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Cosmological Singularities and Bouncing Universes

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General Introduction and Motivations

- Appearance of singularities is one of the most important phenomena in General Relativity and its generalizations and modifications.
- The singularities were first discovered in such simple geometries as those of Friedmann and Schwarzschild and later their general character was established (Penrose, Hawking).
- The investigation of the oscillatory approach to the cosmological singularity (Belinsky, Khalatnikov, Lifshitz) known also as Mixmaster universe (Misner) has opened the way to the birth of a new branch of the mathematical physics chaotic cosmology and hyperbolic Kac-Moody algebras (Damour, Henneaux, Nicolai).

General Introduction and Motivations

- Is it possible to avoid singularity due to a bounce?
- How often the bounces are realized?
- What happens when a universe approaching the cosmological singularity?
- One can try to study the opportunity to cross the singularity.
- Very interesting effects arise at the singularities crossing.
- Can quantum cosmology eliminate the singularities arising in classical General Relativity?
- What happens with quantum particles at the singularity crossing?
- When is it possible and when it is not possible to describe the transition through the singularity?
  An attempt of a general approach.

3 Lectures

1. Bouncing universes
2. Oscillatory approach to the cosmological singularity
3. Singularities and their crossing

Lecture 1. Bouncing Universes

1. Bounces in simple cosmological models
2. Bounces and chaos in cosmology
3. Chaos and semiclassical quantization

Bounces in simple cosmological models

Let us consider a closed Friedmann universe

$$ds^2 = dt^2 - a^2(t)d^2\Omega^{(3)},$$

filled with a minimally coupled massive scalar field with the action

$$S = \int d^4x\sqrt{-g}\left\{\frac{m_P^2}{16\pi}R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}m^2\phi^2\right\},$$

where m_P is the Planck mass.

$$\frac{m_P^2}{16\pi}\left(\ddot{a}+\frac{\dot{a}^2}{2a}+\frac{1}{2a}\right)+\frac{a\dot{\phi}^2}{8}-\frac{m^2\phi^2 a}{8}=0,$$

$$\ddot{\phi}+\frac{3\dot{\phi}\dot{a}}{a}+m^2\phi=0.$$

The first integral of motion of our system (the first Friedmann equation) is

$$-\frac{3}{8\pi}m_P^2(\dot{a}^2+1)+\frac{a^2}{2}\left(\dot{\phi}^2+m^2\phi^2\right)=0.$$

All the trajectories (cosmological evolutions) in this model have points of the maximal expansion. In the plane (a,ϕ) these points find themselves in the so called Euclidean region

$$\phi^2 a^2 \leq \frac{3}{4\pi}\frac{m_P^2}{m^2}.$$

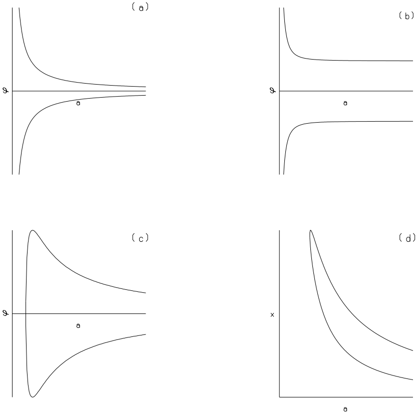


Figure 1: In Fig. 1 the shapes of Euclidean regions for different isotropic cosmological models with the scalar field are presented; a) simplest model with the scalar field, b) model with negative cosmological constant, c) model with hydrodynamical matter, d) model with complex scalar field.

More precisely, the points of maximal expansion of the universe find themselves in the internal part of the Euclidean region

$$\phi^2 a^2 \leq \frac{1}{2\pi}\frac{m_P^2}{m^2}.$$

Some trajectories can have the points of minimal contraction or bounces in the external part of the Euclidean region

$$\frac{1}{2\pi}\frac{m_P}{m^2} \leq \phi^2 a^2 \leq \frac{3}{4\pi}\frac{m_P^2}{m^2}.$$

Probability to have a bounce is very low. (Parker and Fulling, 1973, Starobinsky, 1978). Hawking, 1984 has shown that the set of infinitely bouncing trajectories has measure zero. Page, 1984 argued that this set has a fractal nature.

The fractality of the set of infinitely bouncing trajectories were further studied in our papers Kamenshchik, Khalatnikov and Toporensky, 1997-1998.

- We have classified all possible trajectories, starting from their points of maximal expansion and using numerical simulations.
- The region of the localization of these points is divided into intermingled zones. There are zones, where the trajectory immediately falls to the singularity, there are zones, where the trajectory falls to the singularity after one oscillation of the scalar field (the crossing of the value $\phi = 0$), there are zones where the trajectory falls to the singularity after n oscillations of the scalar field and so on.

- Between these zones there are zones where the universe has a bounce (a point of minimal contraction) after n oscillations.
- Note that the points, where the scalar field achieves the extremal values can be found only in the Lorentzian region

$$\phi^2 a^2 \geq \frac{3}{4\pi}\frac{m_P^2}{m^2}.$$

- Then studying the substructure of the zones where a bounce has taken place from the point of view of the possibility of having two bounces, one can see that this substructure repeats on the qualitative level the structure of the whole region of the possible points of maximal expansion.
- Continuing this procedure *ad infinitum* one can see that as the result one has the fractal set of infinitely bouncing trajectories.
- The same scheme gives us an opportunity to see that there is also a set of periodical trajectories. All these periodical trajectories contains bounces intermingled with a series of oscillations of the scalar field ϕ.
- If this set is large enough then the dynamics of the system is chaotic.

Bounces and chaos in cosmology

A simple method to prove that a dynamical system is chaotic is to calculate its topological entropy.
Cornish and Shellard, 1998 have calculated it for this simple cosmological model, using the symbolical dynamics and have find that it is positive.

- Let us denote by the letter A a bounce, and by the letter B an oscillation of the scalar field.
- For the simplest model with the scalar field there is the only prohibition rule: two letters A cannot stay together, which means that it is impossible to have two bounces one after another without oscillations between them.
- The strings of the letters B can be arbitrarily long.

- $Q(k)$ - the number of "words" (trajectories) of length k, which begin with A and end with A.
- $P(k)$ - the number of words which begin with A and end with B.
- The recurrent relations:

$$Q(k+1) = P(k),$$
$$P(k+1) = Q(k) + P(k).$$

- They give

$$P(k+1) = P(k) + P(k-1).$$

-

$$P(2) = 1, \ (AB), \ P(3) = 1, \ (ABB).$$

.
- This is the Fibonacci sequence!

The formula for the general term of the Fibonacci series.

One can look for $P(k)$ as a linear combination of terms λ^k, where λ is the solution of equation

$$\lambda^{k+1} = \lambda^k + \lambda^{k-1}$$

or, equivalently, because we are interested only in nonzero roots

$$\lambda^2 - \lambda - 1 = 0.$$

Looking for $P(k)$ in the form

$$P(k) = c_1\lambda_1^k + c_2\lambda_2^k,$$

where λ_1 and λ_2 are the solutions of the equation above and satisfying the conditions for $k = 2, k = 3$, we obtain

$$P(k) = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} + (-1)^{k-2}\left(\frac{\sqrt{5}-1}{2}\right)^{k-1}\right].$$

The definition of the topological entropy.

We denote by $N(k)$ the number of periodic trajectories of period or length k.
The definition of topological entropy:

$$H_T = \limsup_{k\to\infty} \frac{1}{k} \ln N(k).$$

If $H_T > 0$, one can conclude that the dynamics is chaotic.
In our case

$$H_T = \limsup_{k\to\infty} \frac{1}{k} \ln P(k) = \ln\left(\frac{1+\sqrt{5}}{2}\right) > 0,$$

where $\frac{1+\sqrt{5}}{2}$ is the famous golden ratio.

In the paper
A.Yu. Kamenshchik, I. M. Khalatnikov, S.V. Savchenko and A.V. Toporensky,
Topological entropy for some isotropic cosmological models,
Phys. Rev. D 59 (1999) 123516
the topological entropy was calculated for some more complicated models.

We shall have the same letters A - bounces, and B - oscillations of the scalar field ϕ, but the "grammatical" rules will be more complicated.

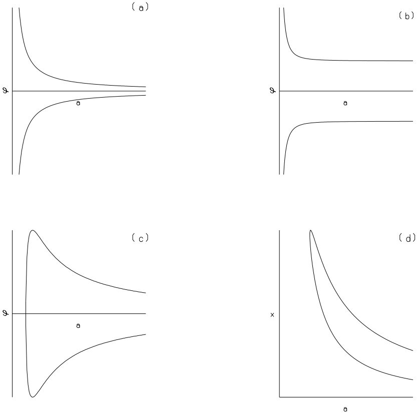


Figure 1: In Fig. 1 the shapes of Euclidean regions for different isotropic cosmological models with the scalar field are presented; a) simplest model with the scalar field, b) model with negative cosmological constant, c) model with hydrodynamical matter, d) model with complex scalar field.

In the case of the cosmological model with scalar field and negative cosmological constant the periodical trajectories can have only a restricted number of oscillations of the scalar field ϕ. This rule can be encoded in the prohibition to have more that n letters B staying together.
The recurrent relations are

$$Q(k+1) = P(k),$$
$$P(k+1) = Q(k) + P(k) - Q(k-n)\theta(k-n).$$

In the limit $k \to \infty$ one can write down the recurrent relation:

$$P(k+1) = P(k) + P(k-1) - P(k-n-1),$$

which in turn implies the following equation for topological entropy:

$$\lambda^{n+2} - \lambda^{n+1} - \lambda^{n} + 1 = 0,$$

where topological entropy is equal to the logarithm of its biggest solution.

In the model with the scalar field and matter or in the model with the complex scalar field and nonzero classical charge, the rules governing the structure are rather complicated.

We shall consider one particular example, however, the algorithm which shall be presented could be used for different sets of rules as well.

The rules

(1) It is impossible to have more than 19 letters B;
(2) after a series with 19 letters B (and letter A), one can have the next series only with 1 letter B;
(3) after a series with 18 letters B, one can have the next series with 1 or 2 letters B;
(4) after a series with 17 letters B, one can have the next series with 1, 2 or 3 letters B:
$\dots$
(20) after a series with 1 letter B, one can have the series with n letters B, where $0 \leq n \leq 19$.

Let us now introduce the following notation:

$Q(k)$ is the number of words of length k which begin with letter A and end with letter A,
$Q_1(k)$ is the number of words of length k beginning with letter A and ending with a series with 1 letter B,
$Q_2(k)$ is the number of words beginning with letter A and ending with the series with 2 letters B,
$\dots$,
$Q_{19}(k)$ is the number of words beginning with letter A and ending with the series with 19 letters B.

