Quantization is Deformation

Mathematisches Forschungsinstitut Oberwolfach
Workshop 0603, Monday 16 January 2006
Deformations and contractions
in mathematics and physics

Daniel Sternheimer
Institut de Mathématiques de Bourgogne, CNRS U.M.R. 5584
Université de Bourgogne, Dijon, France
& Dept. of Mathematics, Keio University, Yokohama, Japan,
email: Daniel.Sternheimer@u-bourgogne.fr
1. Epistemological Introduction

Mathematics and Physics are two communities separated by a common language (the mathematical language).

Three questions to a scientist: why, what, how

Work: 1% inspiration, 99% perspiration

Mathematicians, physicists, and mathematical physicists: different approaches to the 3 questions, even when dealing with physics.

Mathematicians translate into their own language, to something entirely different.

Physicists make very liberal use of the mathematical language.

Mathematical physicists aim at precise formulations and solutions of physical problems.

Example: Quantization.

Physics: Planck, Einstein, Bohr, de Broglie, Schrödinger, Heisenberg, Dirac...

Maths: Weyl, von Neumann, Jauch–Piron, Berezin, geometric quantization (Kostant...)

Mathematical physics: Deformation quantization.
2. The context

The apple: Adam & Eve, Sir Isaac Newton (Eureka when apple fell)
In equations: \( F = ma \), Lagrange, Hamilton. More precisely:

- Phase space \((n \text{ particles}): \mathbb{R}^{2n} \) with variables \((q_i, p_i), i = 1, 2, 3\)
- Canonical Poisson bracket: \( \{ F, G \} = \sum_i \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i} \frac{\partial F}{\partial p_i} \)
- Hamiltonian: real-valued function \( H(q, p) \) on the phase space
- Dynamics: Hamilton’s equations of motion \( \dot{p} = \{ H, p \} \), ditto \( q \).

Symmetries and Geometry (1 particle: Galileo group \( SO(3) \cdot \mathbb{R}^3 \cdot \mathbb{R}^4 \)).

Deformations (in physics) of geometry: Earth non flat (curvature \( > 0 \)).
Einstein \((c^{-1} > 0, \text{deform Galileo symmetry to Poincaré } SO(3, 1) \cdot \mathbb{R}^4 \)).

Flato’s deformation philosophy: Physics evolves in stages.
When need appears, “extrapolate” (smartly), i.e deform mathematical structures associated with a formulation of physical phenomena, in some appropriate category.
3. Quantization and the three questions.

Why Quantization? In physics, experimental need. In mathematics, because physicists need it (and it gives nice maths). In mathematical physics, deformation philosophy.

What is quantization? In (theoretical) physics, expression of “quantum” phenomena appearing (usually) in the microworld. In mathematics, passage from commutative to noncommutative. In (our) mathematical physics, deformation quantization.

How do we quantize? In physics, correspondence principle. For many mathematicians (Weyl, Berezin, Kostant, . . .), functor (between categories of algebras of “functions” on phase spaces and of operators in Hilbert spaces; take physicists’ formulation for God’s axiom; but stones. . .). In mathematical physics, deformation (of composition laws)
4. Some mathematics: Deformation theory

Riemann surface theory (19th century), then from 1958, Kodaira–Spencer: deformations of complex analytic structures (cohomology important tool). “To study geometric structures look at algebras of functions over them.”

A (Gerstenhaber 1964) deformation of an algebra $A$ over a field $\mathbb{K}$ is an algebra $\tilde{A}$ (flat) over $\mathbb{K}[\![t]\!]$ such that $\tilde{A}/t\tilde{A} \approx A$. Two deformations $\tilde{A}$ and $\tilde{A}'$ are said equivalent if they are isomorphic over $\mathbb{K}[\![t]\!]$ and $\tilde{A}$ is trivial if it is isomorphic to the original algebra $A$ considered by base field extension as a $\mathbb{K}[\![t]\!]$-algebra.

For associative algebras with ordinary product $\cdot$ (resp. Lie with bracket $\{\cdot, \cdot\}$) this means that there exists a new product $\ast$ (resp. bracket $[\cdot, \cdot]$) such that the new (deformed) algebra is again associative (resp. Lie). For $u_1, u_2 \in A[\![t]\!]$ we have $u_1 \ast u_2 = u_1 \cdot u_2 + \sum_{r=1}^{\infty} t^r C_r(u_1, u_2)$ and $[u_1, u_2] = \{u_1, u_2\} + \sum_{r=1}^{\infty} t^r B_r(u_1, u_2)$ with bilinear maps $C_r$ and $B_r$ such that for $u_1, u_2, u_3 \in A$, $(u_1 \ast u_2) \ast u_3 = u_1 \ast (u_2 \ast u_3)$ and $\mathcal{I}[[u_1, u_2], u_3] = 0$ (where $\mathcal{I}$ denotes summation over cyclic permutations). Similar expressions for bialgebras (with coproduct $\Delta$) etc.
More general deformations; contractions and deformations

