BH perturbations and stability

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I- Test field approximation

As a first example, we will study a scalar field in a spherically symmetric spacetime. The scalar field is considered weak enough not to produce any backreaction (a test field as we use test particles)

Considering a static spherically symmetric spacetime, we have

$$ds^2 = -A(r)\,dt^2 + dr^2/B(r) + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Also we will consider a massless scalar field $\quad \Box\Phi = 0 \quad$ or $\quad \frac{1}{\sqrt{-g}}\partial_\mu\left[\sqrt{-g}\,g^{\mu\nu}\partial_\nu\right]\Phi = 0$

Because of the symmetries of the background spacetime, the problem will simplify if we decompose Φ into spherical harmonics (same is used in quantum mechanics when we study a spherically symmetric problem such as the hydrogen atom).

$$\Phi(t,r,\theta,\phi) = \underbrace{\frac{1}{r}}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} u_{\ell m}(t,r)\, Y_\ell^m(\theta,\phi)$$

it's only to simplify the equations

$$\Box\Phi = \frac{1}{\sqrt{-g}}\partial_0\left[\sqrt{-g}\left(-\frac{1}{A}\right)\partial_0\right]\Phi + \frac{1}{\sqrt{-g}}\partial_1\left[\sqrt{-g}\,B\,\partial_1\right]\Phi + \frac{1}{\sqrt{-g}}\partial_2\left[\sqrt{-g}\,\frac{1}{r^2}\,\partial_2\right]\Phi$$

$$+ \frac{1}{\sqrt{-g}}\partial_3\left[\sqrt{-g}\,\frac{1}{r^2\sin^2\theta}\,\partial_3\right]\Phi \qquad \text{with} \qquad \sqrt{-g} = \sqrt{\frac{A}{B}}\,r^2\sin\theta$$

$$= -\frac{1}{A}\ddot{\Phi} + \sqrt{\frac{B}{A}}\frac{1}{r^2}\partial_r\left(\sqrt{AB}\,r^2\,\Phi'\right) + \frac{1}{r^2\sin\theta}\partial_\theta\left(\sin\theta\,\partial_\theta\Phi\right) + \frac{1}{r^2\sin^2\theta}\partial_\phi^2\Phi$$

$$= \sum_{\ell,m}\left[-\frac{1}{rA}\ddot{u}_{\ell m}\,Y_\ell^m - \sqrt{\frac{B}{A}}\frac{1}{r^2}\left(\partial_r\sqrt{AB}\right)u_{\ell m}Y_\ell^m + \sqrt{\frac{B}{A}}\frac{1}{r}\partial_r\left[\sqrt{AB}\,u'_{\ell m}\right]Y_\ell^m\right.$$

$$\left. + \frac{u_{\ell m}}{r^3}\underbrace{\left[\frac{\cos\theta}{\sin\theta}\partial_\theta Y_\ell^m + \partial_\theta^2 Y_\ell^m + \frac{1}{\sin^2\theta}\partial_\phi^2 Y_\ell^m\right]}_{-\ell(\ell+1)\,Y_\ell^m}\right]$$

$$= \sum_{\ell, m} \frac{1}{rA} Y_\ell^m(\theta,\phi) \Big[-\ddot{u}_{\ell m}(t,r) - \frac{\sqrt{AB}}{r} \partial_r(\sqrt{AB})\, u_{\ell m}(t,r) + \sqrt{AB}\,\partial_r\left[\sqrt{AB}\, u'_{\ell m}(t,r)\right]$$

$$- \frac{A(r)}{r^2} \ell(\ell+1)\, u_{\ell m}(t,r) \Big] = 0$$

The eq. could be separated because we have decomposed on the basis of the 2-sphere space which corresponds to the symmetries of the problem.

$\forall (\ell, m)$

$$-\ddot{u}_{\ell m}(t,r) - \frac{\sqrt{AB}}{r}\partial_r(\sqrt{AB})\, u_{\ell m}(t,r) + \sqrt{AB}\,\partial_r\left[\sqrt{AB}\, u'_{\ell m}(t,r)\right] - \frac{A(r)}{r^2}\ell(\ell+1)\, u_{\ell m}(t,r) = 0$$

Let's define the tortoise coordinate $\quad ds^2 = A\left[-dt^2 + \frac{dr^2}{AB}\right] + r^2 d\Omega^2$

that is to say $\quad dr^* = \frac{dr}{\sqrt{AB}} \quad \Rightarrow \quad ds^2 = A(-dt^2 + dr_*^2) + r^2 d\Omega^2$

we have $\partial_r u_{\ell m} = \frac{1}{\sqrt{AB}} \partial_{r^*} u_{\ell m} \Rightarrow \sqrt{AB}\,\partial_r\left[\sqrt{AB}\, u'_{\ell m}\right] = \partial^2_{r_*} u_{\ell m}$

$$\Rightarrow \quad \partial^2_* u_{\ell m} - \partial^2_t u_{\ell m} - \sqrt{AB}\left[\frac{\partial_r \sqrt{AB}}{r} + \frac{\ell(\ell+1)}{r^2}\sqrt{\frac{A}{B}}\right] u_{\ell m} = 0$$

Let's consider the Schwarzschild spacetime, viz. $A = B = 1 - r_s/r$

$$\boxed{-\partial^2_t u_{\ell m} + \partial^2_* u_{\ell m} - V_\ell(r)\, u_{\ell m} = 0}$$

with $V_\ell(r) = \left(1 - \frac{r_s}{r}\right)\left[\frac{\ell(\ell+1)}{r^2} + \frac{r_s}{r^3}\right] \quad ; \quad \partial_* \equiv \frac{\partial}{\partial r_*}$

We have found a wave equation propagating at the speed of light.
Also the equation is independent of the azimuthal number "m", because of the spherical symmetry (exactly as the hydrogen atom, which is independent of the direction contrary to the Zeeman effect).

Passing to the Fourier space $u_{\ell m}(t,r) = \int d\omega \, \hat{u}_{\ell m}(\omega, r)\, e^{-i\omega t}$

$$\Rightarrow \quad \left[-\frac{d^2}{dr_*^2} + V_\ell(r) \right] \hat{u}_{\ell m} = \omega^2 \hat{u}_{\ell m}$$

Which is as the Schrödinger eq. in 1 dimension in a potential $V_\ell(r)$.

Perturbations around BH will have the same form, only the potential $V_\ell(r)$ will be different.

⚠ Later, we will see how to conclude about stability from this eq.

II - General perturbation formalism

We consider a background spacetime $\bar{g}_{\mu\nu}$ and a perturbation $h_{\mu\nu}$ such as $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$

with $\bar{g}_{\mu\nu}$ the metric of a spherically symmetric spacetime

Because $\bar{g}_{\mu\nu}$ is spherically symmetric, we can make a transformation

$$\begin{cases} \theta \mapsto h_1(\theta,\phi) \\ \phi \mapsto h_2(\theta,\phi) \end{cases} \iff x^i \mapsto h_i(\theta,\phi) \quad \text{with } x^i = (\theta,\phi)$$

and see how $h_{\mu\nu}$ changes

We have $ds^2 = g_{00}\,dt^2 + 2g_{01}\,dt\,dr + g_{11}\,dr^2 + 2g_{0i}\,dt\,dx^i + 2g_{1i}\,dr\,dx^i + g_{ij}\,dx^i dx^j$ with $(i,j) = (\theta,\phi)$

Under this $\mathcal{T}$ $\quad g_{\mu\nu}(t,r,\theta,\phi) \longmapsto g_{\mu\nu}(t,r,h_1(\theta,\phi),h_2(\theta,\phi))$

y $\quad dt \mapsto dt \qquad dr \mapsto dr \qquad dx^i \mapsto \partial_j h^i\,dx^j$

For example, because $\theta \mapsto h_1(\theta,\phi)$ $\quad d\theta \mapsto \partial_\theta h_1 . d\theta + \partial_\phi h_1\, d\phi$

So in the line element, we have the following $\mathcal{T}$.

$$\left.\begin{aligned} g_{00}(t,r,\theta,\phi)\,dt^2 &\longmapsto g_{00}(t,r,h_1,h_2)\,dt^2 \\ g_{01}(t,r,\theta,\phi)\,dt\,dr &\longmapsto g_{01}(t,r,h_1,h_2)\,dt\,dr \\ g_{11}(t,r,\theta,\phi)\,dr^2 &\longmapsto g_{11}(t,r,h_1,h_2)\,dr^2 \end{aligned}\right\} \text{(g_{00}, g_{01}, g_{11}) transform as scalars under this $\mathcal{T}$.}$$

$$\left.\begin{aligned} g_{0i}(t,r,\theta,\phi)\,dt\,dx^i &\longmapsto g_{0i}(t,r,h_1,h_2)\,\partial_j h^i . dt\,dx^j \\ g_{1i}(t,r,\theta,\phi)\,dr\,dx^i &\longmapsto g_{1i}(t,r,h_1,h_2)\,\partial_j h^i\,dr\,dx^j \end{aligned}\right\} \text{(g_{0i}, g_{1i}) transform as vectors}$$

$$g_{ij}(t,r,\theta,\phi)\,dx^i dx^j \longmapsto g_{ij}(t,r,h_1,h_2)\,\partial_k h^i \partial_l h^j\,dx^k dx^l \Big\}\ \text{g_{ij} transforms as a 2-tensor}$$

Therefore, under this $\mathcal{G}_c$ we have 3 scalars (g_{00}, g_{01}, g_{11}), 2 vectors of dim. 2 (g_{0i}, g_{1i}) and a tensor (g_{ij}) of 3 components which means $3 + 2\times 2 + 3 = 10$ functions

According to the Helmholtz theorem, each vector can be descomposed in $V_i = \partial_i \omega + \omega_i$ with $\nabla_i \omega^i = 0$

$\Rightarrow$ Each vector can be written as a scalar and a "pure" vector.