The system of recurrence relations for these quantities is

$$
\begin{aligned}
&Q(k+1) = Q_1(k) + Q_2(k) + \cdots + Q_{19}(k),\\
&Q_1(k) = Q(k-1),\\
&Q_{19}(k) = Q_1(k-20),\\
&Q_d(k) = Q_{d-1}(k-1) - Q_{21-d}(k-d-1),\ 2 \le d \le 10\\
&Q_d(k) = Q_{d+1}(k+1) + Q_{20-d}(k-d-1),\ 11 \le d \le 18
\end{aligned}
$$

Resolving this system with respect to $Q(k)$, one get the following recurrent relation:

$$Q(k+1) = Q(k-1) + Q(k-2) + Q(k-3) + Q(k-4)$$
$$+Q(k-5) + Q(k-6) + Q(k-7) + Q(k-8) + Q(k-9)$$
$$+Q(k-10) + Q(k-13) + 2Q(k-14) + 3Q(k-15) + 4Q(k$$
$$+5Q(k-17) + 6Q(k-18) + 7Q(k-19) + 8Q(k-20) + 9Q($$
$$-9Q(k-24) - 16Q(k-25) - 21Q(k-26) - 24Q(k-27)$$
$$-25Q(k-28) - 24Q(k-29) - 21Q(k-30) - 16Q(k-31)$$
$$-9Q(k-32) - 8Q(k-36) - 21Q(k-37) - 36Q(k-38)$$
$$-50Q(k-39) - 60Q(k-40) - 63Q(k-41) - 56Q(k-42)$$
$$-36Q(k-43) + 36Q(k-47) + 84Q(k-48) + 126Q(k-49)$$
$$+150Q(k-50) + 150Q(k-51) + 126Q(k-52) + 84Q(k-5$$
$$+36Q(k-54) + 28Q(k-59) + 84Q(k-60) + 150Q(k-61)$$

$$+200Q(k-62)+210Q(k-63)+168Q(k-64)+84Q(k-6$$
$$-84Q(k-70)-224Q(k-71)-350Q(k-72)-400Q(k-7$$
$$-350Q(k-74)-224Q(k-75)-84Q(k-76)-56Q(k-82$$
$$-175Q(k-83)-300Q(k-84)-350Q(k-85)-280Q(k-$$
$$-126Q(k-87)+126Q(k-93)+350Q(k-94)+525Q(k-$$
$$+525Q(k-96)+350Q(k-97)+126Q(k-98)+70Q(k-1$$
$$+210Q(k-106)+315Q(k-107)+280Q(k-108)+126Q(k$$
$$-126Q(k-116)-336Q(k-117)-441Q(k-118)-336Q(k$$
$$-126Q(k-120)-56Q(k-128)-147Q(k-129)-168Q(k$$
$$-84Q(k-131)+84Q(k-139)+196Q(k-140)+196Q(k-$$
$$+84Q(k-142)+28Q(k-151)+56Q(k-152)+36Q(k-1$$
$$-36Q(k-162)-64Q(k-163)-36Q(k-164)-8Q(k-17$$
$$-9Q(k-175)+9Q(k-185)+9Q(k-186)+Q(k-197)$$
$$-Q(k-208).$$

It gives the following equation for the topological entropy:

$$x^{209} - x^{207} - x^{206} - x^{205} - x^{204} - x^{203} - x^{202} - x^{201} - x^{200}$$
$$-x^{199} - x^{198} - x^{195} - 2x^{194} - 3x^{193} - 4x^{192} - 5x^{191} - 6x^{190}$$
$$-7x^{189} - 8x^{188} - 9x^{187} + 9x^{184} + 16x^{183} + 21x^{182} + 24x^{181}$$
$$+25x^{180} + 24x^{179} + 21x^{178} + 16x^{177} + 9x^{176} + 8x^{172} + 21x^{171}$$
$$+36x^{170} + 50x^{169} + 60x^{168} + 63x^{167} + 56x^{166} + 36x^{165} - 36x^{161}$$
$$-84x^{160} - 126x^{159} - 150x^{158} - 150x^{157} - 126x^{156} - 84x^{155}$$
$$-36x^{154} - 28x^{149} - 84x^{148} - 150x^{147} - 200x^{146} - 210x^{145}$$
$$-168x^{144} - 84x^{143} + 84x^{142} + 224x^{137} + 350x^{136} + 400x^{135}$$
$$+350x^{134} + 224x^{133} + 84x^{132} + 56x^{126} + 175x^{125} + 300x^{124}$$
$$+350x^{123} + 280x^{122} + 126x^{121} - 126x^{115} - 350x^{114} - 525x^{113}$$
$$-525x^{112} - 350x^{111} - 126x^{110} - 70x^{103} - 210x^{102} - 315x^{101}$$
$$-280x^{100} - 126x^{99} + 126x^{92} + 336x^{91} + 441x^{90} + 336x^{89}$$

$$
\begin{aligned}
&+126x^{88} + 56x^{80} + 147x^{79} + 168x^{78} + 84x^{77} - 84x^{69}\\
&-196x^{68} - 196x^{67} - 84x^{66} - 28x^{57} - 56x^{56} - 36x^{55}\\
&+36x^{46} + 64x^{45} + 36x^{44} + 8x^{34} + 9x^{33} - 9x^{23}\\
&-9x^{22} - x^{11} + 1 = 0.
\end{aligned}
$$

Resolving numerically this equation we find the biggest root

$$\lambda \approx 1.61771.$$

Correspondingly the topological entropy is given by the logarithm of the biggest root.

Chaos and semiclassical quantization

A. Einstein, Zur Quantuensatz von Zommerfeld and Epstein [On the Quantum Theorem of Sommerfeld and Epstein], Deutsche Physicalische Gesellschaft, Verhandlungen 19, 82-92 (1917).

See also,
A.D. Stone, Einstein's unknown insights and the problem of quantizing chaos, Physics Today 58 (8), 37-43 (2005).

The paper by Einstein contained an elegant reformulation of the Bohr-Sommerfeld quantization rules of the old quantum theory.
Besides, it offered an insight into the limitations of the old quantum theory when applied to a mechanical system that is nonintegrable-or in modern terminology, chaotic.

For the mechanical degree of freedom the quantization condition is

$$\int pdq = \int_0^T p\frac{dq}{dt}dt = nh,$$

where T is the period of one-dimensional bound motion.

For a multidimensional separable bound motion

$$\int p_i dq^i = \int_0^{T_i} p_i \frac{dq^i}{dt}dt = n_i h, \;\; i = 1, 2, \cdots, d.$$

Generally, the integral

$$\int \sum p_i dq^i$$

is invariant.
This integral is connected with the action

$$\int \sum p_i dq^i = S(r, r', t) = \int_0^t L dt - Et.$$

The momentum

$$p = \nabla S$$

can be nonsingle-valued due to the presence of turning points. Einstein noticed that it becomes single-valued if the motion is treated as a motion on a d-dimensional torus.

Einstein proposed the generalized quantization rule:

$$\oint_{C_i} \vec{p}d\vec{q} = n_i h, \ i = 1, \cdots, d,$$

where C_i are d independent loops on the d-dimensional torus, n_i are the quantum numbers associated with the energetic levels of the system.

The main point

The periodical trajectories is an exceptional case.
The bounded trajectories are non-periodical but they visit any volume element dr an infinite number of times.
"A priori, two types of orbits are possible, obviously of fundamentally different characteristics." To paraphrase Einstein:
An orbit of type (a) passes through dr an infinite number of times with only a finite number of different momentum directions,
an orbit of type (b) passes through dr an infinite number of times with an infinite number of different momentum directions.
In the latter case, the momentum p cannot be represented as a multivalued function of r as it was in the central-force example.

Einstein: “One notices immediately that the type (b) motion excludes the quantum condition we have formulated”.
When do we have the motion of the type (b)?
“ If there exist fewer than d constants of motion, as is the case, for example, according to Poincaré in the three-body problem, then the p_i are not expressible by the q_i and the quantum condition of Sommerfeld–Epstein fails also in the slightly generalized form that has been given here”.

It is not the complexity or the number of degrees of freedom of the system that matters. The dynamics of systems with fewer constants of motion than degrees of freedom are fundamentally different from those of systems with at least as many constants as degrees of freedom. In modern terminology, the former type of system is nonintegrable and the latter integrable.

By 1971, although he was unaware of Einstein's work, Gutzwiller had come to understand that it is not possible to use a Bohr–Sommerfeld type of quantization to deal with chaotic systems. He introduced an entirely new semiclassical approach that abandoned the attempt to find individual chaotic states. Instead, his approach yielded an equation to calculate the density of states of a chaotic system from knowledge of the unstable periodic orbits of the system.

Studying the quantization of the classically chaotic systems is called Quantum Chaos.

Which lesson for cosmology follows from all that?
Studying the quantum state of the Universe and its dynamics (quantum cosmology) we have to use the semiclassical methods with caution.

Einstein-Brillouin-Keller quantization

$$I_i = \frac{1}{2\pi} \oint p_i dq_i = \hbar \left(n_i + \frac{\mu_i}{4} + \frac{b_i}{2} \right),$$

where n_i is a positive integer, μ_i - the number of classical turning points and b_i - the number of reflections from a hard wall.

Together, μ_i and b_i are called Maslov indices.

Lecture 2. Oscillatory approach to the cosmological singularity

1. Introduction
2. Killing vector fields and symmetries
3. Bianchi classification of the three-dimensional Lie algebras and of the homogeneous cosmologies
4. Bianchi-I universe
5. Bianchi-II universe
6. Bianchi-IX universe, the oscillatory approach to the singularity (BKL) and the Mixmaster universe
7. Chaos in cosmology
8. Mixmaster universes and infinite-dimensional Lie algebras
9. Concluding remarks

Introduction

- The Friedmann - Lemaître cosmology

$$ds^2 = dt^2 - a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right),$$

describing an expanding or contracting spatially homogeneous and isotropic universe is very successful in the description of its global evolution from the times of inflation until the present epoch of the cosmic acceleration.
A.A. Friedmann, 1922,1924; G. Lemaître, 1927, 1931.

- For the description of the inhomogeneities of the universe one uses the theory of the cosmological perturbations on the Friedmann background:

$$g_{ij} = g_{ij}^{(0)} + h_{ij}.$$

E.M. Lifshitz, 1946

One can explain the origin of the large-scale structure of the contemporary universe starting from the quantum fluctuations in the very early universe.

V.F. Mukhanov and G.V. Chibisov, 1981

- However, there are very interesting things in gravity and cosmology beyond Friedmann models and the perturbations on their background.
- The study of spatially homogeneous, but anisotropic models takes its origin from the work by Luigi Bianchi, written as early as in the year 1898.
- In this work the complete classification of the three-dimensional homogeneous Riemann spaces and the three-dimensional Lie groups was presented long before the creation of the General Relativity by Einstein in 1916.
- Later, in 50th and 60th the Bianchi classification was modernised, simplified and applied to the cosmology. A. Taub, O. Heckmann and E. Schucking, C.G. Behr and others

- In 1963 I.M. Khalatnikov and E.M. Lifshitz have begun apply the Bianchi universes for the study of the problem of singularity in cosmology.
- At the end of sixties V.A. Belinski, I.M. Khalatnikov and E.M. Lifshitz have discovered the oscillatory approach to singularity.
- Using the Hamiltonian formalism this phenomenon was described by C. Misner as the Mixmaster Universe.
- When universe tends to the singularity arises chaos.
- At the beginning of the new millennium the connection between the chaotic behaviour in the cosmological models and the infinite-dimensional Lie algebras was discovered T. Damour, M. Henneaux and H. Nicolai.