Nambu mechanics quantization: \[ \sum a_n t^n \ast \sum b_p t^p = a_0 \ast b_0 \] ("\( t^2 = 0 \)"")

Nadaud-Pinczon generalized deformations (Nadaud, RMP 10 (1998), LMP 58 (2001) and Thèse; or Pinczon LMP 41 (1997)): \( t \) acts on \( A \) to left and right (or left only), \( t \cdot a = \sigma(a)t \) and \( a \cdot t = \tau(a)t \) with \( \sigma, \tau \in \text{End}(A, \ast) \),
\[ \sum a_n t^n \ast \sum b_p t^p = \sum (\tau^p(a_n) \ast \sigma^n(b_p))t^{n+p} \]. Here (Pinczon) the “rigid” enveloping algebra of Weyl (CCR) can be deformed to that of \( \mathfrak{osp}(1, 2) \).

Idea in both: “deformed” algebra \( A_t \); when “contracts,” \( t = 0 \), gets \( A \).

**Arnal-Cortet** Geometrical theory of contractions of groups and representations (JMP 1979): a set of constraints on \( T^\ast(S^{n+1}) \) giving \( T^\ast(S^n) \) make the Dirac bracket appear as a deformation of the Poisson bracket and give a contraction of \( \mathfrak{so}(n+1) \) onto \( \mathfrak{so}(n) \cdot \mathbb{R}^n \).

Deformations and contractions are not always inverse one to another (cf. e.g. “\( \frac{1}{t} \) problem”) even if the “classical limit” \( t = 0 \) is intuitive.

\( \text{CCR } \left[ \frac{P}{i\hbar}, \frac{Q}{i\hbar} \right] = \frac{I}{i\hbar} \) i.e. (abstract) Lie algebra \( \mathfrak{g} \ni X \rightarrow \text{function } \frac{X}{i\hbar} \).
5. Classical Mechanics and around

Non trivial phase spaces → Symplectic manifolds → Poisson manifolds

**Definition: Symplectic manifold**

Differentiable manifold $M$ with nondegenerate closed 2-form $\omega$ on $M$. $M$ is necessarily even dimensional: $\dim M = 2n$;

Locally: $\omega = \omega_{ij} dx^i \wedge dx^j$; $\omega_{ij} = -\omega_{ji}$; $\det \omega_{ij} \neq 0$; $\text{Alt}(\partial_i \omega_{jk}) = 0$.

and one can find coordinates $(q_i, p_i)$ so that $\omega$ is constant:

$\omega = \sum_{i=1}^{n} dq^i \wedge dp^i$.

Define $\pi^{ij} = \omega_{ij}^{-1}$, then $\{F, G\} = \pi^{ij} \partial_i F \partial_j G$ is a Poisson bracket.

**Examples:**

1) $\mathbb{R}^{2n}$ with $\omega = \sum_{i=1}^{n} dq^i \wedge dp^i$;

2) Cotangent bundle $T^*N$, $\omega = d\alpha$, where $\alpha$ is the canonical one-form on $T^*N$ (Locally, $\alpha = -p_i dq^i$)
**Definition: Poisson manifold**

Differentiable manifold $M$, and skewsymmetric contravariant 2-tensor (not necessarily nondegenerate) $\pi = \sum_{i,j} \pi^{ij} \partial_i \wedge \partial_j$ (locally) such that

$$\{F, G\} = i(\pi)(dF \wedge dG) = \sum_{i,j} \pi^{ij} \partial_i F \wedge \partial_j G$$

is a Poisson bracket, i.e. the bracket $\{\cdot, \cdot\}: C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ is a skewsymmetric ($\{F, G\} = -\{G, F\}$) bilinear map satisfying:

- **Jacobi identity:** $\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0$
- **Leibniz rule:** $\{FG, H\} = \{F, H\}G + F\{G, H\}$

**Examples:**
1) Symplectic manifolds ($d\omega = 0 = [\pi, \pi] \equiv $ Jacobi identity)
2) Lie algebra with structure constants $C_k^{ij}$ and $\pi^{ij} = \sum_k x^k C_k^{ij}$.
3) $\pi = X \wedge Y$, where $(X, Y)$ are two commuting vector fields on $M$.

**Facts:** Every Poisson manifold is “foliated” by symplectic manifolds. If $\pi$ is nondegenerate, then $\omega_{ij} = (\pi^{ij})^{-1}$ is a symplectic form.

