ (*isotony*);
- $O_1 \subset O_2' \implies H(O_1) \subset H(O_2)'$ (*locality*);
- $\exists$ a unitary, positive energy representation U of $\mathcal{P}_+^\uparrow$ on $\mathcal{H}$ s.t. $U(g)H(O) = H(gO)$ (*Poincaré covariance*);
- $\overline{\sum_{x\in\mathbb{R}^{d+1}} H(O+x)} = \mathcal{H}$ (*non-degeneracy*).

Set $H(C) \equiv \mathrm{lin.span.}\{H(O) : O \subset C\}$ for any region C

The dual net

H a local Poincaré covariant net the Bisognano-Wichmann property. The *dual net* H^d is

$$H^d(O) = H(O')' , \quad O \in \mathcal{O} ,$$

$$H^d(O) = \bigcap_{W \supset O} H(W), \qquad W \text{ wedge}$$

By locality, $H(O') \subset H(O)'$, therefore

$$H(O) \subset H^d(O) , \quad O \in \mathcal{O}, \quad H^d(W) = H(W).$$

H^d is local, Poincaré covariant, with BW property and satisfies Haag duality

$$H^d(O)' = H^d(O') , \quad O \in \mathcal{O} .$$

H^d is the maximal extension of H on $\mathcal{H}$ that is relatively local with respect to H

Universal bound (V. Morinelli, R. L.)

H a local, Poincaré covariant net of standard subspaces on t $\mathcal{H}$, with covariance unitary representation U. Given $h \in \mathcal{H}$, the *entropy of h with respect the region* $C \subset \mathbb{R}^{d+1}$ *relative to H* is

$$S_H(h\|C) \equiv S\big(h\|H(C)\big)\,.$$

Given an anti-unitary, positive energy representation V of $\mathcal{P}_+$, we then define *the entropy of* φ *with respect to C associated with V* as

$$S_V(\varphi\|C) \equiv S_K(\varphi\|C)\,,$$

where K is the local net of real linear spaces associated with V

With U the covariance unitary representation of $\mathcal{P}_+^\uparrow$ of H, let $\tilde{U}$ be the canonical extension of U to an anti-unitary rep. of $\mathcal{P}_+$.
For every region $C \subset \mathbb{R}^{d+1}$ and vector $\varphi \in \mathcal{H}$, the bound

$$S_H(h\|C) \leq S_{\tilde{U}}(h\|C)$$

holds and depends only on $\tilde{U}$, not on H.

Nets of standard subspaces on $\mathbf{S}^1$.

A local Möbius covariant net H of real linear subspaces on S^1 is a map

$$I \in \mathcal{I} \to H(I) \subset \mathcal{H}$$

$\mathcal{I} \equiv$ intervals of S^1, $H(I)$ closed, real linear, that satisfies:

- **A.** *Isotony*. $I_1 \subset I_2 \implies H(I_1) \subset H(I_2)$
- **B.** *Locality*. $I_1 \cap I_2 = \varnothing \implies H(I_1) \subset H(I_2)'$
- **C.** *Möbius covariance*. $\exists$ unitary rep. U of the Möbius group Möb $= PSL(2,\mathbb{R})$ on $\mathcal{H}$ s.t.

$$U(g)H(I) \;=\; H(gI), \quad g \in \text{Möb},\ I \in \mathcal{I}.$$

- **D.** *Positivity of the energy*. Generator L_0 of rotation subgroup of U is positive.
- **E.** *Irreducibility*. $\overline{lin.span}\{H(I), I \in \mathcal{I}\} = \mathcal{H}$

Some consequences

- *Reeh-Schlieder theorem*: Each $H(I)$ is a standard subspace.
- *Bisognano-Wichmann property*: Tomita-Takesaki modular operator Δ_I and conjugation J_I of $(\mathcal{A}(I), \Omega)$, are

$$U(\Lambda_I(2\pi t)) = \Delta_I^{it}, \; t \in \mathbb{R}, \qquad \text{dilations}$$
$$U(r_I) = J_I \qquad \text{reflection}$$

(if $I \sim \mathbb{R}^+$ by stereographic map, then $\Lambda_I(t) : x \mapsto e^{-t}x$)

- *Haag duality*: $H(I)' = H(I'), \qquad I' \equiv S^1 \setminus I$.

Converse construction (Brunetti, Guido, L.)

Given a positive energy unitary representation U of Möb on $\mathcal{H}$ we set

$$H(I) \equiv \{\xi \in \mathcal{H} : S_I \xi = \xi\}, \qquad S_I \equiv J_I \Delta_I^{1/2}$$

where Δ_I is *by definition* given by $\Delta_I^{-it} = U(\Lambda_I(2\pi t))$ with Λ_I one-parameter group of "dilations" associated with I.