- The study of the oscillatory approach to the cosmological singularity and of its relation with some advanced mathematical structures has become an important branch of mathematical physics.
- It can become useful for the description of some physical effects in the very early universe.
- It is not excluded that this connection is important for quantum gravity.

Killing vector fields and symmetry

On the manifold $\mathcal{M}$ it is defined a vector field $X(P)$. In any point this field defines a curve to which this field is tangent. One can represent this field as

$$X = \frac{d}{d\lambda},$$

where λ parametrises the curve. Then, making infinitesimal shifts along the curves, one can define the Lie derivative L_X. For a function f:

$$L_X f = Xf = \frac{df}{d\lambda}.$$

For a vector field Y:

$$L_X Y = [X, Y].$$

If

$$L_X g = 0,$$
$$g_{ij,k} X^k + g_{ik} X^k_{,j} + g_{kj} X^k_{,i} = 0,$$
$$X_{i;j} + X_{j:i} = 0,$$

then X is called Killing vector field and the equation is called Killing equation.

Here ; signifies the covariant derivative.

The transformations of the manifold generated by the Killing fields do not change the form of the metric and are called isometries.

An n-dimensional manifold can have no more than

$$\frac{n(n+1)}{2}$$

Killing fields.
If the manifold has n Killing vector fields it is called homogeneous.

Bianchi classification of the three-dimensional Lie algebras and of the homogeneous cosmologies

Let us suppose that we have three spatial Killing fields X_a. In this case, our spacetime is spatially homogenous.
In this case the structure functions of these fields are constants. It is not convenient to choose these three vector fields as a basis, because the metric components in this basis depend on coordinates.
We can find other three vector fields e_a, which commute with the Killing fields:

$$[e_a, X_b] = 0.$$

Then we can construct the dual basis of one-forms ω^a:

$$\omega^a(e_b) = \delta^a_b.$$

Now, we can construct the cosmological models with the metric

$$ds^2 = dt^2 - g_{ab}(t)\omega^a \otimes \omega^b,$$

where the metric coefficients depend only on the time parameter.
This metric is invariant with respect to the Killing vectors X_a.
The components of the curvature tensors depend only on the coefficients $g_{ab}(t)$, their time derivatives and the structure constants of the Lie algebra, generated by the vectors e_a.

All the three-dimensional Bianchi Lie algebras have the form

$$
\begin{aligned}
[X_1, X_2] &= -aX_2 + n_3 X_3, \\
[X_2, X_3] &= n_1 X_1, \\
[X_3, X_1] &= n_2 X_2 + aX_3,
\end{aligned}
$$

where

$$n_1 a = 0.$$

Einstein equations in the synchronous reference system for the Bianchi universes

$$\kappa_{ab} = \dot{\gamma}_{ab}, \kappa_a^b = \dot{\gamma}_{ac}\gamma^{cb}.$$

$$P_a^b = \frac{1}{2\gamma}[2C^{bd}C_{ad} + C^{db}C_{ad} + C^{bd}C_{da} - C^d{}_d(C^b{}_a + C_a{}^b)$$
$$+\delta_a^b((C^d{}_d)^2 - 2C^{df}C_{df})],$$

$$C_a{}^b = \gamma_{ac}C^{cb}, \;\; C_{ab} = \gamma_{ac}\gamma_{bd}C^{cd}.$$

$$[X_a, X_b] = C_{ab}^c X_c, \;\; C_{ab}^c = \epsilon_{abd}C^{dc}.$$

$$R^0_0 = -\frac{1}{2}\dot{\kappa}^a_a - \frac{1}{4}\kappa^b_a\kappa^a_b,$$

$$R^0_a = -\frac{1}{2}\kappa^a_b(C^b_{ca} - \delta^b_a C^d_{dc}),$$

$$R^b_a = -\frac{1}{2\sqrt{\gamma}}(\sqrt{\gamma}\kappa^b_a)^{\cdot} - P^b_a.$$

Bianchi-I universe

$$X_a = \frac{\partial}{\partial x^a}, \;\; a = 1, 2, 3$$
$$e_a = X_a,$$
$$\omega^a = dx^a,$$
$$ds^2 = dt^2 - a^2(t) \sum_{a=1}^{3} e^{2\beta_a(t)} (dx^a)^2,$$
$$\sum_{a=1}^{3} \beta_a = 0.$$

where β_a are the anisotropy parameters.

$$
\begin{aligned}
\kappa_{ab} &= 2(\dot{a}a + \dot{\beta}_a a^2)e^{2\beta_a}\delta_{ab}, \\
\kappa_a^b &= 2\left(\frac{\dot{a}}{a} + \dot{\beta}_a\right)\delta_a^b, \\
\kappa &= 6\frac{\dot{a}}{a}, \\
P_a^b &= 0, \\
R_0^0 &= -3\frac{\ddot{a}}{a} - \sum_{a=1}^{3}\dot{\beta}_a^2, \\
R_a^0 &= 0, \\
R_a^b &= -\left(\frac{\ddot{a}}{a} + 2\frac{\dot{a}^2}{a^2} + 3\frac{\dot{a}}{a}\dot{\beta}_a - \ddot{\beta}_a\right)\delta_a^b, \\
R &= -\left(6\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \sum_{a=1}^{3}\dot{\beta}_a^2\right).
\end{aligned}
$$

Let us suppose that the universe is filled with some fluid with an isotropic pressure. That means that

$$R_1^1 = R_2^2 = R_3^3.$$

Then

$$R_2^2 + R_3^3 - 2R_1^1 = 0.$$

Hence,

$$\ddot{\beta}_1 + 3\dot{\beta}_1 \frac{\dot{a}}{a} = 0,$$

and

$$\dot{\beta}_1 = \frac{\beta_{10}}{a^3}.$$

Analogously,

$$\dot{\beta}_2 = \frac{\beta_{20}}{a^3}, \ \dot{\beta}_3 = \frac{\beta_{30}}{a^3},$$

where

$$\sum_{a=1}^{3} \beta_{a0} = 0.$$

The 00 - component of the Einstein equations

$$R^0_0 - \frac{1}{2} R = \rho,$$

where ρ is the energy density of the fluid, is now

$$3\frac{\dot{a}^2}{a^2} - \frac{1}{2}\frac{\bar{\beta}^2}{a^6} = \rho,$$

where

$$\bar{\beta}^2 \equiv \sum_{a=1}^{3} \beta^2_{a0}.$$

Empty universe. Kasner solution (1921)

If $\rho = 0$, then

$$a(t) = \left(\frac{3}{2}\right)^{\frac{1}{6}} (\bar{\beta} t)^{\frac{1}{3}}.$$

Correspondingly, three scale factors have the form

$$a_a(t) = a(t) e^{\beta_a(t)} = a_{a0} t^{p_a},$$

where

$$p_a = \frac{1}{3} + \sqrt{\frac{2}{3}} \frac{\beta_{a0}}{\bar{\beta}}$$

and

$$p_1 + p_2 + p_3 = 1,$$
$$p_1^2 + p_2^2 + p_3^2 = 1.$$

p_a are the Kasner indices.

Lifshitz-Khalatnikov parametrization of the Kasner indices (1963)

$$p_1 = -\frac{u}{1+u+u^2},$$
$$p_2 = \frac{1+u}{1+u+u^2},$$
$$p_3 = \frac{u(1+u)}{1+u+u^2}.$$

A remarkable property:

$$p_1\left(\frac{1}{u}\right) = p_1(u),$$
$$p_2\left(\frac{1}{u}\right) = p_3(u),$$
$$p_3\left(\frac{1}{u}\right) = p_2(u).$$

Bianchi-II universe

$$
\begin{aligned}
[X_2, X_3] &= X_1, \\
X_1 &= \frac{\partial}{\partial z}, \\
X_2 &= \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}, \\
X_3 &= \frac{\partial}{\partial x}, \\
e_1 &= \frac{\partial}{\partial z}, \\
e_2 &= \frac{\partial}{\partial y}, \\
e_3 &= \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}.
\end{aligned}
$$

$$\omega^1 = dz + ydx,$$
$$\omega^2 = dy,$$
$$\omega^3 = dx.$$

The metric is

$$dt^2 - a^2(t)(\omega^1)^2 - b^2(t)(\omega^2)^2 - c^2(t)(\omega^3)^2.$$

It is convenient to introduce a new time parameter τ

$$dt = a(t)b(t)c(t)d\tau,$$

and to represent the scale factors as

$$a = e^{\alpha(\tau)},$$
$$b = e^{\beta(\tau)},$$
$$c = e^{\gamma(\tau)}.$$

The spatial components of the Einstein equations in the empty spacetime are

$$\ddot{\alpha} = -\frac{1}{2}e^{4\alpha},$$

$$\ddot{\beta} = \frac{1}{2}e^{4\alpha},$$

$$\ddot{\gamma} = \frac{1}{2}e^{4\alpha},$$

while the temporal component of the Einstein equations is

$$\ddot{\alpha} + \ddot{\beta} + \ddot{\gamma} = 2(\dot{\alpha}\dot{\beta} + \dot{\alpha}\dot{\gamma} + \dot{\beta}\dot{\gamma}).$$

These equations can be integrated. The result is as follows. Consider a contraction of the universe. An asymptotic regime at $\tau \to \infty$, the Bianchi-II universe behaves as the Kasner solution of the Bianchi-I universe

$$a(t) \sim t^{p_1},$$
$$b(t) \sim t^{p_2},$$
$$c(t) \sim t^{p_3},$$

where

$$p_1 = -\frac{u}{1+u+u^2},$$
$$p_2 = \frac{1+u}{1+u+u^2},$$
$$p_3 = \frac{u(1+u)}{1+u+u^2}.$$

When the universe moves closer to the singularity $\tau \to -\infty$ we have

$$a(t) \sim t^{p_1'},$$
$$b(t) \sim t^{p_2'},$$
$$c(t) \sim t^{p_3'},$$

where the Kasner indices are given by

$$p_1' = \frac{u}{1-u+u^2} = p_2(u-1),$$
$$p_2' = \frac{1-u}{1-u+u^2} = p_1(u-1),$$
$$p_3' = \frac{u(u-1)}{1-u+u^2} = p_3(u-1).$$

Thus, we have reproduced the known law of the transition from one Kasner epoch to another.

Bianchi-IX universe, the oscillatory approach to the singularity (BKL) and the Mixmaster universe

Geometry

$$[X_1, X_2] = X_3,\ [X_2, X_3] = X_1,\ [X_3, X_1] = X_2.$$

This is the algebra Lie of the Lie group $SU(2)$ or $SO(3)$.
The elements of the $SU(3)$ can be represented as

$$g = x^0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + x^1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + x^2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + x^3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where

$$(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = 1,$$

i.e. the group manifold of $SU(3)$ is the three-dimensional sphere of the unit radius, embedded into the four-dimensional Euclidean space $\mathbb{R}^4$.