A **Classical System** is a Poisson manifold $(M, \pi)$ with a distinguished smooth function, the Hamiltonian $H : M \to \mathbb{R}$. 
6. Some physical background

Louis de Broglie [1924]: “wave mechanics” (waves and particles are two manifestations of the same physical reality).

Traditional quantization (Schrödinger, Heisenberg) of classical system \((\mathbb{R}^{2n}, \{\cdot, \cdot\}, H)\): Hilbert space \(\mathcal{H} = L^2(\mathbb{R}^n) \ni \psi\) where acts “quantized” Hamiltonian \(H\), energy levels \(H\psi = \lambda \psi\), and von Neumann representation of CCR. Define \(\hat{q}_\alpha(f)(q) = q_\alpha f(q)\) and \(\hat{p}_\beta(f)(q) = -i\hbar \frac{\partial f(q)}{\partial q_\beta}\) for \(f\) differentiable in \(\mathcal{H}\). Then (CCR) \([\hat{p}_\alpha, \hat{q}_\beta] = i\hbar \delta_{\alpha\beta} I\) \((\alpha, \beta = 1, \ldots, n)\).

The couple \((\hat{q}, \hat{p})\) quantizes the coordinates \((q, p)\). A polynomial classical Hamiltonian \(H\) is quantized once chosen an operator ordering, e.g. (Weyl) complete symmetrization of \(\hat{p}\) and \(\hat{q}\). In general the quantization on \(\mathbb{R}^{2n}\) of a function \(H(q, p)\) with inverse Fourier transform \(\tilde{H}(\xi, \eta)\) can be given by (Hermann Weyl [1927] with weight \(\sigma = 1\)): \[ H \mapsto \mathbf{H} = \Omega_\sigma(H) = \int_{\mathbb{R}^{2n}} \tilde{H}(\xi, \eta) \exp(i(\hat{p}.\xi + \hat{q}.\eta)/\hbar) \sigma(\xi, \eta) d^n \xi d^n \eta. \]
E. Wigner [1932] inverse $H = (2\pi \hbar)^{-n} \text{Tr}[\Omega_1(H) \exp((\xi \cdot \hat{p} + \eta \cdot \hat{q})/i\hbar)]$. 

$\Omega_1$ defines an isomorphism of Hilbert spaces between $L^2(\mathbb{R}^{2n})$ and Hilbert–Schmidt operators on $L^2(\mathbb{R}^n)$. Can extend e.g. to distributions.

The correspondence $H \mapsto \Omega(H)$ is not an algebra homomorphism, neither for ordinary product of functions nor for the Poisson bracket $P$ (the latter, “Van Hove theorem”). Take two functions $u_1$ and $u_2$, then (Groenewold [1946]): $\Omega_1^{-1}(\Omega_1(u_1)\Omega_1(u_2)) = u_1 u_2 + \frac{i\hbar}{2} \{u_1, u_2\} + O(\hbar^2)$. More precisely $\Omega_1$ maps into product and bracket of operators (resp.):

$u_1 *_M u_2 = \exp(tP)(u_1, u_2) = u_1 u_2 + \sum_{r=1}^{\infty} \frac{t^r}{r!} P^r(u_1, u_2)$ (with $2t = i\hbar$),

$M(u_1, u_2) = t^{-1} \sinh(tP)(u_1, u_2) = P(u_1, u_2) + \sum_{r=1}^{\infty} \frac{t^{2r}}{(2r+1)!} P^{2r+1}(u_1, u_2)$

We recognize previous formulas for deformations of algebras (+ below).

**Deformation quantization:** forget the correspondence principle $\Omega$ and work in an *autonomous* manner with “functions” on phase spaces.
7. Some other mathematicians’ approaches

Geometric quantization (Kostant, Souriau). [1970’s. Mimic correspondence principle for general phase spaces $M$. Look for generalized Weyl map from functions on $M$:] Start with “prequantization” on $L^2(M)$ and tries to halve the number of degrees of freedom using (complex, in general) polarizations to get Lagrangian submanifold $\mathcal{L}$ of dimension half of that of $M$ and quantized observables as operators in $L^2(\mathcal{L})$. Fine for representation theory ($M$ coadjoint orbit, e.g. solvable group) but few observables can be quantized (linear or maybe quadratic, preferred observables in def.q.).