Then H *is a local* Möb-*covariant net of standard subspaces*
Therefore:

Möb-covariant, local net of standard subspaces

$$\updownarrow$$

Unitary, positive energy representation of $PSL(2, \mathbb{R})$.

Entropy for $U(1)$ higher derivatives

$H_{(k)}$ net assocciated with the k-derivative of the $U(1)$- current.

The entropy of $f \in C_0^\infty(\mathbb{R})$ w.r.t. $H_{(k)}(B)$ on $\mathcal{H}^{(1)}$ is given by

$$S\big(f\|H_{(k)}(B)\big) = \pi \int_B (1-x^2)f'(x)^2 dx - \pi k(k-1)\int_B f(x)^2 dx\,;$$

if $\int_B x^n f(x)dx = 0$, $n = 0, 1, \ldots, k-2$. $(B = (-1, 1))$

Hence

$$S\big(f\|H_{(k)}(B)\big) \leq S\big(f\|H_{(1)}(B)\big)\,,\ k = 1, 2, \ldots$$

with $S\big(f\|H_{(1)}(B)\big)$ the universal bound.

Bekenstein inequality for wave packets

Let Φ be a (massive or massless) wave packet. If Φ is localized in B, that is the Cauchy data are supported in B, then

$$S(\Phi \| B) \leq 2\pi R \cdot E(\Phi \| B)$$

with

$$E(\Phi \| B) \equiv (\Phi, H\Phi)$$

the energy of Φ (in B), where H is the Hamiltonian and R is the radius of B.

(S. Hollands, G. Morsella, R. L.)

Corollary: bounds on the massive modular Hamiltoniam

As real-linear operators $M, L : L^2(B) \to L^2(B)$, we have the bounds

$$0 \leq M \leq R\,,$$
$$0 \leq -L \leq -R(\nabla^2 - m^2)$$

on $C_0^\infty(B)$.

Entropy formulas for wave packets

H the local, Poincaré covariant net of standard subspaces. Set $H(x) \equiv H(W(x))$, and $S(x) := S(\Phi|H(x))$ the entropy of $\Phi \in \mathcal{H}$ with respect to $H(x)$.

Entropy balance formula

$$S(x) - S(y) = \bar{S}(x) - \bar{S}(y) + 2\pi(y_1 - x_1)(\Phi, P\Phi) + 2\pi(y_0 - x_0)(\Phi, P_1\Phi)$$

Ant formula Set $S(a) = S(a, a)$, we have

$$S(a) - S(b) = \bar{S}(a) - \bar{S}(b) + 2\pi(b - a)(\Phi, P_+\Phi)$$

so, if $b > a$, we have

$$-\frac{S(a) - S(b)}{a - b} \leq 2\pi(\Phi, P_+\Phi)$$

by the monotonicity of the relative entropy. A

$$-\partial_a S(a) \leq 2\pi \inf_{\Psi \sim_a \Phi} (\Psi, P_+ \Psi) ,$$

Indeed

$$-\partial_a S(a) = 2\pi \inf_{\Psi \sim_a \Phi} (\Psi, P_+ \Psi) ,$$

at every point a such that $\partial_a S(a)$ exists.
As a consequence, $\partial_a S(a)$ is increasing, so S is a *convex function*

Crossed product and black hole entropy. Chandrasekaran, Penington, Witten, L., '23

The algebra of observables for a static patch in de Sitter spacetime, with operators gravitationally dressed to the worldline of an observer. $\mathcal{M} = \mathcal{A}(dS)$, K the Hamiltonian (proportional to the modular Hamiltonian), Q the Hamiltonian of the observer

$$e^{itH} = e^{itK} \otimes e^{itQ} = \text{global evolution}$$

$$\hat{\mathcal{M}} = \{\mathcal{M} \otimes 1, e^{itH}\}'' \text{ global algebra (crossed product)}$$

$\hat{\mathcal{M}}$ is a type II_∞ factor. Take E positive spectral projection of H

$$\mathrm{trace}(E) = 1 < \infty \implies E\hat{\mathcal{M}}E \text{ is a factor of type } II_1 \text{ (finite trace)}$$

(Crossed product and reduction)

There is a natural notion of entropy for a state of such an algebra. There is a maximum entropy state (the trace), which corresponds to an empty de Sitter space

$$\Psi_{max} = \Psi_{dS}\sqrt{\beta_{dS}}e^{\beta_{dS}x/2}$$

and the entropy of any semiclassical state of the type II_1 algebras agrees, up to an additive constant independent of the state, with the expected generalized entropy $S_{gen} = (A/4G_N) + S_{out}$