We can choose the three Killing vectors in the space $\mathbb{R}^4$:

$$X_1 = \frac{1}{2}\left(x^0\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^0} - x^2\frac{\partial}{\partial x^3} + x^3\frac{\partial}{\partial x^2}\right),$$
$$X_2 = \frac{1}{2}\left(x^0\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^0} + x^1\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^1}\right),$$
$$X_3 = \frac{1}{2}\left(x^0\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^0} - x^1\frac{\partial}{\partial x^2} + x^2\frac{\partial}{\partial x^1}\right).$$

This vectors constitute the Bianchi-IX Lie algebra in $\mathbb{R}^4$ and being restricted on the sphere $\mathbb{S}^3$ as well.

The triple of the reciprocal vector fields is

$$e_1 = \frac{1}{2}\left(x^0\frac{\partial}{\partial x^1} - x^1\frac{\partial}{\partial x^0} + x^2\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^2}\right),$$

$$e_2 = \frac{1}{2}\left(x^0\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^0} - x^1\frac{\partial}{\partial x^3} + x^3\frac{\partial}{\partial x^1}\right),$$

$$e_3 = \frac{1}{2}\left(x^0\frac{\partial}{\partial x^3} - x^3\frac{\partial}{\partial x^0} + x^1\frac{\partial}{\partial x^2} - x^2\frac{\partial}{\partial x^1}\right).$$

We shall use the Hopf parametrization for the three-dimensional unit sphere:

$$x^0 = \cos\chi\cos\phi,$$
$$x^1 = \cos\chi\sin\phi,$$
$$x^2 = \sin\chi\cos\psi,$$
$$x^3 = \sin\chi\sin\psi,$$
$$0 \le \phi < 2\pi,\ \ 0 \le \psi < 2\pi,\ \ 0 \le \chi \le \frac{\pi}{2}.$$

The fields e_i being restricted on the $\mathbb{S}^3$ have the form:

$$e_1 = \frac{1}{2}\left(\frac{\partial}{\partial\phi} + \frac{\partial}{\partial\psi}\right),$$

$$e_2 = \frac{1}{2}\left(\tan\chi\sin(\phi+\psi)\frac{\partial}{\partial\phi} - \cot\chi\sin(\phi+\psi)\frac{\partial}{\partial\psi}\right.$$

$$\left.+\cos(\phi+\psi)\frac{\partial}{\partial\chi}\right),$$

$$e_3 = \frac{1}{2}\left(-\tan\chi\cos(\phi+\psi)\frac{\partial}{\partial\phi} + \cot\chi\cos(\phi+\psi)\frac{\partial}{\partial\psi}\right.$$

$$\left.+\sin(\phi+\psi)\frac{\partial}{\partial\chi}\right).$$

The dual basis of one-forms is

$$\omega^1 = 2(\cos^2\chi d\phi + \sin^2\chi d\psi),$$
$$\omega^2 = \sin 2\chi \sin(\phi+\psi) d\phi - \sin 2\chi \sin(\phi+\psi) d\psi$$
$$+2\cos(\phi+\psi) d\chi,$$
$$\omega^3 = -\sin 2\chi \cos(\phi+\psi) d\phi + \sin 2\chi \cos(\phi+\psi) d\psi$$
$$+2\sin(\phi+\psi) d\chi.$$

Using these forms we can construct the metric of a Bianchi-IX universe.
It is easy to check that

$$\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3$$

is the metric of the three-dimensional sphere with the radius equal to 2.

Dynamics

We have seen that the presence of the term $\frac{a^2}{b^2c^2}$ in the spatial curvature of the Bianchi-II universe, where $a \to \infty$, while $b, c \to 0$ when the universe tends to the singularity, induces the transition from one Kasner regime (epoch) to another. As a result, the functions $a(t)$ and $b(t)$ exchange their roles, while the parameter

$$u \to u - 1.$$

The term $\frac{a^2}{b^2c^2}$ becomes small.

However, in the Bianchi-IX universe three terms are present:

$$\frac{a^2}{b^2c^2}, \quad \frac{b^2}{a^2c^2}, \quad \frac{c^2}{a^2b^2}.$$

When $\frac{a^2}{b^2c^2}$ becomes small the term $\frac{b^2}{a^2c^2}$ becomes large. It implies the next change of the Kasner epoch (a and b exchange their roles) and again

$$u \rightarrow u - 1$$

If we start from some value

$$u > 1$$

then we have $[u]$ changes of the Kasner epochs, where $[u]$ is the integer part of the number u.

Then, the parameter

$$u < 1.$$

That means that

$$p_2 > p_3$$

and the functions $b(t)$ and $c(t)$ exchange their roles.
It is called the change of the Kasner era and can be described by the transformation

$$u \to \frac{1}{u}.$$

The number $\frac{1}{u} > 1$ and the new series of the changes of the Kasner epochs inside the new Kasner era begins.
All this is called Oscillatory approach to the cosmological singularity or BKL.

A particular case

If

$$u = \frac{p}{q}, \text{ a rational number,}$$

then the process of the changes of Kasner epochs and eras will bring us to

$$u = 1$$

and, then to

$$u = 0.$$

It means that

$$p_1 = 0,$$
$$p_2 = 1,$$
$$p_3 = 0.$$

$$ds^2 = dt^2 - t^2 dx^2 - dy^2 - dz^2.$$

What does it mean ?
It is simply a disguised Minkowski spacetime.
If we take the Minkowski spacetime

$$ds^2 = dT^2 - d\xi^2 - dy^2 - dz^2$$

and make the change of variables:

$$T = t \cosh x,$$
$$\xi = t \sinh x,$$

we shall obtain it.

However, the rational numbers is the measure zero subset of the set of real numbers.

Mixmaster universe
Hamiltonian formalism
C. Misner, (1969)

Let us parametrize the scale factors as

$$a = e^{\Omega + \frac{\beta_+}{2} + \frac{\sqrt{3}}{2}\beta_-},$$
$$b = e^{\Omega + \frac{\beta_+}{2} - \frac{\sqrt{3}}{2}\beta_-},$$
$$c = e^{\Omega - \beta_+}.$$

Introducing the conjugate momenta, one arrives to the gauge-fixed Hamiltonian

$$H = \frac{1}{2}\left(p_+^2 + p_-^2 - \frac{1}{4}p_\Omega^2\right) + V(\beta_+, \beta_-, \Omega).$$

If we consider a Bianchi-II universe, then the potential V is

$$V = V_0 e^{4(\Omega - \beta_+)}.$$

In the case of the Bianchi-IX universe the structure of the potential is more complicated and the dynamics of the system can be represented as a motion of a ball in the billiard with moving walls. The billiard is two-dimensional (β_+, β_-) and the parameter Ω plays the role of a time variable. The dependence of the potential on the Ω means that the walls of the billiard are moving.

A bounce of the ball from one of the walls is the change of the Kasner regime.

The relations connecting the Kasner indices with the conjugate momenta:

$$p_1 = \frac{p_\Omega + 4p_+}{3p_\Omega},$$
$$p_2 = \frac{p_\Omega - 2p_+ - 2\sqrt{3}p_-}{3p_\Omega},$$
$$p_3 = \frac{p_\Omega - 2p_+ + 2\sqrt{3}p_-}{3p_\Omega}.$$

Inversely,

$$\frac{p_+}{p_\Omega} = \frac{3p_1 - 1}{4},$$
$$\frac{p_-}{p_\Omega} = \frac{\sqrt{3}(p_3 - p_2)}{4}.$$

The Kasner indices satisfy the condition
$p_1^2 + p_2^2 + p_3^2 = 1$
due to the Hamiltonian constraint

$$p_\Omega^2 = 4(p_+^2 + p_-^2),$$

which is valid far away from the wall, where the potential is negligible.

Misner parametrization of the Kasner indices :

$$p_1 = -\frac{(s-3)(s+3)}{3(s^2+3)},$$
$$p_2 = \frac{2s(s-3)}{3(s^2+3)},$$
$$p_3 = \frac{2s(s+3)}{3(s^2+3)}.$$

We can make a change of variables, eliminating the dependence of the potential on the spatial volume Ω:

$$\bar{\beta}_+ = \frac{1}{\sqrt{3}}(\beta_+ - \Omega),$$
$$\bar{\Omega} = \frac{1}{2\sqrt{3}}(4\Omega - \beta_+).$$

The corresponding conjugate momenta are

$$\bar{p}_+ = \frac{1}{\sqrt{3}}(4p_+ + p_\Omega),$$
$$\bar{p}_\Omega = \frac{2}{\sqrt{3}}(p_+ + p_\Omega).$$

Now, the Hamiltonian looks as

$$H = \frac{1}{2}\left(\frac{1}{4}\bar{p}_+^2 + p_-^2 - \frac{1}{4}\bar{p}_\Omega^2\right) + V_0 e^{-4\sqrt{3}\bar{\beta}_+}.$$

The conjugate momenta p_- and $\bar{p}_\Omega$ are constant because the potential does not depend on β_- and $\bar{\Omega}$.
Then

$$\frac{\bar{p}_\Omega}{p_-} = \frac{2}{\sqrt{3}} \frac{\frac{p_+}{p_\Omega} + 1}{\frac{p_-}{p_\Omega}} = const.$$

This relation connect the values of the momenta at two Kasner regimes.

$$\frac{s}{3} + \frac{3}{s} = const.$$

The only change of the variable s, leaving the left-hand side of this relation intact is

$$s \to \frac{9}{s}.$$

The change of the Kasner regime described in terms of the Misner parameter s coincides with those, described in terms of the Khalatnikov-Lifshitz parameter u.

Chaos in cosmology

The evolution of the universe towards a singular point consists of successive periods (called eras) in which distances along two axes oscillate and along the third axis decrease monotonically, the volume decreases according to a law which is near to $\sim t$. In the transition from one era to another, the axes along which the distances decrease monotonically are interchanged. The order in which the pairs of axes are interchanged and the order in which eras of different lengths follow each other acquires a stochastic character.

To every (sth) era corresponds a decreasing sequence of values of the parameter u. This sequence has the form $u^{(s)}_{max}, u^{(s)}_{max} - 1, \ldots, u^{(s)}_{min}$, where $u^{(s)}_{min} < 1$.
Let us introduce the following notation:

$$u^{(s)}_{min} = x^{(s)},\ u^{(s)}_{max} = k^{(s)} + x^{(s)}$$

i.e. $k^{(s)} = [u^{(s)}_{max}]$.
The number $k^{(s)}$ defines the era length.
For the next era we obtain

$$u^{(s+1)}_{max} = \frac{1}{x^{(s)}},\ k^{(s+1)} = \left[\frac{1}{x^{(s)}}\right].$$

The ordering with respect to the length of $k^{(s)}$ of the successive eras (measured by the number of Kasner epochs contained in them) acquires asymptotically a stochastic character .

The random nature of this process arises because of the rules which define the transitions from one era to another in the infinite sequence of values of u.