Similarly, for a 2-tensor

$$T_{ij} = \lambda g_{ij} + \nabla_{ij} \Psi + \frac{1}{2}\left[\nabla_i S_j + \nabla_j S_i\right] + \mathcal{G}_{ij}$$

λ: trace of T_{ij}

S_i: pure vector $\nabla_i S^i = 0$

$\mathcal{G}_{ij}$: pure tensor : traceless $(\mathcal{G}_i{}^i = 0)$ and transverse $\nabla_i \mathcal{G}^i{}_j = 0$

$\Rightarrow$ We have 2 additional scalars (λ, Ψ) and an additional vector (S_i)

In 4 dimensions, $\mathcal{G}_{ij} = 0$ because it has 3 components and 3 constraints $(\mathcal{G}_i{}^i = 0 ; \nabla_i \mathcal{G}^i{}_j = 0)$ (More formally, it can be seen in Higuchi (1986) J. math. phys 28, 1553)

$\Rightarrow$ We have 7 scalars $\oplus$ 3 vectors

7 scalars : g_{00}, g_{01}, g_{11} - the scalar part of g_{0i}, g_{1i} and finally λ, Ψ

Vectors are of dimension 2 but with 1 constraint, $\partial_i V^i = 0$

So each vector has only 1 free function

7 scalars $\oplus$ 3 vectors = 10 functions

All scalar funct° can be written, $\phi = \sum_{\ell, m} \phi_{\ell m}(t,r)\, Y_\ell^m(\theta, \phi)$

For the vectors, we saw that $V_i = \nabla_i w + w_i$ with $\nabla_i w^i = 0$

Because w is a scalar, $\nabla_i w = \sum_{\ell m} w_{\ell m}(t,r)\, \nabla_i Y_\ell^m(\theta, \phi)$ $\quad (i = (\theta, \phi))$

w_i is the rotational part $(\vec{w} = \vec{\nabla} \times \vec{A})$ - Using the Levi-Civita tensor, we have $(\vec{\nabla} \times \vec{A})^i = \varepsilon^{ijk}\, \partial_j A_k$

In our case, we are in 2D $\quad (i = \theta, \phi) \Rightarrow w^i = \varepsilon^{ij} \partial_j \Psi$

In a curved space, the Levi-Civita tensor is

$$E_{ij} = \sqrt{\det \gamma}\; \varepsilon_{ij}$$

γ_{ij} is the metric of the 2D space $\gamma_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$

$$\varepsilon_{\theta\phi} = -\varepsilon_{\phi\theta} = 1$$

$$\Rightarrow V_i = \sum_{\ell, m} \left[w_{\ell m}(t,r)\, \underline{\nabla_i Y_\ell^m} + \chi_{\ell m}(t,r)\, \underline{E_i{}^j\, \nabla_j Y_\ell^m} \right]$$

which are known as the vectorial spherical harmonics

As a summary, we have for vector perturbations

$$h_{ta} = \sum_{\ell m} h^{(0)}_{\ell m}(t,r)\, E_{ab}\, \partial^b Y_\ell^m$$

$$h_{ra} = \sum_{\ell, m} h^{(1)}_{\ell m}(t,r)\, E_{ab}\, \partial^b Y_\ell^m$$

$$h_{ab} = \frac{1}{2} \sum_{\ell, m} h^{(2)}_{\ell m}(t,r) \left[E_a{}^c\, \nabla_c \nabla_b Y_\ell^m + E_b{}^c\, \nabla_c \nabla_a Y_\ell^m \right]$$

with $a = (\theta, \phi)$

and for scalar perturbations

$$h_{tt} = A(r) \sum_{l,m} H^{(0)}_{lm}(t,r) Y_l^m$$

$$h_{tr} = \sum_{l,m} H^{(1)}_{lm}(t,r) Y_l^m$$

$$h_{rr} = \frac{1}{B(r)} \sum_{l,m} H^{(2)}_{lm}(t,r) Y_l^m$$

$$h_{ta} = \sum_{l,m} \beta_{lm}(t,r)\, \partial_a Y_l^m$$

$$h_{ra} = \sum_{l,m} \alpha_{lm}(t,r)\, \partial_a Y_l^m$$

$$h_{ab} = \sum_{l,m} K_{lm}(t,r)\, \bar{g}_{ab} Y_l^m + \sum_{l,m} G_{lm}(t,r) \nabla_{ab} Y_l^m$$

We have a total of 10 functions ($h^{(0)}_{lm}, h^{(1)}_{lm}, h^{(2)}_{lm}, H^{(0)}_{lm}, H^{(1)}_{lm}, H^{(2)}_{lm}, \alpha_{lm}, \beta_{lm}, K_{lm}, G_{lm}$)

which are not independent - We can use the freedom to change coordinates, to eliminate 4 functions.

/!\ The split into scalars and vectors has been made, because they don't mix, indeed

scalar + vector $= 0$

rotation matrix

And we perform a rotation, we get scalar + R vector $= 0$

$$\Rightarrow \begin{cases} \text{scalar} = 0 \\ \text{vector} = 0 \end{cases}$$

Perturbations do not mix - We can study them separately.

III - Gauge freedom

Let's consider the transf. $\mathscr{C}$. $x^\mu \longmapsto x^\mu + \xi^\mu$ with ξ^μ infinitesimal

ξ^μ can be decomposed :

$$\xi_t = \sum_{l,m} T_{lm}(t,r)\, Y_l^m(\theta,\phi)$$

$$\xi_r = \sum_{l,m} R_{lm}(t,r)\, Y_l^m(\theta,\phi)$$

$$\xi_a = \underbrace{\sum_{l,m} \Theta_{lm}(t,r)\,\partial_a Y_l^m(\theta,\phi)}_{\text{scalar}} + \underbrace{\sum_{l,m} \Lambda_{lm}(t,r)\, E_a^{\ b}\,\partial_b Y_l^m(\theta,\phi)}_{\text{vectorial}}$$

Under this $\mathscr{C}$, the metric transforms as

$$h_{\mu\nu} \longmapsto h_{\mu\nu} - \nabla_\mu \xi_\nu - \nabla_\nu \xi_\mu = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + 2\,\bar{\Gamma}^\sigma_{\mu\nu}\,\xi_\sigma$$

Let's first consider only vector perturbations $\Rightarrow$ only $\Lambda_{lm}(t,r)$

We have

$$h_{ta} \longmapsto h_{ta} - \partial_t \xi_a - \partial_a \underbrace{\cancel{\xi_t}}^{\text{scalar}} + 2\,\bar{\Gamma}^\mu_{ta}\,\xi_\mu \quad \text{con } \bar{\Gamma}^\mu_{ta} = \frac{1}{2}\bar{g}^{\mu\nu}\left(\partial_t \cancel{\bar{g}_{a\nu}} + \partial_a \cancel{\bar{g}_{\nu t}} - \partial_\nu \cancel{\bar{g}_{ta}}\right) = 0$$

$$\longmapsto h_{ta} - \sum_{l,m} \dot{\Lambda}_{lm}(t,r)\, E_a^{\ b}\,\partial_b Y_l^m$$

Because $h_{ta} = \sum_{lm} h^{(0)}_{lm}(t,r)\, E_{ab}\,\partial^b Y_l^m$

$$\Rightarrow \boxed{h^{(0)}_{lm} \longmapsto h^{(0)}_{lm} - \dot{\Lambda}_{lm}}$$

$$h_{ra} \longmapsto h_{ra} - \partial_r \xi_a - \partial_a \xi_r + 2\,\bar{\Gamma}^b_{ar}\,\xi_b \qquad \bar{\Gamma}^b_{ar} = \frac{1}{2}\bar{g}^{bc}\left(\partial_r \bar{g}_{ac} + \cancel{\partial_a \bar{g}_{cr}} - \cancel{\partial_c \bar{g}_{ra}}\right) = \delta^b_a \frac{1}{r}$$

$$\longmapsto h_{ra} - \sum_{l,m} \Lambda'(t,r)\, E_a^{\ b}\,\partial_b Y_l^m + \frac{2}{r}\sum_{l,m} \Lambda_{lm}\, E_a^{\ b}\,\partial_b Y_l^m$$

$$\Rightarrow \boxed{h^{(1)}_{lm} \longmapsto h^{(1)}_{lm} - \Lambda'_{lm} + \frac{2}{r}\Lambda_{lm}}$$

$$h_{ab} \longmapsto h_{ab} - \partial_a \xi_b - \partial_b \xi_a + 2\bar{\Gamma}^c_{ab}\xi_c = h_{ab} - \nabla_a \xi_b - \nabla_b \xi_a$$

$$\longmapsto h_{ab} - \sum_{l,m} \Lambda_{lm} E_b^{\ c} \nabla_a \nabla_c Y_l^m - \sum_{l,m} E_a^{\ c} \nabla_b \nabla_c Y_l^m$$

$$\Rightarrow \quad \boxed{h^{(2)}_{lm} \longmapsto h^{(2)}_{lm} - 2\Lambda_{lm}}$$

We can use a coordinate system, namely ξ_μ, which means Λ_{lm} such that $h^{(2)}_{lm} = 0$. In that case, we have fixed completely the gauge because it is an algebraic eq. while to fix $h^{(0)}_{lm} = 0$ or $h^{(1)}_{lm} = 0$ one need to solve a differential equation, which means that the gauge would not be totally fixed

$h^{(2)}_{lm}(t,r) = 0$ is known as the Regge-Wheeler gauge

For scalar perturbations, we have

$$h_{tt} \longmapsto h_{tt} - 2\partial_0 \xi_0 + 2\bar{\Gamma}^\mu_{00}\xi_\mu \qquad \bar{\Gamma}^\mu_{00} = \frac{1}{2}\bar{g}^{\mu\sigma}\left(2\cancel{\partial_0 \bar{g}_{0\sigma}} - \partial_\sigma \bar{g}_{00}\right)$$

$$\longmapsto h_{tt} - 2\sum_{l,m} \dot{T}_{lm} Y_l^m + BA' \sum_{l,m} R_{lm} Y_l^m \qquad = \frac{1}{2}\bar{g}^{\mu 1} A'(r)$$

$$\Rightarrow \quad A H^{(0)}_{lm} \longmapsto A H^{(0)}_{lm} - 2\dot{T}_{lm} + BA' R_{lm}$$

$$\Rightarrow \quad \boxed{H^{(0)}_{lm} \longmapsto H^{(0)}_{lm} - 2\frac{\dot{T}_{lm}}{A} + \frac{BA'}{A} R_{lm}}$$

$$h_{tr} \longmapsto h_{tr} - \partial_0 \xi_r - \partial_r \xi_0 + 2\bar{\Gamma}^\mu_{0r}\xi_\mu \qquad \bar{\Gamma}^\mu_{0r} = \frac{1}{2}\bar{g}^{\mu\sigma}\left(\partial_0 \cancel{\bar{g}_{r\sigma}} + \partial_r \bar{g}_{\sigma 0} - \cancel{\partial_\sigma \bar{g}_{0r}}\right)$$

$$\longmapsto h_{tr} - \sum_{l,m} \dot{R}_{lm} Y_l^m - \sum_{l,m} T'_{lm} Y_l^m + \frac{A'}{A}\sum_{l,m} T_{lm} Y_l^m \qquad = -\frac{1}{2}\bar{g}^{\mu 0} A'$$

$$\Rightarrow \quad \boxed{H^{(1)}_{lm} \longmapsto H^{(1)}_{lm} - \dot{R}_{lm} - T'_{lm} + \frac{A'}{A} T_{lm}}$$