If this infinite sequence begins since some initial value

$$u_{max}^{(0)} = k^{(0)} + x^{(0)},$$

then the lengths of series $k^{(0)}, k^{(1)}, \ldots$ are numbers included into an expansion of a continuous fraction:

$$k^{(0)} + x^{(0)} = k^{(0)} + \frac{1}{k^{(1)} + \frac{1}{k^{(2)}+\cdots}}.$$

We can describe statistically this sequence of eras if we consider instead of a given initial value $u_{max}^{(0)} = k^{(0)} + x^{(0)}$ a distribution of $x^{(0)}$ over the interval $(0, 1)$ governed by some probability law (E.M. Lifshitz, I.M. Lifshitz, I.M. Khalatnikov 1971).
Then we also obtain some distributions of the values of $x^{(s)}$ which terminate every sth series of numbers. It can be shown that with increasing s, these distributions tend to a stationary (independent of s) probability distribution $w(x)$ in which the initial value $x^{(s)}$ is completely "forgotten":

$$w(x) = \frac{1}{(1+x)\ln 2}.$$

The probability distribution of the lengths of series k is given by

$$W(k) = \frac{1}{\ln 2} \ln \frac{(k+1)^2}{k(k+2)}.$$

The source of stochasticity arising at the oscillatory approach to the cosmological singularity can be described in such terms: the transition from one Kasner era to another is described by the transformation of the interval $[0, 1]$ into itself by the formula

$$Tx = \left\{ \frac{1}{x} \right\}, \quad \text{i.e.,} \; x_{s+1} = \left\{ \frac{1}{x_s} \right\}.$$

This transformation is expanding and possesses the property of exponential instability.

It is not one to one transformation. Its inverse is not unique. In other words, fixing the value of the parameter u we can predict the evolution towards singularity, but we cannot describe the past.

Mixmaster universes and infinite-dimensional Lie algebras

Presence of matter

If one considers the universe filled with a perfect fluid with the equation of state $p = w\rho, w < 1$, then the presence of this matter cannot change the dynamics in the vicinity of the singularity.

$$\rho = \frac{\rho_0}{(abc)^{w+1}} = \frac{\rho_0}{t^{w+1}}.$$

This term is is weaker than the terms of geometrical origin coming from the time derivatives of the metric, which behaves like $1/t^2$, let alone the perturbations due to the presence of spatial curvature, responsible for changes of a Kasner regime, which behave like $1/t^{2+4|p_1|}$.

The situation changes, if the parameter $w = 1$. Such kind of matter is called "stiff matter" and can be represented by a massless scalar field. In this case $\rho \sim 1/t^2$ and the contribution of matter is of the same order as main terms of geometrical origin.

$$p_1 + p_2 + p_3 = 1, \quad p_1^2 + p_2^2 + p_3^2 = 1 - q^2,$$

where the number q^2 reflects the presence of the stiff matter and

$$q^2 \leq \frac{2}{3}.$$

If $q^2 > 0$, then exist combinations of the positive Kasner indices, satisfying the above relations.
If $q^2 \geq \frac{1}{2}$ only sets of three positive Kasner indices are acceptable.

If a universe finds itself in a Kasner regime with three positive indices, the perturbative terms, existing due to the spatial curvatures, are too week to change this Kasner regime, and thus, it becomes stable. In the presence of the stiff matter, the universe after a finite number of changes of Kasner regimes finds itself in a stable regime and oscillations stop. The massless scalar field plays “anti-chaotizing” role in the process of the cosmological evolution.(Belinski and Khalatnikov, 1973).

After one bounce

$$q^2 \to q'^2 = q^2 \times \frac{1}{(1+2p_1)^2} > q^2.$$

The value of the parameter q^2 grows and the probability to find all the three Kasner indices to be positive increases.

Multidimensional cosmology

The multidimensional analog of a Bianchi-I universe:
a generalized Kasner metric:

$$ds^2 = dt^2 - \sum_{i=1}^{d} t^{2p_i} dx^{i2},$$

$$\sum_{i=1}^{d} p_i = \sum_{i=1}^{d} p_i^2 = 1.$$

In the presence of spatial curvature terms the transition from one Kasner epoch to another occurs:

$$p'_1, p'_2, \ldots, p'_d = \text{ordering of } (q_1, q_2, \ldots, q_d),$$

$$q_1 = \frac{-p_1 - P}{1 + 2p_1 + P}, \; q_2 = \frac{p_2}{1 + 2p_1 + P}, \dots,$$

$$q_{d-2} = \frac{p_{d-2}}{1 + 2p_1 + P}, \; q_{d-1} = \frac{2p_1 + P + p_{d-1}}{1 + 2p_1 + P},$$

$$q_d = \frac{2p_1 + P + p_d}{1 + 2p_1 + P},$$

$$P = \sum_{i=2}^{d-2} p_i .$$

However, such a transition from one Kasner epoch to another occurs if at least one of the numbers α_{ijk} is negative. These numbers are defined as

$$\alpha_{ijk} \equiv 2p_i + \sum_{l \neq j,k,i} p_l, \ (i \neq j, i \neq k, j \neq k).$$

For the spacetimes with $d < 10$ one of the factors α is always negative and one change of Kasner regime is followed by another one, implying the oscillatory behaviour of the universe in the neighbourhood of the cosmological singularity.

For the spacetimes with $d \geq 10$ exist such combinations of Kasner indices, for which all the numbers α_{ijk} are positive. If a universe enters into the Kasner regime with such indices, (so called "Kasner stability region") its chaotic behaviour disappears and this Kasner regime conserves itself. (Demaret, Hanquin, Henneaux 1985).

The discovery of the fact that the chaotic character of the approach to the cosmological singularity disappears in the spacetimes with $d \geq 10$ was unexpected and looked as an accidental result of a game between real numbers satisfying the generalized Kasner relations.

Later it became clear that behind this fact there is a deep mathematical structure, namely, the hyperbolic Kac-Moody algebras.
(Damour, Henneaux, Nicolai, 2000).

In the vicinity of the singularity the models based on superstring theories, living in 10-dimensional spacetime and the $d+1=11$ supergravity model reveal oscillating behaviour of the BKL type.

The important new feature of the dynamics in these models is the role played by non-gravitational bosonic fields (p-forms) which are also responsible for transitions from one Kasner regime to another.

The Hamiltonian formalism becomes very convenient.

The configuration space of the Kasner parameters describing the dynamics of the universe could be treated as a billiard while the curvature terms in Einstein theory and also p-form's potentials in superstring theories play role of walls of these billiards.

The transition from one Kasner epoch to another is the reflection from one of the walls.

There is a correspondence between rather complicated dynamics of a universe in the vicinity of the cosmological singularity and the motion of an imaginary ball on the billiard table.

However, there exists a more striking and unexpected correspondence between the chaotic behaviour of the universe in the vicinity of the singularity and such an abstract mathematical object as the hyperbolic Kac-Moody algebras.

Every Lie algebra is defined by its generators $h_i, e_i, f_i, i = 1, \ldots, r$, where r is the rank of the Lie algebra, i.e. the maximal number of its generators h_i which commute amongst them (these generators constitute the Cartan subalgebra). The commutation relations between generators are

$$
\begin{aligned}
[e_i, f_j] &= \delta_{ij} h_i, \\
[h_i, e_j] &= A_{ij} e_j, \\
[h_i, f_j] &= -A_{ij} f_j, \\
[h_i, h_j] &= 0.
\end{aligned}
$$

The coefficients A_{ij} constitute the generalized Cartan $r \times r$ matrix such that $A_{ii} = 2$, its off-diagonal elements are non-positive integers and $A_{ij} = 0$ for $i \neq j$ implies $A_{ji} = 0$.

One can say that the e_i are rising operators, similar to well-known operator $L_+ = L_x + iL_y$ in the theory of angular momentum, while f_i are lowering operators like $L_- = L_x - iL_y$. The generators h_i of the Cartan subalgebra could be compared with the operator L_z. The generators should also obey the Serre's relations

$$(\mathrm{ad}\ e_i)^{1-A_{ij}} e_j = 0,$$
$$(\mathrm{ad}\ f_i)^{1-A_{ij}} f_j = 0,$$

where $(\mathrm{ad} A)B \equiv [A, B]$.

The Lie algebras $\mathcal{G}(A)$ constructed on a symmetrizable Cartan matrix A have been classified according to the properties of their eigenvalues:

if A is positive definite, $\mathcal{G}(A)$ is a finite-dimensional Lie algebra;

if A admits one null eigenvalue and the others are all strictly positive, $\mathcal{G}(A)$ is an Affine Kac-Moody algebra;

if A admits one negative eigenvalue and all the others are strictly positive, $\mathcal{G}(A)$ is a Lorentz KM algebra.

There exists a correspondence between the structure of a Lie algebra and a certain system of vectors in the r-dimensional Euclidean space, which simplifies the classification of the Lie algebras.

These vectors called roots represent the rising and lowering operators of the Lie algebra.

The vectors corresponding to the generators e_i and f_i are called simple roots.

The system of simple positive roots (i.e. the roots, corresponding to the rising generators e_i) can be represented by nodes of their Dynkin diagrams, while the edges connecting (or non connecting) the nodes give an information about the angles between simple positive root vectors.

An important subclass of Lorentz KM algebras can be defined as follows: A KM algebra such that the deletion of one node from its Dynkin diagram gives a sum of finite or affine algebras is called an hyperbolic KM algebra. These algebras are all known. In particular, there exists no hyperbolic algebra with a rank higher than 10.

Weyl group

The reflections with respect to hyperplanes orthogonal to simple roots leave the systems of roots invariant.

The corresponding finite-dimensional group is called Weyl group.

The hyperplanes mentioned above divide the r-dimensional Euclidean space into regions called Weyl chambers.

The Weyl group transform one Weyl chamber into another.

Correspondence

The links between the billiards describing the evolution of the universe in the neighbourhood of singularity and its corresponding Kac-Moody algebra can be described as follows:

the Kasner indices describing the “free” motion of the universe between the reflections from the wall correspond to the elements of the Cartan subalgebra of the KM algebra;

the dominant walls, i.e. the terms in the equations of motion responsible for the transition from one Kasner epoch to another, correspond to the simple roots of the KM algebra;

the group of reflections in the cosmological billiard is the Weyl group of the KM algebra;

the billiard table can be identified with the Weyl chamber of the KM algebra.

One can imagine two types of billiard tables:

such where the linear motion without collisions with walls is possible (non-chaotic regime) and

those where reflections from walls are inevitable and the regime can be only chaotic.

Remarkably, the Weyl chambers of the hyperbolic KM algebras are designed in such a way that infinite repeating collisions with walls occur.

It was shown that all the theories with the oscillating approach to the singularity such as Einstein theory in dimensions $d < 10$ and superstring cosmological models correspond to hyperbolic KM algebras.

Conclusion

- The existence of links between the BKL approach to the singularities and the structure of some infinite-dimensional Lie algebras has inspired some authors to declare a new program of development of quantum gravity and cosmology. They propose “to take seriously the idea that near the singularity (i.e. when the curvature gets larger than the Planck scale) the description of a spatial continuum and spacetime based (quantum) field theory breaks down, and should be replaced by a much more abstract Lie algebraic description.

- Thereby the information previously encoded in the spatial variation of the geometry and of the matter fields gets transferred to an infinite tower of Lie-algebraic variables depending only on "time". In other words we are led to the conclusion that space–and thus, upon quantization also spacetime – actually disappears (or "de-emerges") as the singularity is approached."
- There is a ramified activity in this field.
- Study of the role of the fermion fields.
- Quantum Bianchi-IX (mixmaster) cosmology
- Quantum supersymmetric Bianchi-IX cosmology
- Is it possible to cross the singularity ?

Lecture 3. Singularities and their crossing

1. The Big Brake cosmological singularity, more general soft singularities and their crossing
2. Transformations of matter fields at a singularity crossing
3. Is it possible to cross the “hard” Big Bang - Big Crunch singularity?
4. Quantum cosmology and singularities
5. Particles, fields and singularities
6. Covariant approach to the singularities
7. Conclusions

Description of the tachyon model

The flat Friedmann universe

$$ds^2 = dt^2 - a^2(t)dl^2.$$

The tachyon Lagrange density

$$L = -V(T)\sqrt{1-\dot{T}^2}.$$

The energy density

$$\rho = \frac{V(T)}{\sqrt{1-\dot{T}^2}}.$$

The pressure

$$p = -V(T)\sqrt{1-\dot{T}^2}.$$

The Friedmann equation

$$H^2 \equiv \frac{\dot{a}^2}{a^2} = \rho.$$

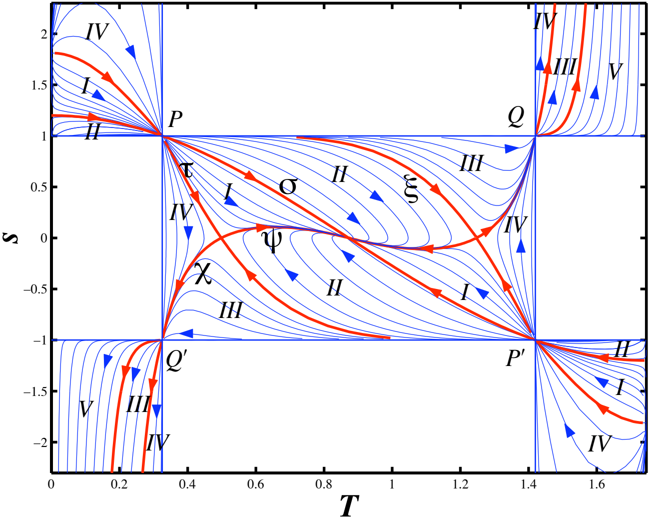
The equation of motion for the tachyon field

$$\frac{\ddot{T}}{1-\dot{T}^2} + 3H\dot{T} + \frac{V_{,T}}{V} = 0.$$

In our model

$$V(T) = \frac{\Lambda}{\sin^2\left[\frac{3}{2}\sqrt{\Lambda(1+k)}\ T\right]} \times\sqrt{1-(1+k)\cos^2\left[\frac{3}{2}\sqrt{\Lambda(1+k)}\ T\right]},$$

where k and $\Lambda > 0$ are the parameters of the model. The case $k > 0$ is more interesting.



Phase portrait of the model for a positive k.

Some trajectories (cosmological evolutions) finish in an infinite de Sitter expansion. In other trajectories the tachyon field transforms into a pseudotachyon field with the Lagrange density, energy density and positive pressure:

$$L = W(T)\sqrt{\dot{T}^2 - 1},$$
$$\rho = \frac{W(T)}{\sqrt{\dot{T}^2 - 1}},$$
$$p = W(T)\sqrt{\dot{T}^2 - 1},$$
$$W(T) = \frac{\Lambda}{\sin^2\left[\frac{3}{2}\sqrt{\Lambda(1+k)}\ T\right]}$$
$$\times\sqrt{(1+k)\cos^2\left[\frac{3}{2}\sqrt{\Lambda(1+k)}\,T - 1\right]}$$

What happens to the Universe after the transformation of the tachyon into the pseudotachyon ?

It encounters the Big Brake

cosmological singularity.

The Big Brake cosmological singularity and other soft singularities

$$t \to t_{BB} < \infty$$

$$a(t \to t_{BB}) \to a_{BB} < \infty$$

$$\dot{a}(t \to t_{BB}) \to 0$$

$$\ddot{a}(t \to t_{BB}) \to -\infty$$

$$R(t \to t_{BB}) \to +\infty$$

$$\rho(t \to t_{BB}) \to 0$$

$$p(t \to t_{BB}) \to +\infty$$

If $\dot{a}(t_{BB}) \neq 0$ it is a more general soft singularity.

Crossing the Big Brake singularity and the future of the universe

At the Big Brake singularity the equations for geodesics are regular, because the Christoffel symbols are regular (moreover, they are equal to zero).

Is it possible to cross the Big Brake ?

Let us study the regime of approaching to the Big Brake.

On analyzing the equations of motion we find that on approaching the Big Brake singularity the tachyon field behaves as

$$T = T_{BB} + \left(\frac{4}{3W(T_{BB})}\right)^{1/3}(t_{BB} - t)^{1/3}.$$

Its time derivative $s \equiv \dot{T}$ behaves as

$$s = -\left(\frac{4}{81W(T_{BB})}\right)^{1/3}(t_{BB} - t)^{-2/3},$$

the cosmological radius is

$$a = a_{BB} - \frac{3}{4}a_{BB}\left(\frac{9W^2(T_{BB})}{2}\right)^{1/3}(t_{BB} - t)^{4/3},$$

its time derivative is

$$\dot{a} = a_{BB}\left(\frac{9W^2(T_{BB})}{2}\right)^{1/3}(t_{BB} - t)^{1/3}$$

and the Hubble variable is

$$H = \left(\frac{9W^2(T_{BB})}{2}\right)^{1/3}(t_{BB} - t)^{1/3}.$$

All these expressions can be continued in the region where $t > t_{BB}$,which amounts to crossing the Big Brake singularity. Only the expression for s is singular at $t = t_{BB}$ but this singularity is integrable and not dangerous.

Once reaching the Big Brake, it is impossible for the system to stay there because of the infinite deceleration, which eventually leads to a decrease of the scale factor. This is because after the Big Brake crossing the time derivative of the cosmological radius and Hubble variable change their signs. The expansion is then followed by a contraction, culminating in the Big Crunch singularity.

Crossing of the soft singularity in the model with the anti-Chaplygin gas and dust

One of the simplest cosmological models revealing a Big Brake singularity is the model based on the anti-Chaplygin gas with an equation of state

$$p = \frac{A}{\rho}, \quad A > 0$$

Such an equation of state arises in the theory of wiggly strings (B. Carter, 1989, A. Vilenkin, 1990).

$$\rho(a) = \sqrt{\frac{B}{a^6} - A}$$

At $a = a_* = \left(\frac{B}{A}\right)^{1/6}$ the universe encounters the Big Brake singularity.

The anti-Chaplygin gas plus dust

The energy density and the pressure are

$$\rho(a) = \sqrt{\frac{B}{a^6} - A} + \frac{M}{a^3}, \quad p(a) = \frac{A}{\sqrt{\frac{B}{a^6} - A}}.$$

Due to the dust component, the Hubble parameter has a non-zero value at the encounter with the singularity, therefore the dust implies further expansion. With continued expansion however, the energy density and the pressure of the anti-Chaplygin gas would become ill-defined.

Change of the equation of state at soft singularity crossings

The abrupt transition from the expansion to the contraction of the universe does not look natural. There is an alternative/complementary way of resolving the paradox.

One can try to change the equation of state of the anti-Chaplygin gas on passing the soft singularity.

There is some analogy between the transition from an expansion to a contraction of a universe and the perfectly elastic bounce of a ball from a wall in classical mechanics. There is also an abrupt change of the direction of the velocity (momentum).

However, we know that in reality the velocity is changed continuously due to the deformation of the ball and of the wall.

The pressure of the anti-Chaplygin gas

$$p = \frac{A}{\sqrt{\frac{B}{a^6} - A}}$$

tends to $+\infty$ when the universe approaches the soft singularity.

Requiring the expansion to continue into the region $a > a_S$, while changing minimally the equation of state, we assume

$$p = \frac{A}{\sqrt{|\frac{B}{a^6} - A|}},$$

$$p = \frac{A}{\sqrt{A - \frac{B}{a^6}}}, \text{ for } a > a_S.$$

This implies the energy density

$$\rho = -\sqrt{A - \frac{B}{a^6}}.$$

The anti-Chaplygin gas transforms itself into Chaplygin gas with negative energy density.
The pressure remains positive, expansion continues.
The spacetime geometry remains continuous.
The expansion stops at $a = a_0$, where

$$\frac{M}{a_0^3} - \sqrt{A - \frac{B}{a_0^6}} = 0.$$

Then the contraction of the universe begins.

At the moment when the energy density of the Chaplygin gas becomes equal to zero

(again a soft singularity), the Chaplygin gas transforms itself into the anti-Chaplygin gas

and the contraction continues culminating in an encounter with the Big Crunch singularity

$a = 0$.

Analogous effects arise in the model with the tachyon field and dust. The Lagrangian of the Born-Infeld like field changes its form.

Big Bang – Big Crunch crossing ?

- The idea that the Big Bang - Big Crunch singularity can be crossed appears very counterintuitive.
- Some approaches to the description of this crossing were elaborated during the recent years (I. Bars, S.H. Chen, P.J. Steinhardt and N. Turok, C. Wetterich, P. Dominis Prester).
- There is an analogy with the horizon which arises due to a certain choice of the spacetime coordinates: the singularity arises because of some choice of the field parametrization.

- On choosing some convenient field parametrization one can provide a matching between the characteristics of the universe before and after the singularity crossing.
- Analogy to the Kruskal coordinates for the Schwarzschild metric.
- On choosing appropriate combinations of the field variables we can describe the passage through the Big Bang - Big Crunch singularity, but this does not mean that the presence of such a singularity is not essential. Indeed, extended objects cannot survive this passage.

Friedmann cosmology in the presence of a scalar field: Einstein frame versus Jordan frame

$$S = \int d^4x\sqrt{-g}\left[U(\sigma)R - \frac{1}{2}g^{\mu\nu}\sigma_{,\mu}\sigma_{,\nu} + V(\sigma)\right]$$

Conformal coupling

$$U(\sigma) = U_0 - \frac{1}{12}\sigma^2$$

A conformal transformation of the metric

$$g_{\mu\nu} = \frac{U_1}{U}\tilde{g}_{\mu\nu},$$

A new scalar field ϕ:

$$\frac{d\phi}{d\sigma} = \frac{\sqrt{U_1(U + 3U'^2)}}{U} \Rightarrow \phi = \int \frac{\sqrt{U_1(U + 3U'^2)}}{U} d\sigma.$$

$$\phi = \sqrt{3U_1} \ln \left[\frac{\sqrt{12U_0} + \sigma}{\sqrt{12U_0} - \sigma} \right]$$

$$\sigma = \sqrt{12U_0} \tanh \left[\frac{\phi}{\sqrt{12U_1}} \right].$$

The action then becomes the action for a minimally coupled scalar field:

$$S = \int d^4x \sqrt{-\tilde{g}} \left[U_1 R(\tilde{g}) - \frac{1}{2}\tilde{g}^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + W(\phi) \right],$$

$$W(\phi) = \frac{U_1^2 V(\sigma(\phi))}{U^2(\sigma(\phi))}.$$

This is called the transformation from the Jordan frame to the Einstein frame.