$$h_{rr} \longmapsto h_{rr} - 2\partial_r \xi_r + 2\bar{\Gamma}^{\mu}_{11}\xi_\mu \qquad \bar{\Gamma}^{\mu}_{11} = \frac{1}{2}\bar{g}^{\mu\sigma}\left(2\partial_1 \bar{g}_{1\sigma} - \partial_\sigma \bar{g}_{11}\right) = -\frac{B'}{2B^2}\bar{g}^{\mu 1}$$

$$\longmapsto h_{rr} - 2\sum_{\ell,m} R'_{\ell m} Y^m_\ell - \frac{B'}{B} R_{\ell m} Y^m_\ell$$

$$\boxed{H^{(2)}_{\ell m} \longmapsto H^{(2)}_{\ell m} - 2B(r)R'_{\ell m} - B'(r)R_{\ell m}}$$

$$h_{ta} \longmapsto h_{ta} - \partial_0 \xi_a - \partial_a \xi_0 + 2\bar{\Gamma}^{\mu}_{0a}\xi_b \qquad \bar{\Gamma}^{\mu}_{0a} = \frac{1}{2}\bar{g}^{\mu\nu}\left(\cancel{\partial_0 \bar{g}_{a\nu}} + \cancel{\partial_a \bar{g}_{\nu 0}} - \cancel{\partial_\nu \bar{g}_{0a}}\right) = 0$$

$$\longmapsto h_{ta} - \sum_{\ell,m}\dot{\Theta}_{\ell m}\partial_a Y^m_\ell - \sum_{\ell,m} T_{\ell m}\partial_a Y^m_\ell$$

$$\Rightarrow \quad \boxed{\beta_{\ell m} \longmapsto \beta_{\ell m} - \dot{\Theta}_{\ell m} - T_{\ell m}}$$

$$h_{ra} \longmapsto h_{ra} - \partial_r \xi_a - \partial_a \xi_r + 2\bar{\Gamma}^{\mu}_{ar}\xi_\mu \qquad \bar{\Gamma}^{\mu}_{ar} = \frac{1}{2}\bar{g}^{\mu\nu}\left(\cancel{\partial_a \bar{g}_{r\nu}} + \partial_r \bar{g}_{\nu a} - \cancel{\partial_\nu \bar{g}_{ar}}\right) = \frac{1}{r}\delta^{\mu}_{a}$$

$$\longmapsto h_{ra} - \sum_{\ell,m}\Theta'_{\ell m}\partial_a Y^m_\ell - \sum_{\ell,m} R_{\ell m}\partial_a Y^m_\ell + \frac{2}{r}\sum_{\ell,m}\Theta_{\ell m}\partial_a Y^m_\ell$$

$$\Rightarrow \quad \boxed{\alpha_{\ell m} \longmapsto \alpha_{\ell m} - \Theta'_{\ell m} - R_{\ell m} + \frac{2}{r}\Theta_{\ell m}}$$

$$h_{ab} \longmapsto h_{ab} - \partial_a \xi_b - \partial_b \xi_a + 2\underbrace{\bar{\Gamma}^{\mu}_{ab}\xi_\mu}_{\to\ \bar{\Gamma}^{c}_{ab}\xi_c + \cancel{\bar{\Gamma}^{0}_{ab}}\xi_0 + \bar{\Gamma}^{1}_{ab}\xi_1}$$

$$\bar{\Gamma}^{1}_{ab} \to -\frac{B}{2}\partial_1 \bar{g}_{ab} = -\frac{B}{r}\bar{g}_{ab}$$

$$\longmapsto h_{ab} - \underbrace{\bar{\nabla}_a \xi_b - \bar{\nabla}_b \xi_a} - \frac{2B}{r}\bar{g}_{ab}\sum_{\ell,m} R_{\ell m} Y^m_\ell$$

$$- 2\sum_{\ell,m}\Theta_{\ell m}\bar{\nabla}_a\bar{\nabla}_b Y^m_\ell$$

Covariant derivative in the space (θ, φ)

$$\bar{\nabla}_a \xi_b = \partial_a \xi_b - \bar{\Gamma}^{c}_{ab}\xi_c$$

$$\Rightarrow \quad \boxed{K_{\ell m} \longmapsto K_{\ell m} - \frac{2B}{r}R_{\ell m}}$$

$$\boxed{G_{\ell m} \longmapsto G_{\ell m} - 2\Theta_{\ell m}}$$

We can choose $(T_{lm}, R_{lm}, \Theta_{lm})$ to fix the gauge

$$\begin{cases} \text{Choice of } \Theta_{lm} \text{ gives } G_{lm} = 0 \\ \text{Choice of } R_{lm} \text{ gives } K_{lm} = 0 \text{ or } \alpha_{lm} = 0 \text{ or } H^{(0)}_{lm} = 0 \\ \text{Choice of } T_{lm} \text{ gives } \beta_{lm} = 0 \end{cases}$$

We will choose the following gauge $G_{lm} = K_{lm} = \beta_{lm} = 0$

As we have seen it for a massless scalar field, the final eq. doesn't depend on m, because of the spherical symmetry of $\bar{g}_{\mu\nu}$

Therefore, we can choose $m=0$, whitout lost of generality

$$Y_l^{m=0} = \sqrt{\frac{2l+1}{4\pi}}\ \underbrace{P_l(\cos\theta)}_{\text{Legendre polynomials}}$$

Because Y_l^0 doesn't depend on $\phi \Rightarrow \partial_\phi Y_l^0 = 0$

For example, $h_{t\theta} = \sum_l h_l^{(0)}(t,r)\, E_{\theta\phi}\, \partial^\phi Y_l^0 = 0$

$$h_{t\phi} = \sum_l h_l^{(0)}(t,r)\, \underbrace{E_{\phi\theta}}_{-r^2\sin\theta}\, \underbrace{\partial^\theta Y_l^0}_{\frac{1}{r^2}\sqrt{\frac{2l+1}{4\pi}}P_l'(\cos\theta)}$$

$$= -\sum_l h_l^{(0)}(t,r) \sin\theta \sqrt{\frac{2l+1}{4\pi}}\, \partial_\theta P_l(\cos\theta)$$

$h_{r\theta} = 0$ and $h_{r\phi} = -\sum_l h_l^{(1)}(t,r) \sin\theta \sqrt{\frac{2l+1}{4\pi}}\, \partial_\theta P_l(\cos\theta)$

$$\boxed{h_{\mu\nu}^{(\text{vectorial})} = -\sum_l \sqrt{\frac{2l+1}{4\pi}} \sin\theta\, \partial_\theta P_l(\cos\theta) \cdot \begin{pmatrix} 0 & 0 & 0 & h_l^{(0)} \\ 0 & 0 & 0 & h_l^{(1)} \\ 0 & 0 & 0 & 0 \\ h^{(0)} & h^{(1)} & 0 & 0 \end{pmatrix}}$$

Similarly, for scalar perturbations, $h_{t\phi} = h_{r\phi} = 0$

$$h_{\mu\nu}^{(scalar)} = \sum_{\ell} \sqrt{\frac{2\ell+1}{4\pi}} \begin{pmatrix} A\, H_\ell^{(0)} P_\ell(\cos\theta) & H_\ell^{(1)} \cdot P_\ell(\cos\theta) & 0 & 0 \\ H_\ell^{(1)} P_\ell(\cos\theta) & \frac{1}{B} H_\ell^{(2)} \cdot P_\ell(\cos\theta) & \alpha_\ell P_\ell'(\cos\theta) & 0 \\ 0 & \alpha_\ell P_\ell'(\cos\theta) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

IV - Regge-Wheeler equation

Let's consider the Einstein eq.

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} \Rightarrow R_{\mu\nu} = 8\pi G \left(T_{\mu\nu} - \frac{1}{2} T g_{\mu\nu}\right)$$

In the vacuum $R_{\mu\nu} = 0$

If $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} \Rightarrow g^{\mu\nu} = \bar{g}^{\mu\nu} - h^{\mu\nu} + O(|h|^2)$

$$\Rightarrow \Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right)$$

$$= \bar{\Gamma}^{\rho}_{\mu\nu} - \frac{1}{2} h^{\rho\sigma}\left(\partial_\mu \bar{g}_{\nu\sigma} + \partial_\nu \bar{g}_{\sigma\mu} - \partial_\sigma \bar{g}_{\mu\nu}\right) + \frac{1}{2}\bar{g}^{\rho\sigma}\left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\right) + O(h^2) \quad x$$

$$= \bar{\Gamma}^{\rho}_{\mu\nu} + \underbrace{\frac{1}{2}\bar{g}^{\rho\sigma}\left(\bar{\nabla}_\mu h_{\nu\sigma} + \bar{\nabla}_\nu h_{\sigma\mu} - \bar{\nabla}_\sigma h_{\mu\nu}\right)}_{\delta\Gamma^{\rho}_{\mu\nu}} + O(h^2)$$

Indeed $\quad \delta\Gamma^{\rho}_{\mu\nu} \equiv \frac{1}{2}\bar{g}^{\rho\sigma}\left(\bar{\nabla}_\mu h_{\nu\sigma} + \bar{\nabla}_\nu h_{\sigma\mu} - \bar{\nabla}_\sigma h_{\mu\nu}\right)$

$$= \frac{1}{2}\bar{g}^{\rho\sigma}\Big(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu} - \bar{\Gamma}^{\alpha}_{\mu\nu} h_{\alpha\sigma} - \cancel{\bar{\Gamma}^{\alpha}_{\mu\sigma} h_{\nu\alpha}}$$
$$- \cancel{\bar{\Gamma}^{\alpha}_{\nu\sigma} h_{\alpha\mu}} - \bar{\Gamma}^{\alpha}_{\nu\mu} h_{\sigma\alpha} + \cancel{\bar{\Gamma}^{\alpha}_{\sigma\mu} h_{\alpha\nu}} + \cancel{\bar{\Gamma}^{\alpha}_{\sigma\nu} h_{\mu\alpha}}\Big)$$

$$= \frac{1}{2}\bar{g}^{\rho\sigma}\left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\right) - \bar{g}^{\rho\sigma}\bar{\Gamma}^{\alpha}_{\mu\nu} h_{\alpha\sigma}$$

$$= \frac{1}{2}\bar{g}^{\rho\sigma}\left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\sigma\mu} - \partial_\sigma h_{\mu\nu}\right) - \frac{1}{2}\underbrace{\bar{g}^{\rho\sigma}\bar{g}^{\alpha\beta} h_{\alpha\sigma}}_{h^{\beta\rho}}\left(\partial_\mu \bar{g}_{\nu\beta} + \partial_\nu \bar{g}_{\beta\mu} - \partial_\beta \bar{g}_{\mu\nu}\right) \quad x$$

$$R_{\mu\nu} = \partial_\alpha \Gamma^{\alpha}_{\mu\nu} - \partial_\mu \Gamma^{\alpha}_{\alpha\nu} + \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\alpha\mu} \quad \text{con } \Gamma^{\rho}_{\mu\nu} = \bar{\Gamma}^{\rho}_{\mu\nu} + \delta\Gamma^{\rho}_{\mu\nu}$$

$$= \underbrace{\partial_\alpha \bar{\Gamma}^{\alpha}_{\mu\nu} - \partial_\mu \bar{\Gamma}^{\alpha}_{\alpha\nu} + \bar{\Gamma}^{\alpha}_{\mu\nu}\bar{\Gamma}^{\beta}_{\alpha\beta} - \bar{\Gamma}^{\alpha}_{\beta\nu}\bar{\Gamma}^{\beta}_{\alpha\mu}}_{\bar{R}_{\mu\nu}} + \partial_\alpha \delta\Gamma^{\alpha}_{\mu\nu} - \partial_\mu \delta\Gamma^{\alpha}_{\alpha\nu} + \bar{\Gamma}^{\alpha}_{\mu\nu}\delta\Gamma^{\beta}_{\alpha\beta}$$
$$+ \delta\Gamma^{\alpha}_{\mu\nu}\bar{\Gamma}^{\beta}_{\alpha\beta} - \bar{\Gamma}^{\alpha}_{\beta\nu}\delta\Gamma^{\beta}_{\alpha\mu} - \delta\Gamma^{\alpha}_{\beta\nu}\cdot\bar{\Gamma}^{\beta}_{\alpha\mu} + O(h^2)$$

$$= \bar{R}_{\mu\nu} + \bar{\nabla}_\alpha \left(\delta\Gamma^\alpha_{\mu\nu}\right) - \bar{\nabla}_\mu \left(\delta\Gamma^\alpha_{\alpha\nu}\right) + O(\lambda^2)$$

We Know that $\delta\Gamma^\alpha_{\mu\nu} = \frac{1}{2}\bar{g}^{\alpha\sigma}\left(\bar{\nabla}_\mu h_{\nu\sigma} + \bar{\nabla}_\nu h_{\sigma\mu} - \bar{\nabla}_\sigma h_{\mu\nu}\right)$

$$\Rightarrow \quad \delta\Gamma^\alpha_{\alpha\nu} = \frac{1}{2}\bar{g}^{\alpha\sigma}\left(\cancel{\bar{\nabla}_\alpha h_{\nu\sigma}} + \bar{\nabla}_\nu h_{\sigma\alpha} - \cancel{\bar{\nabla}_\sigma h_{\alpha\nu}}\right) = \frac{1}{2}\bar{\nabla}_\nu h$$

$$R_{\mu\nu} = \bar{R}_{\mu\nu} + \bar{\nabla}_\alpha \left(\delta\Gamma^\alpha_{\mu\nu}\right) - \bar{\nabla}_\mu \left(\delta\Gamma^\alpha_{\alpha\nu}\right) + O(\lambda^2)$$

$$= \bar{R}_{\mu\nu} + \frac{1}{2}\bar{g}^{\alpha\sigma}\bar{\nabla}_\alpha\left(\bar{\nabla}_\mu h_{\nu\sigma} + \bar{\nabla}_\nu h_{\sigma\mu} - \bar{\nabla}_\sigma h_{\mu\nu}\right) - \frac{1}{2}\bar{\nabla}_{\mu\nu} h + O(h^2)$$

$$= \bar{R}_{\mu\nu} + \frac{1}{2}\left[\bar{\nabla}_{\alpha\mu} h_\nu{}^\alpha + \bar{\nabla}_{\alpha\nu} h_\mu{}^\alpha - \bar{\Box} h_{\mu\nu} - \bar{\nabla}_{\mu\nu} h\right] + O(h^2)$$

and $\bar{R}_{\mu\nu} = 0 \Rightarrow \boxed{\delta R_{\mu\nu} = \frac{1}{2}\left[\bar{\nabla}_{\alpha\mu} h_\nu{}^\alpha + \bar{\nabla}_{\alpha\nu} h_\mu{}^\alpha - \bar{\Box} h_{\mu\nu} - \bar{\nabla}_{\mu\nu} h\right] = 0}$

Considering vector pert., we have

$$\delta R_{00} = \delta R_{11} = \delta R_{01} = 0$$

$$\delta R_{23} = \sum_\ell \frac{2\cos\theta \frac{d}{d\theta}P_\ell(\cos\theta) + \lambda P_\ell(\cos\theta)\sin\theta}{4A(r)}\left[AB' h_1 + BA' h_1 + 2AB h_1' - 2\dot{h}_0\right] + O(\lambda^2)$$

$$\delta R_{13} = \sum_\ell \frac{\sin\theta \cdot \frac{d}{d\theta}P_\ell(\cos\theta)}{2r^2 A(r)}\left[A\left(rB' + 2B - \lambda\right)h_1 + rBA'h_1 - 2r\dot{h}_0 + r^2\dot{h}_0' - r\ddot{h}_1\right] + O(\lambda^2)$$

con $\lambda = \ell(\ell+1)$

Other eq. are redundant

From $\delta R_{23} = 0 \rightsquigarrow \dot{h}_0 = \frac{AB'}{2}h_1 + \frac{BA'}{2}h_1 + AB h_1'$

That we can use in eq. $\delta R_{13} = 0$

$$\Rightarrow \quad \alpha h_1 + \beta h_1' + \gamma h_1'' + \delta \ddot{h}_1 = 0$$

with
$$\begin{cases} \alpha = r^2\left(BA'' + 2A'B'\right) + A\left(r^2 B'' + 4B - 2\lambda\right) \\ \beta = r\left(3rBA' + A\left(3rB' - 4B\right)\right) \qquad \gamma = 2r^2 AB \qquad \delta = -2r^2 \end{cases}$$

We can always make a change of variables to eliminate h_1'

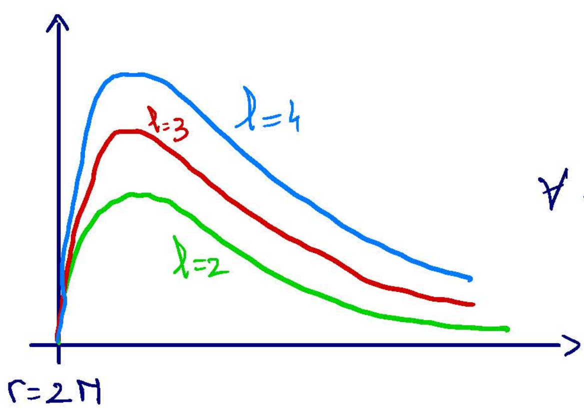
Considering $h_1 = \frac{r\cdot Q}{\sqrt{AB}}$ and the tortoise coord. $dr_* = \frac{dr}{\sqrt{AB}}$

$$\Rightarrow \boxed{\frac{\partial^2 Q}{\partial t^2} - \frac{\partial^2 Q}{\partial r_*^2} + V(r)\, Q = 0} \quad \text{with} \quad V = \frac{2\lambda A - 3rBA' - 3rAB'}{2r^2}$$

Known as the Regge-Wheeler equation.

In the case $A = B = 1 - \frac{2M}{r} \Rightarrow V = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right]$

Known as the Regge-Wheeler eq.



$\forall l, \quad V(r) \geq 0$ for $r \in [r_H, \infty[$

V. Stability

1. First way to see the stability

We work in the Fourier space, $Q(t,r) = \int d\omega\, \hat{Q}(\omega,r)\, e^{-i\omega t}$

$$\Rightarrow \frac{d^2\hat{Q}}{dr_*^2} + (\omega^2 - V)\,\hat{Q} = 0$$

That eq. has many solut°, but we are looking for an eq. with particular boundary conditions

For $r \simeq r_s$ and $r \to \infty$ $\quad V(r) \longmapsto 0$

$$\Rightarrow \frac{d^2\hat{Q}}{dr_*^2} + \omega^2 \hat{Q} = 0 \quad \Rightarrow \hat{Q} \simeq A e^{i\omega r_*} + B e^{-i\omega r_*}$$

$$\Rightarrow Q = \int d\omega \Big[\underbrace{A e^{-i\omega(t - r_*)}}_{\text{outgoing wave}} + \underbrace{B e^{-i\omega(t + r_*)}}_{\text{ingoing wave}} \Big]$$

For $r \simeq r_s$, we want only an ingoing wave

$$\Rightarrow \hat{Q} \sim e^{-i\omega r_*}$$

For $r \mapsto \infty$, we want only an outgoing wave

$$\Rightarrow \hat{Q} \sim e^{i\omega r_*}$$

Because the wave goes to infinity or inside the BH $\Rightarrow$ the system is dissipative

$$\Rightarrow \omega = \omega_R + i\,\omega_I$$

The perturbation decreases with time if $\omega_I < 0$ because $e^{-i\omega t} = e^{-i\omega_R t} \cdot e^{\omega_I t}$

if $\omega_I < 0$: the BH is stable

if $\omega_I > 0$: the BH is unstable

We have the equation $-\hat{Q}'' + V\hat{Q} = \omega^2 \hat{Q}$

Multiplied by $\bar{\hat{Q}}$ (the conjugate) and integrated over the exterior region

$$\int_{-\infty}^{\infty} dr_* \Big(\underbrace{-\bar{\hat{Q}}\,\hat{Q}''}_{-\frac{d}{dr_*}\left(\bar{\hat{Q}}\,\hat{Q}'\right) + \bar{\hat{Q}}'\hat{Q}'} + V|\hat{Q}|^2 \Big) = \omega^2 \int dr_* \, |\hat{Q}|^2$$

$$\rightsquigarrow \; -\bar{\hat{Q}}\,\hat{Q}'\Big|_{\text{horizon}}^{\infty} + \int dr_* \left[\left|\frac{d\hat{Q}}{dr_*}\right|^2 + V|\hat{Q}|^2 \right] = \omega^2 \underbrace{\int dr_* \, |\hat{Q}|^2}_{\equiv A^2 \;(>0)}$$

At infinity $\hat{Q} \sim e^{i\omega r_*}$

At the horizon $\hat{Q} \sim e^{-i\omega r_*}$ $\Rightarrow$ $\bar{\hat{Q}}\,\hat{Q}'\Big|_{\text{horizon}} = -i\omega \,|\hat{Q}(\text{horizon})|^2$

$$\bar{\hat{Q}}\,\hat{Q}'\Big|_{\infty} = i\omega\,|\hat{Q}(\infty)|^2$$

$$\Rightarrow \; -\bar{\hat{Q}}\,\hat{Q}'\Big|_{\text{horizon}}^{\infty} = -i\omega B^2 \quad \text{with} \quad B^2 = |\hat{Q}(\infty)|^2 + |\hat{Q}(\text{hor.})|^2 \;(>0)$$