In a flat Friedmann universe

$$ds^2 = N^2 d\tau^2 - a^2 dl^2,$$

$$d\tilde{s}^2 = \tilde{N}^2 d\tau^2 - \tilde{a}^2 dl^2.$$

$$\tilde{N} = \sqrt{\frac{U}{U_1}} N, \;\; \tilde{a} = \sqrt{\frac{U}{U_1}} a, \;\; t = \int \sqrt{\frac{U_1}{U}} d\tilde{t},$$

where t and $\tilde{t}$ are the cosmic time parameters in the Jordan and the Einstein frames.

$$a = \tilde{a} \sqrt{\frac{U_1}{U_0}} \cosh\left(\frac{\phi}{\sqrt{12U_1}}\right).$$

In the vicinity of the singularity in the Einstein frame:

$$\tilde{a} \sim \tilde{t}^{\frac{1}{3}} \to 0, \ \text{when} \ \tilde{t} \to 0.$$

However, in the Jordan frame:

$$a \sim \tilde{t}^{\frac{1}{3}} \left(\tilde{t}^{\frac{1}{3}} + \tilde{t}^{-\frac{1}{3}} \right) \to \text{const} \neq 0.$$

Meanwhile, the scalar field σ crosses the value $\pm\sqrt{12U_0}$ and the coupling function U changes its sign.

Thus, the evolution in the Jordan frame is regular, and we can use this fact to describe the crossing of the Big Bang - Big Crunch singularity in the Einstein frame.

If one considers the expansion of the universe from the Big Bang with normal gravity driven by the standard scalar field, the continuation backward in time shows that it was preceded by the contraction towards a Big Crunch singularity in the antigravity regime, driven by a phantom scalar field with a negative kinetic term.

The possibility of a change of sign of the effective gravitational constant in the model with a conformably coupled scalar field was analyzed in 1981 by A. Starobinsky.

It was shown that in a homogeneous and isotropic universe, one can indeed cross the point where the effective gravitational constant changes sign.

However, the presence of anisotropies changes the situation: these anisotropies grow indefinitely when this constant is equal to zero.

To describe the Big Bang - Big Crunch singularity crossing in anisotropic universes it is necessary to use another methods. We have done it for the Bianchi-I universe.

Singularity crossing in a Bianchi - I universe

$$d\tilde{s}^2 = \tilde{N}(\tau)^2 d\tau^2 - \tilde{a}^2(\tau)(e^{2\beta_1(\tau)}dx_1^2 + e^{2\beta_2(\tau)}dx_2^2 + e^{2\beta_3(\tau)}dx_3^2),$$

$$ds^2 = N(\tau)^2 d\tau^2 - a^2(\tau)(e^{2\beta_1(\tau)}dx_1^2 + e^{2\beta_2(\tau)}dx_2^2 + e^{2\beta_3(\tau)}dx_3^2),$$

$$\beta_1 + \beta_2 + \beta_3 = 0.$$

$$\dot{\beta}_i = \frac{\beta_{i0}}{\tilde{a}^3},\ \ \theta_0 = \beta_{10}^2 + \beta_{20}^2 + \beta_{30}^2.$$

$$\dot{\phi} = \frac{\phi_0}{\tilde{a}^3},\ \ \phi = \frac{\phi_0}{\left(\frac{3\theta_0}{2} + \frac{3\phi_0^2}{4U_1}\right)^{\frac{1}{2}}} \ln \tilde{t}.$$

In the vicinity of the singularity in the Einstein frame

$$\tilde{a} \sim \tilde{t}^{\frac{1}{3}}.$$

In the Jordan frame

$$a \sim \tilde{t}^{\frac{1}{3}}(\tilde{t}^{\gamma} + \tilde{t}^{-\gamma}) \to 0,$$

because

$$\gamma = \frac{\phi_0}{3\sqrt{\phi_0^2 + 2\theta_0 U_1}} < \frac{1}{3}.$$

Thus, one also encounters the Big Bang singularity in the Jordan frame.

Mixing between geometrical and matter degrees of freedom and the singularity crossing

The Friedmann-Lemaîthe model with a massless scalar field can be described by the Lagrangian

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\dot{y}^2,$$

where

$$x = \frac{4\sqrt{U_1}}{\sqrt{3}}\tilde{a}^{\frac{3}{2}} \cosh \frac{\sqrt{3}}{4\sqrt{U_1}}\phi, \;\; y = \frac{4\sqrt{U_1}}{\sqrt{3}}\tilde{a}^{\frac{3}{2}} \sinh \frac{\sqrt{3}}{4\sqrt{U_1}}\phi,$$

and the Friedmann equation is

$$\dot{x}^2 - \dot{y}^2 = 0.$$

Inversely,

$$\tilde{a}^3 = \frac{3(x^2 - y^2)}{16U_1},$$

$$\phi = \frac{4\sqrt{U_1}}{\sqrt{3}} \operatorname{arctanh} \frac{x}{y}.$$

Initially

$$x > |y|.$$

The solution

$$x = x_1 \tilde{t} + x_0, \;\; y = y_1 \tilde{t} + y_0, \;\; x_1^2 = y_1^2.$$

Choosing the constants as

$$x_0 = y_0 = A > 0, \;\; x_1 = -y_1 = B > 0,$$

we have

$$\tilde{a}^3 = \frac{3AB\tilde{t}}{4U_1}.$$

We can make a continuation in the plane (x, y), to $x < |y|$ or, in other words, to $\tilde{t} < 0$. Such a continuation implies an antigravity regime and the transition to the phantom scalar field, just as in the more complicated schemes, discussed before.

How can we generalize these considerations to the case when the anisotropy term is present ?

$$L = \frac{1}{2}\dot{r}^2 - \frac{1}{2}r^2(\dot{\varphi}^2 + \dot{\varphi}_1^2 + \dot{\varphi_2}^2),$$

$$\varphi_1 = \sqrt{\frac{3}{8}}\alpha_1,\ \varphi_2\sqrt{\frac{3}{8}}\alpha_2,$$

$$\beta_1 = \frac{1}{\sqrt{6}}\alpha_1 + \frac{1}{\sqrt{2}}\alpha_2,\ \beta_2 = \frac{1}{\sqrt{6}}\alpha_1 - \frac{1}{\sqrt{2}}\alpha_2,\ \beta_3 = -\frac{2}{\sqrt{6}}\alpha_1.$$

We can again consider the plane (x, y) as

$$x = r\cosh\Phi,$$
$$y = r\sinh\Phi,$$

where a new hyperbolic angle Φ is defined by

$$\Phi = \int d\tilde{t}\sqrt{\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + \dot{\varphi}^2}.$$

We have reduced a four-dimensional problem to the old two-dimensional one, on using the fact that the variables α_1, α_2 and ϕ enter into the equation of motion for the scale factor $\tilde{a}$ only through the squares of their time derivatives.

The behaviour of the scale factor before and after the crossing of the singularity can be matched by using the transition to the new coordinates x and y, which mix geometrical and scalar field variables in a particular way.

To describe the behaviour of the anisotropic factors it is enough to fix the constants β_{i0}.

Kantowski-Sachs universe

$$ds^2 = \frac{a_0^2\left(A_0 \tan\frac{t}{2}+1\right)^2\cos^4\frac{t}{2}}{A_0}(dt^2 - d\theta^2 - \sin^2\theta d\phi^2)$$
$$-\frac{b_0^2\left(A_0 \tan\frac{t}{2}+1\right)^2}{4A_0}dr^2.$$

This metric is regular at $t = 0$ and has singularities at $t = \pm\pi$ and at $t = t_0 = -2\mathrm{arctan}\frac{1}{A_0}$.

At $t \to \pm\pi$ the scale factor $a \to 0$ while $b \to \infty$. At $t \to t_0$ both scale factors vanish.

At $t < 0$, we find ourselves in the region with antigravity because $U_c < 0$.

The expression contains only integer powers of the trigonometrical functions and one can describe the crossing of the singularities in a unique way.

Thus, we can imagine an infinite periodic evolution of the universe.

In the vicinity of the moment $t \to \pi$, the asymptotic expressions for the metric coefficients are

$$ds^2 = dT^2 - c_1^2 T d\theta^2 - c_2^2 T \sin^2\theta d\phi^2 - c_3^2 \frac{1}{T} dr^2.$$

This form has a structure similar to that of the Kasner solution for a Bianchi-I universe, where the Kasner indices have the values

$$p_1 = \frac{1}{2},\ p_2 = \frac{1}{2},\ p_3 = -\frac{1}{2}.$$

These indices do not satisfy the standard Kasner relations

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1,$$

they satisfy the generalized relation

$$\sum_{i=1}^{3} p_i^2 = 2\sum_{i=1}^{3} p_i - \left(\sum_{i=1}^{3} p_i\right)^2.$$

In the vicinity of the singularity at $t = t_0$:

$$ds^2 = dT^2 - c_1^2 T d\theta^2 - c_2^2 T \sin^2\theta d\phi^2 - c_3^2 T dr^2.$$

This behavior is isotropic.

Quantum cosmology and singularities

Speaking about quantum cosmology and singularities people mean two different things:

Modification of the Friedmann equation.

$$\frac{\dot{a}^2}{a^2} + \frac{k}{a^2} = \rho_{\text{matter}} + \rho_{\text{quantum corrections}}.$$

Vanishing of the quantum state of the universe.

$$\Psi(\text{geometry} + \text{matter})_{\text{geometry is singular}} = 0.$$

Wheeler-DeWitt equation

$$\hat{\mathcal{H}}\Psi = 0.$$

Where is the time ?

What is the probability ?

A time can be defined as a certain function of geometrical variables.
After that the wavefunction describing matter variables satisfies an effective Schrödinger equation.
The singularity is associated with such values of the matter variables when this singularity arises in the classical theory.
Our analysis of some simple models tells that the probability of the arising of soft singularities is not suppressed by the wave function of the universe, while the probability of Big Bang – Big Crunch singularity tends to zero.

The suppression of the Big Bang – Big Crunch singularity follows from the requirement of the normalizability of the wave function of the Universe

$$\int d\phi \bar{\Psi}(\phi)\Psi(\phi) < \infty.$$

When $|\phi| \to \infty$, the probability density $\bar{\Psi}\Psi$ should tend to zero rapidly.

If $|\phi| \to \infty$ corresponds to Big Bang – Big Crunch singularity, when this singularity is suppressed.