Therefore, we have

$$-i\omega B^2 + \int dr_* \left[\left|\frac{d\hat{Q}}{dr_*}\right|^2 + V|\hat{Q}|^2 \right] = \omega^2 A^2 \qquad (\text{with } \omega = \omega_R + i\omega_I)$$

Taking the imaginary part

$$-\omega_R B^2 = 2\omega_R \omega_I A^2 \;\rightsquigarrow\; \omega_R \left[B^2 + 2\omega_I A^2 \right] = 0$$

If the BH is unstable, namely $\omega_I > 0 \Rightarrow \omega_R = 0$

$\Rightarrow$ The unstable modes do not oscillate

Taking now the real part

$$\omega_I B^2 + \int dr_* \left[\left| \frac{d\hat{Q}}{dr_*} \right|^2 + V |\hat{Q}|^2 \right] = (\omega_R^2 - \omega_I^2) A^2$$

If the mode is unstable $\Rightarrow$ $\omega_R = 0$ and $\omega_I > 0$

$$\Rightarrow \int dr_* \left[\left| \frac{d\hat{Q}}{dr_*} \right|^2 + V |\hat{Q}|^2 \right] < 0$$

$$\Rightarrow V < 0$$

$V(r) < 0$ is a necessary condition for the unstability

For the Schwarzschild spacetime $V(r) \geq 0 \Rightarrow$ vector modes are stable

The proof is incomplete because we studied only exponential modes, while we could have a linear mode.

A very similar form, is to consider

$$\frac{\partial^2 Q}{\partial t^2} - \frac{\partial^2 Q}{\partial r_*^2} + V(r) Q = 0$$

and define the operator $A \equiv -\dfrac{d^2}{dr_*^2} + V$

which means $-AQ = \ddot{Q}$, multiplying by $\dot{\bar{Q}}$ and integrating

we get $- \int dr_* \dot{\bar{Q}} A Q = \int dr_* \dot{\bar{Q}} \ddot{Q}$

the conjugate is $- \int dr_* \dot{Q} A \bar{Q} = \int dr_* \dot{Q} \ddot{\bar{Q}}$

The sum gives $\dfrac{\partial}{\partial t} \left[\int dr_* |\dot{Q}|^2 + \int dr_* \bar{Q} A Q \right] = 0$

$$\Rightarrow \int dr_* |\dot{Q}|^2 + \int dr_* \bar{Q} A Q = C \quad : \text{constant}$$

Because $\int dr_* \bar{Q} A Q \geqslant 0$ A is positive (A is a positive self-adjoint operator over the Hilbert Space $L^2(r_*)$)

$\Rightarrow$ C is a limit for $\int dr_* |\dot{Q}|^2$

$\Rightarrow$ We can't have an exponential growth of the perturbations

But we can have $Q \sim t$

For the full proof, see (Wald - Note on the stability of the Schwarzschild metric - 1979)

Spectral theory:

An operator A is self-adjoint (or hermitian) if $\forall (\phi, \Psi) \in \mathcal{H}$

Hilbert Space

$$\langle A\phi | \Psi \rangle = \langle \phi | A \Psi \rangle$$ (usually we have $\langle A\phi | \Psi \rangle = \langle \phi | A^* \Psi \rangle$)

Our operator is self-adjoint $A = -\frac{d^2}{dr^2} + V$

Defining the scalar product $\langle \phi | \Psi \rangle = \int dr \, \bar{\phi}(r) \Psi(r)$

$$\langle A\phi | \Psi \rangle = \int dr \, (\overline{A\phi}) \, \Psi = \int dr \left[\left(-\bar{\phi}'' + V \bar{\phi} \right) \Psi \right]$$ because $V(r)$ is real

$$= \int dr \left(-\underbrace{\bar{\phi} \, \Psi''}_{} + V \bar{\phi} \Psi \right) = \int dr \, \bar{\phi} \left(-\frac{d^2}{dr^2} + V \right) \Psi$$

integration by parts

$$= \langle \phi | A \Psi \rangle$$

The spectral theorem says that for an autoadjunct operator A, we have

$$A = \int \lambda \, dE_\lambda$$

with $\{E_\lambda\}$ a family of projection operator

This is the extension to infinite Hilbert space of a well-known theorem

For a vector space of finite dimension, X - $\dim X = n$

and considering a lineal self-adjoint operator $A : X \mapsto X$

① Eigenvalues of A are real

② Eigenvectors of A which correspond to different eigenvalues are orthogonal

1- Considering $Av = \lambda v$ $\quad v \in X \quad (v \neq 0)$

$$\Rightarrow \quad \langle Av | v \rangle = \langle v | Av \rangle \quad \text{(self-adjoint)}$$

$$\Rightarrow \quad \langle \lambda v | v \rangle = \langle v | \lambda v \rangle$$

$$\Rightarrow \quad \bar{\lambda} \langle v | v \rangle = \langle v | v \rangle . \lambda$$

But $\langle v | v \rangle \neq 0 \Rightarrow \lambda = \bar{\lambda} \Rightarrow \lambda \in \mathbb{R}$

2- Considering $Av = \lambda v$ and $Aw = \mu w$ $\quad (v, w) \neq 0$

$$\langle Av | w \rangle = \langle v | Aw \rangle \quad \text{self-adjoint}$$

$$\langle \lambda v | w \rangle = \langle v | \mu w \rangle$$

$$\bar{\lambda} \langle v | w \rangle = \mu \langle v | w \rangle$$

$$\lambda \langle v | w \rangle = \mu \langle v | w \rangle \quad \lambda, \mu \text{ are reals}$$

$$(\lambda - \mu) \langle v | w \rangle = 0$$

If $\lambda \neq \mu$ $\quad v$ is ortogonal to w

$\Rightarrow$ To each eigenvalue λ_i, we associate an eigenvector which define a subspace of X; E_{λ_i}

$$X = E_{\lambda_1} \oplus E_{\lambda_2} \oplus \cdots \oplus E_{\lambda_k} \quad \text{with } k \leq n \equiv \dim X$$

(an eigenvalue can have various eigenvectors)

We can do that descomposition, because according to ② eigenvectors E_{λ_i} are orthogonals to vectors $\in E_{\lambda_j}$ $(j \neq i)$

$$\Rightarrow \quad \forall x \in X \quad x = x^{(1)} + x^{(2)} + \cdots + x^{(k)} \quad \text{with } x^{(i)} \in E_{\lambda_i}$$

$$\Rightarrow \quad Ax = Ax^{(1)} + Ax^{(2)} + \cdots + Ax^{(k)}$$
$$= \lambda_1 x^{(1)} + \cdots + \lambda_k x^{(k)}$$

If we define the projection operator P_i

$$P_i : X \longmapsto X$$
$$x \longmapsto x^{(i)}$$

We can check that $\quad P_i P_j = 0 \quad$ if $\; i \neq j$

$$P_i^2 = P_i$$

So we have the spectral resolution

$$A = \sum_{i=1}^{k} \lambda_i P_i$$

and also a resolution of the identity $\quad I = \sum_{i=1}^{k} P_i$

Indeed $\quad Ax = \sum_{i=1}^{k} \lambda_i P_i x = \sum_{i=1}^{k} \lambda_i x^{(i)}$

In quantum mechanics $\mathbb{1} = \sum_i |\psi_i\rangle\langle\psi_i|$ with $|\psi_i\rangle\langle\psi_i|$ projectors

$$A = \sum_i A|\psi_i\rangle\langle\psi_i| = \sum_i \lambda_i |\psi_i\rangle\langle\psi_i|$$

Theorem: If $A : X \mapsto X$ is self-adjoint, there exist a spectral resolution of the identity and A.

For infinite Hilbert spaces, we have $A = \int \lambda \, dE_\lambda$ with $\{E_\lambda\}$ the projectors (if A is self-adjoint)

The most important with this decomposition, is to calculate functions of A. For example, how to calculate $\sqrt{A}$ or A^2 ... or $f(A)$

$$\forall x, \quad Ax = \sum_i \lambda_i x^{(i)}$$

$$\Rightarrow A^2 x = \sum_i \lambda_i A x^{(i)} = \sum_i \lambda_i^2 x^{(i)}$$

which means $A^2 = \sum_i \lambda_i^2 P_i$

Generically, we have $f(A) = \sum_i f(\lambda_i) P_i$

For exple, if $A = \begin{pmatrix} 5 & -3 \\ -3 & 5 \end{pmatrix}$ which is self-adjoint $A^+ = A$

We can diagonalize the matrix $A = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$$A = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 8 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \left[2 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 8 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right] \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= 2 \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix} + 8 \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix} = 2P_1 + 8P_2$$

It is easy to check that (P_1, P_2) are projector, namely $P_1^2 = P_1$, $P_2^2 = P_2$ and $P_1 P_2 = 0$

$$\Rightarrow \sqrt{A} = \sqrt{2} P_1 + \sqrt{8} P_2$$

In the same way, we have $f(A) = \int f(\lambda)\, dE_\lambda$ for a self-adjoint operator over a Hilbert space (λ are not the eigenvalues but just a parameter)

Returning to our problem, we have $\frac{\partial^2 Q}{\partial t^2} = A Q$ with $A = -\frac{\partial^2}{\partial r_*^2} + V$

Let's define $Q(t_0, r_*) = Q_0(r_*)$ and $\dot{Q}_0(r_*)$ initial conditions of the perturbations and we define $\Psi(t, r_*) = \cos(A^{1/2} t)\, Q_0 + A^{-1/2} \sin(A^{1/2} t)\, \dot{Q}_0$

We want to demonstrate $\Psi(t, r_*) = Q(t, r_*)$ which means that it is the unique solution to our problem

Using the spectral representation, we have

$$\Psi(t, r_*) = \int \cos(\sqrt{\lambda}\, t)\, dE_\lambda\, Q_0 + \int \frac{\sin \sqrt{\lambda}\, t}{\sqrt{\lambda}}\, dE_\lambda\, \dot{Q}_0$$

$$\Rightarrow \dot{\Psi}(t, r_*) = -\int \sqrt{\lambda} \sin(\sqrt{\lambda}\, t)\, dE_\lambda\, Q_0 + \int \cos(\sqrt{\lambda}\, t)\, dE_\lambda\, \dot{Q}_0$$

$$\Rightarrow \ddot{\Psi}(t, r_*) = -\int \lambda \cos(\sqrt{\lambda}\, t)\, dE_\lambda\, Q_0 - \int \sqrt{\lambda} \sin \sqrt{\lambda}\, t\, dE_\lambda\, \dot{Q}_0$$

$$= -A \cos(A^{1/2} t)\, Q_0 - A^{1/2} \sin(A^{1/2} t)\, \dot{Q}_0$$

$$= -A \Psi$$

$\Psi(t, r_*)$ is a solut: of the problem, but $\Psi(t=0, r_*) = Q_0$ and $\dot{\Psi}(t=0, r_*) = \dot{Q}_0 \Rightarrow Q(t, r_*) = \Psi(t, r_*)$

$$\boxed{Q(t, r_*) = \cos(A^{1/2} t)\, Q_0 + A^{-1/2} \sin(A^{1/2} t)\, \dot{Q}_0}$$

A is a positive operator, which means that $\forall\, \phi \quad \langle \phi | A \phi \rangle \geq 0$

Indeed $\langle \phi | A \phi \rangle = \int \bar{\phi}\, (-\phi'' + V\phi)\, dr = \int (|\phi'|^2 + V|\phi|^2)\, dr \geq 0$

Therefore $\cos(A^{1/2} t)$ and $A^{-1/2} \sin(A^{1/2} t)$ are bounded operators namely $\| \cos A^{1/2} t \|^2 \leq 1$ and $\| A^{-1/2} \sin A^{1/2} t \|^2 \leq 1$

We know that $\|a+b\| \leq \|a\| + \|b\|$ (triangle inequality)

and $\|a\|\,\|b\| \leq \frac{1}{2}\left(\|a\|^2 + \|b\|^2\right)$ (because $(\|a\| - \|b\|)^2 \geq 0$)

$$\Rightarrow \|a+b\|^2 \leq (\|a\| + \|b\|)^2 = \|a\|^2 + \|b\|^2 + 2\|a\|\,\|b\|$$

$$\leq 2\|a\|^2 + 2\|b\|^2$$

$$\Rightarrow \|Q\|^2 = \left\|\cos(A^{1/2}t)\,Q_0 + A^{-1/2}\sin(A^{1/2}t)\,\dot{Q}_0\right\|^2 \quad (\|Q\|^2 = \langle Q | Q \rangle)$$

$$\leq 2\|\cos(A^{1/2}t)\|^2\,\|Q_0\|^2 + 2\|A^{-1/2}\sin A^{1/2}t\|^2\,\|\dot{Q}_0\|^2$$

$$\leq 2\|Q_0\|^2 + 2\|\dot{Q}_0\|^2$$

which means $\int dr_* \,|Q(t,r_*)|^2 \leq 2\|Q_0(r_0)\|^2 + 2\|\dot{Q}_0(r_*)\|^2$

while we had before a superior limit on $\dot{Q}$, we have now a limit on $Q(t,r_*)$ $\forall t$.

$\Rightarrow$ the integral of Q over all exterior never diverges

We found a limit over the integral, it would be better to have a limit on the perturbation.

For a function $\Psi(x)$, we have $|\Psi(x)|^2 \leq \frac{1}{2}\int |\Psi|^2 dx + \frac{1}{2}\int |\Psi'|^2 dx$

Indeed $\Psi(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\omega x}\,\hat{\Psi}(\omega)\,d\omega$

$$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\omega x}\,(1+\omega^2)^{-1/2}(1+\omega^2)^{1/2}\,\hat{\Psi}(\omega)\,d\omega$$

But the Cauchy-Schwarz inequality says

$$\left|\int_{\mathbb{R}} \Psi_1(x)\, \Psi_2(x)\, dx\right|^2 \leq \int_{\mathbb{R}} |\Psi_1(x)|^2 dx \cdot \int_{\mathbb{R}} |\Psi_2(x)|^2 dx$$

$$\Rightarrow \quad |\Psi(x)|^2 = \frac{1}{2\pi} \left| \int e^{-i\omega x} (1+\omega^2)^{-1/2} (1+\omega^2)^{1/2}\, \hat{\Psi}(\omega)\, d\omega \right|^2$$

$$\leq \frac{1}{2\pi}\left[\underbrace{\int_{\mathbb{R}} \frac{d\omega}{1+\omega^2}}_{\pi} \cdot \int_{\mathbb{R}} (1+\omega^2) |\hat{\Psi}|^2 d\omega \right]$$

$$\leq \frac{1}{2}\left[\int_{\mathbb{R}} |\hat{\Psi}|^2 d\omega + \int_{\mathbb{R}} |\omega \hat{\Psi}|^2 d\omega \right]$$

Using the Plancherel theorem $\displaystyle\int_{\mathbb{R}} |\hat{\Psi}(\omega)|^2 d\omega = \int_{\mathbb{R}} |\Psi(x)|^2 dx$

and $\displaystyle\int_{\mathbb{R}} |\omega \hat{\Psi}(\omega)|^2 d\omega = \int |\Psi'(x)|^2 dx$

$$\Rightarrow \quad |\Psi(x)|^2 \leq \frac{1}{2}\int |\Psi|^2 dx + \frac{1}{2}\int |\Psi'|^2 dx$$

Using this relation for $Q(t, r_*)$

$$\int |Q'|^2 dr_* = \int \bar{Q}(-Q'')\, dr_* \leq \int \bar{Q}\left(-\frac{d^2}{dr_*^2} + V\right) Q\, dr_* \quad \text{because } V \text{ es positive}$$

$$\int |Q'|^2 dr_* = \int \bar{Q}(-Q'')\, dr_* \leq \int \bar{Q}\left(-\frac{d^2}{dr_*^2} + V\right) Q\, dr_*$$

$$\leq \int \bar{Q} A Q\, dr_*$$

But as we have seen $\displaystyle\int dr_* |\dot{Q}|^2 + \int dr_* \bar{Q} A Q = C$: constant

Which means $\int dr_* \, |\dot{Q}|^2 + \int dr_* \, \overline{Q} A Q = \int dr_* \, |\dot{Q}_0|^2 + \int dr_* \, \overline{Q}_0 A Q_0$

$$\Rightarrow \int dr_* \, \overline{Q} A Q \leq \int dr_* \, |\dot{Q}_0|^2 + \int dr_* \, \overline{Q}_0 A Q_0$$

In conclusion $\int |Q'|^2 dr_* \leq \int dr_* \, |\dot{Q}_0|^2 + \int dr_* \, \overline{Q}_0 A Q_0$

But we also have seen $\int |Q|^2 dr_* \leq 2 \int dr_* \, |Q_0|^2 + 2 \int dr_* \, |\dot{Q}_0|^2$

and $|Q(t, r_*)|^2 \leq \frac{1}{2} \int |Q|^2 dr_* + \frac{1}{2} \int |Q'|^2 dr_*$

$$\Rightarrow |Q(t, r_*)|^2 \leq \int dr_* \, |Q_0|^2 + \frac{3}{2} \int dr_* \, |\dot{Q}_0|^2 + \frac{1}{2} \int dr_* \, \overline{Q}_0 A Q_0$$

$\Rightarrow \quad \forall t \quad |Q(t, r_*)|$ is bounded

VI Zerilli equation

Considering the scalar perturbations

It turns out to be much simpler to work from the beginning in the Fourier space

$$h_{\mu\nu}^{(scalar)} = \sum_\ell \sqrt{\frac{2\ell+1}{4\pi}}\, e^{-i\omega t} \begin{pmatrix} A\, H^{(0)}_\ell(r)\, P_\ell(\cos\theta) & H^{(1)}_{\ell}(r)\cdot P_\ell(\cos\theta) & 0 & 0 \\ H^{(1)}_\ell(r)\, P_\ell(\cos\theta) & \frac{1}{B}\, H^{(2)}_\ell(r)\, P_\ell(\cos\theta) & \alpha_\ell(r) P_\ell'(\cos\theta) & 0 \\ 0 & \alpha_\ell(r)\, P_\ell'(\cos\theta) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We have

$$\delta R_{01} = \sum_\ell \sqrt{\frac{2\ell+1}{4\pi}}\, e^{-i\omega t}\, \frac{P_\ell(\cos\theta)}{2r^2}\left[\lambda H_1 - 2i\omega r H_2 + i\lambda\omega\alpha\right] \quad \text{with } \lambda = \ell(\ell+1)$$

$$\Rightarrow \quad H_1 = \frac{i}{\lambda}\left(2r\omega H_2 - \lambda\omega\alpha\right)$$

$$\delta R_{02} = \sum_\ell \sqrt{\frac{2\ell+1}{4\pi}}\, e^{-i\omega t}\, \frac{P_\ell'(\cos\theta)}{2r^2}\left[r_s H_1 + i\omega r^2 H_2 - 2\omega i(r-r_s)\alpha + r(r-r_s)H_1' + i\omega r\alpha'\right]$$

$$\text{with } A(r) = B(r) = 1 - \frac{r_s}{r}$$

Using the expression H_1, we have

$$\frac{\lambda+2}{2} H_2 + (r-r_s) H_2' - \lambda\cdot\frac{2r-r_s}{2r^2}\alpha - \frac{\lambda}{r}(r-r_s)\alpha' = 0$$

Which is a differential eq. for H_2 and α, but we can't solve it for any of thes 2 variables. Therefore we define a new variable

$$H_2 = \frac{\Psi}{r_s - r} + \frac{\lambda}{r}\alpha$$

$$\Rightarrow \quad \Psi' + \frac{\lambda}{2(r-r_s)}\Psi - \frac{\lambda}{2r^2}\left(3r_s + r(\lambda-2)\right)\alpha = 0$$

Having now an algebraic eq. for α:

$$\alpha(r) = \frac{r^2(\lambda\Psi + 2r\Psi' - 2r_s\Psi')}{(r-r_s)(r\lambda - 2r + 3r_s)\lambda}$$

$\Rightarrow$ (H_1, H_2, α) which are functions of Ψ

We are left with only H_0

From ec. δR_{33} we can find a relation between H_0 and H_0'

$$\delta R_{33} = \sum_l \sqrt{\frac{2l+1}{4\pi}} e^{-i\omega t} \left\{ \frac{P_l(\cos\theta)}{2r^2} \sin^2\theta \left[2i\omega r^3 H_1 + 2r^2 H_2 - 2\lambda(r-r_s)\alpha + r^2(r-r_s)H_0' + r^2(r-r_s)H_2' \right] - \sin\theta\cos\theta \frac{P_l'(\cos\theta)}{2r^2} \left[2r_s\alpha + r^2 H_0 - r^2 H_2 + 2r(r-r_s)\alpha' \right] \right\}$$

$$\Rightarrow \begin{cases} H_0 = H_2 - \dfrac{2r_s}{r^2}\alpha - \dfrac{2}{r}(r-r_s)\alpha' \\[2ex] H_0' = \dfrac{2\lambda}{r^2}\alpha - H_2' - \dfrac{2i\omega r H_1 + 2H_2}{r-r_s} \end{cases}$$