What happens with particles (in quantum field theoretical sense) when the universe passes through the cosmological singularity ?
The scalar field in the flat Friedmann universe satisfies the Klein-Gordon equation:

$$\Box\phi + V'(\phi) = 0.$$

One can consider a spatially homogeneous solution of this equation ϕ_0, depending only on time t as a classical background.
A small deviation from this background solution can be represented as a sum of Fourier harmonics satisfying linearized equations

$$\ddot{\phi}(\vec{k}, t) + 3\frac{\dot{a}}{a}\dot{\phi}(\vec{k}, t) + \frac{\vec{k}^2}{a^2}\phi(\vec{k}, (t)) + V''(\phi_0(t))\phi(\vec{k}, (t)) = 0.$$

The corresponding quantized field is

$$\hat{\phi}(\vec{x}, t) = \int d^3\vec{k}(\hat{a}(\vec{k})u(k,t)e^{i\vec{k}\cdot\vec{x}} + \hat{a}^+(\vec{k})u^*(k,t)e^{-i\vec{k}\cdot\vec{x}}),$$

where the creation and the annihilation operators satisfy the standard commutation relations:

$$[\hat{a}(\vec{k}), \hat{a}^+(\vec{k}')] = \delta(\vec{k} - \vec{k}').$$

The basis functions should be normalized so that the canonical commutation relations between the field ϕ and its canonically conjugate momentum $\hat{\mathcal{P}}$ were satisfied

$$[\hat{\phi}(\vec{x}, t), \hat{\mathcal{P}}(\vec{y}, t')] = i\delta(\vec{x} - \vec{y}).$$

$$u(k,t)\dot{u}^*(k,t) - u^*(k,t)\dot{u}(k,t) = \frac{i}{(2\pi)^3 a^3(t)}.$$

The linearized Klein-Gordon equation has two independent solutions.
To define a particle it is necessary to have two independent non-singular solutions.
it is a non-trivial requirement in the situations when a singularity or other kind of irregularity of the spacetime geometry occurs.
It is convenient also to construct explicitly the vacuum state for quantum particles as a Gaussian function of the corresponding variable. Let is introduce an operator

$$\hat{f}(\vec{k},t) = (2\pi)^3(\hat{a}(\vec{k})u(k,t) + \hat{a}^+(-\vec{k})u^*(k,t)).$$

Its canonically conjugate momentum is

$$\hat{p}(\vec{k},t) = a^3(t)(2\pi)^3(\hat{a}(\vec{k})\dot{u}(k,t) + \hat{a}^+(-\vec{k})\dot{u}^*(k,t)).$$

We can express the annihilation operator as

$$\hat{a}(\vec{k}) = i\hat{p}(\vec{k}, t)u^*(k, t) - ia^3(t)\hat{f}(\vec{k}, t)\dot{u}^+(k, t).$$

Representing the operators $\hat{f}$ and $\hat{p}$ as

$$\hat{f} \to f, \quad \hat{p} \to -i\frac{d}{df},$$

one can write down the equation for the corresponding vacuum state in the following form:

$$\left(u^*\frac{d}{df} - ia^3\dot{u}^*f\right)\Psi_0(f) = 0.$$

$$\Psi_0(f) = \frac{1}{\sqrt{|u(k, t)|}}\exp\left(\frac{ia^3(t)\dot{u}^*(k, t)f^2}{2u^*(k, t)}\right).$$

In the case of the Big Bang - Big Crunch singularity, one of the basis functions in the vicinity of the singularity becomes singular and it is impossible to construct a Fock space.

In the case of the Big Rip singularity, when in finite interval of time the universe achieves an infinite volume and infinite time derivative of the scale factor, the Fock space can be constructed for a spectator scalar field, but it does not exist for the phantom scalar field driving the expansion.

In the case of the model with tachyon field, presented above, we have considered three situations.

The non-singular transformation of the tachyon into pseudo-tachyon. In this case both basis functions are regular and hence the operators of creation and annihilation are well defined.
However, at the moment of the transformation the dispersion of the Gaussian wave function of the vacuum becomes infinite and then becomes finite again.

In the vicinity of the Big Brake singularity it is impossible to define a Fock vacuum.

However, if we add to the universe dust, the character of the soft singularity is slightly changed and then the presence of the Fock vacuum is restored.

Interestingly, if we consider the behavior of the fermion (spinor) fields in the vicinities of the cosmological singularities, it appears that the Fock vacuum can be constructed for all typers of singularities described above.
Thus, the fermions are more “resilient” with respect to singularities than bosons.

Samuel W.P. Oliveira and Alexander Yu. Kamenshchik, Cosmological Singularities and Quantum Particles, arXiv: 2605.22623 [gr-qc].

Covariant approach to singularities

The crossing of the Big Bang - Big Crunch singularities looks rather counterintuitive.
However, it can be sometimes described by using the reparametrization of fields, including the metric.
One can say that to do this, it is necessary to resort to one of two ideas, or a combination thereof.
One of these ideas is to employ a reparameterization of the field variables which makes the singular geometrical invariant non-singular.

Another idea is to find such a parameterization of the fields, including, naturally, the metric, that gives enough information to describe consistently the crossing of the singularity even if some of the curvature invariants diverge.
The application of these ideas looks in a way as an craftsman work.
Our goal is to develop a general formalism to distinguish "dangerous" and "non-dangerous" singularities, considering the field variable space of the model under consideration.
When the spacetime singularities can be removed by a reparametrization of the field variables?

Our hypothesis: when the geometry of the space of the field variables is non-singular.
The field space $\mathcal{S}$ was developed in order to treat on the same (geometrical) footing both changes of coordinates in the spacetime $\mathcal{M}$ and field redefinitions in the functional approach to quantum field theory.
This approach requires introducing a local metric G in field space $\mathcal{S}$ and computing the associated geometric scalars by defining a covariant derivative which is compatible with G.
G is actually determined by the kinetic part of the action and its dimension depends on the field content of the latter.

After some cumbersome calculations in the functional space, we have shown that the Kretschmann scalar

$$\mathcal{K} = \mathcal{R}_{ABCD}\,\mathcal{R}^{ABCD}$$

is finite in every theory of pure gravity

$$\mathcal{K} = \frac{n}{8}\left(\frac{n^3}{4} + \frac{3n^2}{4} - 1\right),$$

where n is the spacetime dimension.
It can be interpreted as a statement that all the singularities in empty universe can be crossed.

Another hypothesis: quantum effective action and to homotopy group

Let us introduce the functional

$$\psi[\varphi] = e^{i\,\Gamma[\varphi]},$$

where $\Gamma[\varphi]$ is the effective action. We shall call $\psi[\varphi]$ the functional order parameter because ψ plays the analogous role of an order parameter in the theory of phase transitions in ordered media or cosmology.
The field space $\mathcal{M}$ can be thought of as the ordered medium itself, whereas functional singularities correspond to topological defects.
The functional order parameter ψ defines the map

$$\psi : \mathcal{M} \to \mathbb{S}^1,$$

from the field space to the unit circle, the latter playing the role of the order parameter space.

The singularities can be characterized by the fundamental group (first homotopy group).

If this group is trivial the singularity can be removed.

We have checked on the example of some simple systems with removable singularity that the corresponding homotopy group is indeed trivial.

Conclusions

- The appearance of the singularities in the cosmological and other gravitational systems is not drawback of models or theories
- It is their distinguishing feature.
- Rather than avoid singularities, it is better to study how their presence influences the non-singular quantities (just like in quantum field theory).

C. W. Misner, Absolute Zero of Time,
Physical Review 186 (1969) 1328.

I prefer a more optimistic viewpoint ("Nature and Einstein are subtle but tolerant") which views the initial singularity in cosmological theory not as a proof of our ignorance, but as a source from which we can much valuable understanding of cosmology.

Thus, while I presume that relativity, like other physical theories, will be improved from time to time, I do not see that these changes need bear directly on the problem of cosmological singularity.

We should stretch our minds, find some more acceptable set of words to describe the mathematical situation, now identified as "singular", and then proceed to incorporate this singularity into our physical thinking until observational difficulties force revision on us.

The concept of a true initial singularity (as distinct from an indescribable early era at extravagant but finite high densities and temperatures) can be a positive and useful element in cosmological theory.

The Universe is meaningfully infinitely old because infinitely many things have happened since the beginning.

Are classical physics and cosmology deterministic?

Let us consider a simple mechanical model: Norton's Dome.
A classical point particle finds itself at some maximum of the potential

$$V = V_0 - V_1 x^{\frac{3}{2}}, \ V_0 > 0, \ V_1 > 0.$$

To such a potential corresponds a surface which is not spherical but has some kind of cusp at the top.
The second Newton law for the particle is

$$\ddot{x} = \frac{3}{2} V_1 x^{\frac{1}{2}}.$$

Obviously, we have a solution:

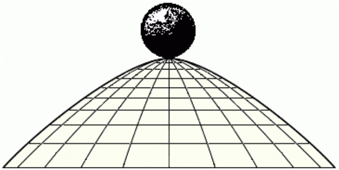
$$x(t) = 0.$$

There is also a whole family of nontrivial solutions:

$$x(t) = \frac{V_1}{64}(t - t_0)^4.$$

Here t_0 is the moment of the beginning of the motion "rolling down" of the particle.
There is no reason to begin falling at $t = t_0$ and there is no probability interpretation here.

John D. Norton (November 2003), "Causation as Folk Science", Philosophers' Imprint. 3 (4): 1-22.



It was possible also to have the potential

$$V = V_0 - V_1 x^{\alpha}, \ V_0 > 0, \ V_1 > 0, \ 1 < \alpha < 2.$$

Then we have two solutions:

$$x(t) = 0$$

and

$$x(t) = \left(\frac{V_1}{\alpha - 1} \right)^{\frac{1}{2-\alpha}} (t - t_0)^{\frac{2}{2-\alpha}}.$$

Norton's Dome and cosmology

V. Husain and V. Tasic,
Indeterminacy in Classical Cosmology with Dark Matter,
Found.Phys. 53 (2023) 2, 42.
The Friedmann equations for a flat universe filled with a perfect fluid with

$$p = w\rho, \ -1 > w > -\frac{2}{3}$$

are written in such a form that they are nonsingular at $a = 0$ and permit both the solutions $a(t) = 0$ and

$$a(t) = a_0(t - t_0)^{\frac{2}{3(1+w)}}.$$

At $w = -\frac{5}{6}$, one has a perfect analogy with the Norton's Dome.

The first Friedmann equation was written in such a form:

$$\dot{a} = Ca^{-\frac{3w+1}{2}}.$$

and not in the form

$$\frac{\dot{a}^2}{a^2} = \frac{C^2}{a^{3(w+1)}}.$$

In the paper
V. Gorini, A. Y. Kamenshchik, U. Moschella and V. Pasquier,
Tachyons, scalar fields and cosmology,
Phys. Rev. D **69**, 123512 (2004)
we have had quite similar differential equation.
In the flat Friedmann universe we have a tachyon field (i.e. Born-Infeld field) with the Lagrangian

$$L = -V(T)\sqrt{1-\dot{T}^2}.$$

The energy density is

$$\rho = \frac{V(T)}{\sqrt{1-\dot{T}^2}}.$$

The pressure is

$$p = -V(T)\sqrt{1-\dot{T}^2}.$$

Let us introduce

$$s \equiv \dot{T}.$$

Then the Klein-Gordon equation is

$$\frac{\dot{s}}{1-s^2} + 3\frac{\dot{a}}{a}s + \frac{V_{,T}}{V(T)} = 0.$$

At $s = 1$, we have a singularity.
In the vicinity of the singularity

$$s = 1 - \tilde{s},$$

where $\tilde{s}$ satisfies the equation

$$\dot{\tilde{s}} = A\tilde{s}^{3/4}, \ \ A = 3 \cdot 2^{3/4} V(T),$$

which has two solutions:

$$\tilde{s} = 0$$

and

$$\tilde{s} = \frac{A^4}{256}(t - t_0)^4.$$

These are exactly the Norton's Dome solutions!