Using these relat° in δR_{12}, we get

$$\delta R_{12} = \sum_l \sqrt{\frac{2l+1}{4\pi}} \frac{P_l'(\cos\theta)}{4r^2(r-r_s)} e^{-i\omega t} \left[2r^2(r-r_s)H_0' - r(2r-3r_s)H_0 + 2i\omega r^3 H_1 + r(2r-r_s)H_2 - 4(r-r_s)\alpha - 2r^3\omega^2\alpha \right]$$

We obtain an eq. for Ψ, of the form $\Psi'' + a_1(r)\Psi' + a_2(r)\Psi = 0$

We can always perform a change of variables to eliminate Ψ'

We also use the tortoise coordinate

$$\Psi = \left(\lambda - 2 + 3\frac{r_s}{r}\right) Q$$

and $$\frac{d}{dr} = \frac{r}{r-r_s}\frac{d}{dr_*}$$

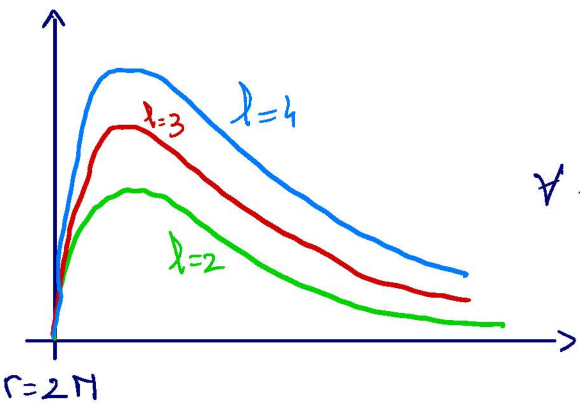
$$\boxed{\frac{d^2Q}{dr_*^2} + \left[\omega^2 - V(r)\right] Q = 0}$$

Zerilli equation

with $$V(r) = \left(1 - \frac{2M}{r}\right) \frac{2\Lambda^2(\Lambda+1)r^3 + 6\Lambda^2 M r^2 + 18\Lambda M^2 r + 18M^3}{r^3(\Lambda r + 3M)^2}$$

Zerilli potential

with $r_s = 2M$ and $\Lambda = \frac{\ell(\ell+1)-2}{2} = \frac{(\ell-1)(\ell+2)}{2}$



$\forall \ell$, $V(r) \geq 0$ for $r \in [r_H, \infty[$

=> scalar pert. are stable

> The Schwarzschild BH is linearly stable under perturbations of the metric

VII - Isospectrality

We found for both perturbations

$$\frac{d^2 Q^{RW}}{dr_*^2} + (\omega^2 - V_{RW}) Q^{RW} = 0 \qquad \text{with} \quad V_{RW} = \left(1 - \frac{2M}{r}\right)\left[\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right]$$

$$\frac{d^2 Q^{Z}}{dr_*^2} + (\omega^2 - V_Z) Q^{Z} = 0 \qquad V_Z(r) = \left(1 - \frac{2M}{r}\right) \frac{2\Lambda^2(\Lambda+1) r^3 + 6\Lambda^2 M r^2 + 18 \Lambda M^2 r + 18 M^3}{r^3 (\Lambda r + 3M)^2}$$

Even if these 2 potentials are different, they have exactly the same spectrum ω

For example, for $l = 2$

$\frac{\omega}{M} = 0.747343 - 0.177925\, i$ Fundamental mode (or first harmonic)

(they are the same frequencies for both potentials)

$\frac{\omega}{M} = 0.693422 - 0.547830\, i$ Second harmonic

$\frac{\omega}{M} = 0.602107 - 0.956554\, i$ ⋮

⋮

It's called isospectrality. It's because both potentials can be derived from a same potential

$$V_{RW} = W^2 - \frac{dW}{dr_*} + \alpha$$

$$V_Z = W^2 + \frac{dW}{dr_*} + \alpha$$

with $W = \frac{3M(r-2M)}{r^2(3M+\Lambda r)} + \frac{\Lambda(\Lambda+1)}{3M}$ and $\alpha = -\frac{\Lambda^2(\Lambda+1)^2}{9M^2}$

Chandrasekhar (1975) ; Chandrasekhar - Detweiler (1975)

(These relations are special cases of the Darboux relation - see e.g. arXiv: 1702.06459)

If 2 operators L_1 and L_2 are such that

$$D L_1 = L_2 D \quad \text{with } D \text{ an operator}$$

$\Rightarrow$ L_1 and L_2 have the same spectrum

If λ is an eigenvalue of L_1 which means $L_1 \Psi = \lambda \Psi$

$\Leftrightarrow \quad D L_1 \Psi = \lambda D \Psi$

$\Leftrightarrow \quad L_2 D \Psi = \lambda D \Psi \Leftrightarrow D\Psi$ is an eigenvector with the same eigenvalue

In our case $\quad L_1 = \dfrac{d^2}{dr_*^2} - V^z \qquad L_2 = \dfrac{d^2}{dr_*^2} - V^{RW}$

and $\quad D = \dfrac{d}{dr_*} - W$

$$D L_1 = L_2 D$$

$$\Updownarrow$$

$$\cancel{\frac{d^3}{dr_*^3}} - \frac{dV^z}{dr_*} - V^z \frac{d}{dr_*} - W \cancel{\frac{d^2}{dr_*^2}} + V^z W = \cancel{\frac{d^3}{dr_*^3}} - \frac{d^2 W}{dr_*^2} - \cancel{W\frac{d^2}{dr_*^2}} - 2\frac{dW}{dr_*}\frac{d}{dr_*} - V^{RW}\frac{d}{dr_*} + V^{RW} W$$

$$\Leftrightarrow -\left(V^z - V^{RW} - 2\frac{dW}{dr_*}\right)\frac{d}{dr_*} + \frac{d^2W}{dr_*^2} - \frac{dV^z}{dr_*} + \left(V^z - V^{RW}\right) W = 0$$

$$\Leftrightarrow \quad V^z - V^{RW} = 2\frac{dW}{dr_*} \quad \text{and} \quad \frac{d^2W}{dr_*^2} - \frac{dV^z}{dr_*} + \left(V^z - V^{RW}\right)W = 0$$

$$\Updownarrow$$

$$\frac{dW}{dr_*} - V^z + W^2 = c^{st} \Leftrightarrow \frac{d^2W}{dr_*^2} - \frac{dV^z}{dr_*} + 2\frac{dW}{dr_*}W = 0$$

$W = \dfrac{(r-2M)3M}{r^2(\Lambda r + 3M)} + \beta$, replacing in we find a constant if $\beta = \dfrac{\Lambda(\Lambda+1)}{3M}$

$$\begin{cases} V^z = W^2 + \dfrac{dW}{dr_*} - \beta^2 \\ V^{RW} = W^2 - \dfrac{dW}{dr_*} - \beta^2 \end{cases} \qquad W = \frac{3M(r-2M)}{r^2(3M+\Lambda r)} + \frac{\Lambda(\Lambda+1)}{3M}$$

For the Schwarzschild spacetime , Reissner-Nordström , Kerr we have isospectrality. If we have a scalar field, extra-dimensions we break the isospectrality.

VIII - Lagrangian approach

There is an other method, more systematic, which we can easily extend to any beyond GR theory. Expanding the action:

$$S = \int \cdots \underbrace{(\cdots\cdots)}_{0^{\text{th}}\ \text{order}} + \int \cdots \underbrace{(\cdots h_0 + \cdots h_1)}_{1^{\text{st}}\ \text{order}} + \int \cdots \underbrace{(h_0^2 + \cdots\cdots)}_{2^{\text{nd}}\ \text{order}}$$

The 0^{th} order is just a constant

The 1^{st} order gives the background metric (it is zero on-shell)

The 2^{nd} order gives first order eq. for $h_{(i)}$, namely linear perturbations.

Therefore, we need to focus namely on the act° at second order: $S^{(2)}$

We can easily integrate over the angles (θ, ϕ) using the relations

$$\int_0^\pi d\theta \sin\theta\, P_\ell^2(\cos\theta) = \frac{2}{2\ell+1} \qquad \int_0^\pi d\theta \cos\theta\, P_\ell(\cos\theta)\frac{d}{d\theta}P_\ell(\cos\theta) = -\ell \cdot \frac{2}{2\ell+1}$$

$$\int_0^\pi d\theta \sin\theta \left(\frac{d}{d\theta}P_\ell\right)^2 = \ell(\ell+1)\frac{2}{2\ell+1} \qquad \int_0^\pi d\theta \frac{1}{\sin\theta}\left(\frac{dP_\ell}{d\theta}\right)^2 = \frac{2}{2\ell+1}\cdot\frac{\ell(\ell+1)(2\ell+1)}{2}$$

and
$$\int_0^\pi d\theta \frac{\cos^2\theta}{\sin\theta}\left(\frac{dP_\ell}{d\theta}\right)^2 = \frac{\ell(\ell+1)(2\ell-1)}{2\ell+1}$$

Therefore, we end with $S^{(2)} = \alpha \int dt\,dr\,\mathcal{L}$

After some integration by parts, we can write

$$\mathcal{L} = a_1 h_0^2 + a_2 h_1^2 + a_3\left(\dot{h}_1^2 - 2\dot{h}_1 h_0' + h_0'^2 + \frac{4}{r}\dot{h}_1 h_0\right)$$

with $a_1 = \dfrac{\ell(\ell+1)}{4r^2}\left(1 + \dfrac{(\ell-1)(\ell+2)}{1-r_s/r}\right)$ $\qquad a_2 = -\dfrac{1-\frac{r_s}{r}}{4r^2}(\ell-1)\ell(\ell+1)(\ell+2)$

$$a_3 = \frac{\ell(\ell+1)}{4}$$

We can see that $\dot{h}_0$ doesn't appear $\Rightarrow$ it's an auxiliary field

But because of the term $h_0'^2$, we can't obtain an algebraic eq. for h_0 and therefore eliminate that variable

We will use the following trick

$$\mathcal{L} = a_1 h_0^2 + a_2 h_1^2 + a_3 \left(\dot{h}_1^2 - 2 \dot{h}_1 h_0' + h_0'^2 + \frac{4}{r} \dot{h}_1 h_0 \right)$$

$$= a_1 h_0^2 + a_2 h_1^2 + a_3 \left[\left(\dot{h}_1 - h_0' + \frac{2}{r} h_0 \right)^2 - \frac{4}{r^2} h_0^2 + \frac{4}{r} h_0 h_0' \right]$$

$$= a_1 h_0^2 - \frac{4}{r^2} a_3 h_0^2 + \underbrace{\frac{4}{r} a_3 h_0 h_0'}_{\frac{2}{r} a_3 (h_0^2)'} + a_2 h_1^2 + a_3 \left(\dot{h}_1 - h_0' + \frac{2}{r} h_0 \right)^2$$

$$\hookrightarrow -\left(\frac{2a_3}{r}\right)' h_0^2 \quad \text{(integrate by parts)}$$

$$= \left[a_1 - \frac{2 (r a_3)'}{r^2} \right] h_0^2 + a_2 h_1^2 + a_3 \left(\dot{h}_1 - h_0' + \frac{2}{r} h_0 \right)^2$$

We define

$$L = \left[a_1 - \frac{2 (r a_3)'}{r^2} \right] h_0^2 + a_2 h_1^2 + a_3 \left[-q^2 + 2q \left(\dot{h}_1 - h_0' + \frac{2}{r} h_0 \right) \right]$$

$(\mathcal{L}, L)$ are equivalents, in the sense that they give the same eq. of motion. Indeed the variat. of L with respect to q gives

$$q = \dot{h}_1 - h_0' + \frac{2}{r} h_0$$

and using this relation in L we obtain $\mathcal{L}$

Using integration by parts, we can eliminate the derivatives over h_0 and h_1

$\Rightarrow$ They are auxiliary fields

Indeed the variation wrt to h_1 gives

$$2 a_2 h_1 - 2 a_3 \dot{q} = 0 \quad \Rightarrow \quad h_1 = \frac{a_3}{a_2} \dot{q}$$

And the variation with respect to h_0 gives

$$2\left[a_1 - \frac{2(r a_3)'}{r^2}\right] h_0 + 2\,(q a_3)' + \frac{2}{r}\, a_3\, q = 0$$

$$\Rightarrow \quad h_0 = \frac{r^2 (q a_3)' + r a_3 q}{2\,(r a_3)' - a_1}$$

Using these expressions in the lagrangian, we obtain

$$L = \alpha(r)\,\dot{q}^2 + \beta(r)\, q'^2 + \gamma(r)\, q^2$$ (after some int. by parts)

Introducing $Q = q \cdot r$ and $dr_* = \dfrac{dr}{1 - B/r}$

$$\Rightarrow \quad S^{(2)} = \frac{l\,(l+1)}{4(l-1)(l+2)} \int dt\, dr_* \left[\dot{Q}^2 - \left(\frac{dQ}{dr_*}\right)^2 - V(r)\, Q^2\right]$$

IX - Reissner-Nordström Stability

We have now, 2 fields $(g_{\mu\nu}, A_\mu)$ which we need to perturb $(\bar{g}_{\mu\nu}, \bar{A}_\mu)$ represent the background field and $(\delta g_{\mu\nu}, \delta A_\mu)$ its perturbations.

We have seen $\bar{g}_{\mu\nu}$ and $\delta g_{\mu\nu}$ ....

For the field A_μ, we have $\bar{A}_\mu = (A_0(r), 0, 0, 0)$ for a spherically symmetric problem (electric charge), in our case $A_0(r) = \frac{Q}{r}$

and $\delta A_\mu = (\underbrace{\delta A_0, \delta A_1}_{\text{scalar pert.}}, \delta A_a)$ with $a = (\theta, \phi)$

scalar/vector pert.

For vector pert., we have $\delta A_\mu^{(\text{vect.})} = (0, 0, \delta A_a)$

with $\delta A_a = \sum_{l,m} \Pi_{lm}(t,r) \, E_{ab} \nabla^b Y_l^m$

For scalar perturbations, we have $\delta A_\mu^{(\text{scal})} = (\delta A_0, \delta A_1, \delta A_a)$

with
$$\delta A_0 = \sum_{l,m} N_{lm}(t,r) \, Y_l^m$$

$$\delta A_1 = \sum_{l,m} P_{lm}(t,r) \, Y_l^m$$

$$\delta A_a = \sum_{l,m} R_{lm}(t,r) \, \nabla_a Y_l^m$$

We have already used the gauge to eliminate some metric components, so we can't reduce more the problem

Performing calculations at the level of action
$$S = \int d^4x \sqrt{-g} \, (R - F_{\mu\nu} F^{\mu\nu})$$

or equations
$$R_{\mu\nu} = 2 F_{\mu\alpha} F_{\nu}{}^{\alpha} - \frac{1}{2} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}$$
$$\nabla_\alpha F^{\alpha\beta} = 0 \quad ; \quad \nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0$$

For vector pert., we have

$$-\frac{\partial^2}{\partial t^2} \vec{\chi} + \frac{\partial^2}{\partial r_*^2} \vec{\chi} - V \vec{\chi} = 0 \quad \text{with } \vec{\chi} = \begin{pmatrix} Q \\ H \end{pmatrix}$$

Q comes from the metric while H from Maxwell field.

with $V = \begin{pmatrix} V_{11} & V_{12} \\ V_{12} & V_{22} \end{pmatrix}$
$$V_{11} = (\lambda - 2) \frac{A}{r^2} - \frac{\partial S}{\partial r_*} + S^2$$
$$V_{22} = \lambda \frac{A}{r^2} + 4 Q^2 \frac{A}{r^4}$$
$$V_{12} = 2 |Q| \sqrt{\lambda - 2} \, \frac{A}{r^3}$$

with $A = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}$, $\lambda = l(l+1)$, $S = \frac{A}{r}$

Passing to Fourier space, we get
$$-\frac{d^2}{dr_*^2} \vec{\chi} + V \vec{\chi} = \omega^2 \vec{\chi}$$
$$\Leftrightarrow \quad L \vec{\chi} = \omega^2 \vec{\chi}$$
$$\text{with } L = -\mathbb{1} \frac{d^2}{dr_*^2} + V$$

Let's define the scalar product
$$(\chi_1, \chi_2) = \int dr_* \left(\bar{Q}_1 Q_2 + \bar{H}_1 H_2 \right) \quad \text{with } \vec{\chi}_i = \begin{pmatrix} Q_i \\ H_i \end{pmatrix}$$

We would like to prove that L is positive, i.e.
$$\forall \chi \quad (\chi, L\chi) \geq 0$$

$$(\mathcal{X}, L\mathcal{X}) = \int dr_* \left[\bar{Q}\left(-\partial^2_{r_*} Q + V_{11} Q + V_{12} H\right) + \bar{H}\left(-\partial^2_{r_*} H + V_{12} Q + V_{22} H\right)\right]$$

$$= \int dr_* \left[\left|\frac{dQ}{dr_*}\right|^2 + \left|\frac{dH}{dr_*}\right|^2 + V_{12}\left(\bar{Q} H + Q\bar{H}\right) + V_{11} |Q|^2 + V_{22} |H|^2 \right]$$

$$\cancel{- \bar{Q}\, \partial_{r_*} Q \Big|_{\mathbb{R}}} - \cancel{\bar{H}\, \partial_{r_*} H\Big|_{\mathbb{R}}}$$

$$= \int dr_* \left[\left| \frac{d}{dr_*} Q + S\, Q\right|^2 + (\lambda - 2)\frac{A}{r^2}|Q|^2 + \left|\frac{d}{dr_*} H\right|^2 + \left(\lambda \frac{A}{r^2} + 4 Q^2 \frac{A}{r^4}\right)|H|^2 \right.$$
$$\left. + 2|Q|\sqrt{\lambda - 2}\, \frac{A}{r^3}\left(\bar{Q} H + Q \bar{H}\right)\right]$$

$$= \int dr_* \left[\left| \frac{d}{dr_*} Q + S\, Q\right|^2 + \left|\frac{d}{dr_*} H\right|^2 + \left|\sqrt{(\lambda-2)\frac{A}{r^2}}\, Q + 2|Q|\frac{\sqrt{A}}{r^2} H\right|^2 + \lambda \frac{A}{r^2}|H|^2\right]$$

$\geq 0 \quad \forall \vec{\mathcal{X}} \Rightarrow$ vectorial stability

/!\ If a potential $V(r)$ is negative but $V + \frac{dS}{dr_*} - S^2 \geq 0$ with S a function of r $\Rightarrow$ we have stability - It's called the S deformation

$$(Q, LQ) = \int dr_* \left(\bar{Q}\left(-\frac{d^2}{dr_*^2} + V\right) Q = \int dr_* \left|\frac{dQ}{dr_*}\right|^2 + V|Q|^2\right.$$

$$= \int dr_* \left[\left|\frac{dQ}{dr_*} + SQ\right|^2 + \left(V + \frac{dS}{dr_*} - S^2\right)|Q|^2\right] + \text{boundary terms}$$

For the scalar pert., we have (Moncrief 1974)

$$\begin{cases} -\dfrac{\partial^2 Q}{\partial t^2} + \dfrac{\partial^2 Q}{\partial r_*^2} - V_Q\, Q = 0 \\ -\dfrac{\partial^2 H}{\partial t^2} + \dfrac{\partial^2 H}{\partial r_*^2} - V_H\, H = 0 \end{cases}$$

(After diagonalizing the system)

with $V_Q = T + \sigma S$

$V_H = T - \sigma S$

and $\sigma = \sqrt{9M^2 + 4Q^2(\ell-1)(\ell+2)}$

$$S = \frac{\ell(\ell+1)}{r^3 P} + \frac{2A(r)}{r^3 P^2}\left[(\ell-1)(\ell+2) + 4\frac{Q^2}{r^2}\right] - \frac{1}{r^3 P}\left[\frac{2M}{r} - \frac{2Q^2}{r^2}\right]$$

$$P = (\ell-1)(\ell+2) + \frac{6M}{r} - \frac{4Q^2}{r^2}$$

$$A(r) = 1 - \frac{2M}{r} + Q^2/r^2$$

$$T = \frac{A(r)}{r^2 P^2}\left(\frac{8Q^2}{r^2} - \frac{6M}{r}\right)^2 + A(r)\cdot\frac{8Q^2}{r^4 P} + \frac{\ell(\ell-1)(\ell+1)(\ell+2)}{r^2 P} + \frac{3M}{r^3} + \frac{4Q^2}{r^4 P}\left(2 - \frac{6M}{r} + \frac{4Q^2}{r^2}\right)$$

We can check that $V_Q \geq 0$ and $V_H \geq 0$

$\Rightarrow$ Scalar pert. are stable

$$\boxed{\text{The Reissner–Nordstrom BH is linearly stable under pert. of the metric and Maxwell field.}}$$