Berezin quantization. (ca.1975). Quantization is an algorithm by which a quantum system corresponds to a classical dynamical one, i.e. (roughly) is a functor between a category of algebras of classical observables (on phase space) and a category of algebras of operators (in Hilbert space). Examples: Euclidean and Lobatchevsky planes, cylinder, torus and sphere, Kähler manifolds and duals of Lie algebras. [Only $(M, \pi)$, no $H$ here.]
8. Deformation quantization: the framework

Poisson manifold \((M, \pi)\), deformations of product of fonctions.

**Definition: Star-product**

[Flato, Lichnerowicz, Sternheimer; and Vey; mid 70’s]


- \(\mathcal{A}_t = C^\infty(M)[[t]]\), formal series in \(t\) with coefficients in \(C^\infty(M) = A\).

Elements: \(f_0 + tf_1 + t^2f_2 + \cdots\) (\(t\) formal parameter, not fixed scalar.)

- \(\star_t: \mathcal{A}_t \times \mathcal{A}_t \to \mathcal{A}_t\); \(f \star_t g = fg + \sum_{r \geq 1} t^r C_r(f, g)\)
  - \(C_r\) are bidifferential operators null on constants: \((1 \star_t f = f \star_t 1 = f)\).
  - \(\star_t\) is associative and \(C_1(f, g) - C_1(g, f) = 2\{f, g\}\), so that
  \([f, g]_t \equiv \frac{1}{2t} (f \star_t g - g \star_t f) = \{f, g\} + O(t)\) is Lie algebra deformation.

**Moyal product** on \(\mathbb{R}^{2n}\) with the canonical Poisson bracket:

\[ P_1(F, G) = \{F, G\} = \sum_i \frac{\partial F}{\partial q_i} \frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i} \frac{\partial F}{\partial p_i} \]
Define
\[
P_k(F, G)(q, p) = \left( \frac{\partial}{\partial q_1} \frac{\partial}{\partial p_2} - \frac{\partial}{\partial p_1} \frac{\partial}{\partial q_2} \right)^k F(q_1, p_1)G(q_2, p_2) \bigg|_{q_1=q_2=q, p_1=p_2=p}.
\]

\[
F \star_M G = \exp \left( \frac{i\hbar}{2} P_1 \right) (f, g) \equiv FG + \sum_{k \geq 1} \frac{1}{k!} \left( \frac{i\hbar}{2} \right)^k P_k(F, G).
\]

Equation of motion (time \( \tau \)): \[
\frac{dF}{d\tau} = [H, F]_M \equiv \frac{1}{i\hbar} (H \star_M F - F \star_M H).
\]

Link with Weyl’s rule of quantization: \( \Omega_1(F \star_M G) = \Omega_1(F)\Omega_1(G) \)

**Equivalence** of two star-products \( \star_1 \) and \( \star_2 \).
- Formal series of differential operators \( T(f) = f + \sum_{r \geq 1} t^r T_r(f) \).
- \( T(f \star_1 g) = T(f) \star_2 T(g) \)

**Remarks:**
- The choice of a star-product fixes a quantization rule.
- Operator orderings can be implemented by good choices of \( T \) (or \( \varpi \)).
- On \( \mathbb{R}^{2n} \), all star-products are equivalent to Moyal product (cf. von Neumann uniqueness theorem on projective UIR of CCR; see below).
9. Existence and classification of star products

Let \((M, \pi)\) be a Poisson manifold. \(f \tilde{\ast} g = fg + t\{f, g\}\) does not define an associative product. But \((f \tilde{\ast} g) \tilde{\ast} h - f \tilde{\ast} (g \tilde{\ast} h) = O(t^2)\).

Is it always possible to modify \(\tilde{\ast}\) in order to get an associative product?

Existence, symplectic case:

– Fedosov [1985,1994]: Construct a flat abelian connection on the Weyl bundle over the symplectic manifold.

For a general Poisson manifold \(M\) with Poisson bracket \(P\):

Solved by Kontsevich [1997, LMP 2003]. “Explicit” local formula:

\[
(f, g) \mapsto f \ast g = \sum_{n \geq 0} t^n \sum_{\Gamma \in G_{n,2}} w(\Gamma) B_\Gamma(f, g),
\]

defines a differential star-product on \((\mathbb{R}^d, P)\); globalizable to \(M\). Here \(G_{n,2}\) is a set of graphs \(\Gamma\), \(w(\Gamma)\) some weight defined by \(\Gamma\) and \(B_\Gamma(f, g)\) some bidifferential operator.
Particular case of Formality Theorem: Hochschild complex $C^\bullet(A,A)$ and its cohomology $H^\bullet(A,A)$ are quasi-isomorphic (same cohomology).

Equivalence. For symplectic manifolds, equivalence classes of star-products are parametrized by the 2\textsuperscript{nd} de Rham cohomology space $H^2_{dR}(M)$.

Precisely: $\{\star_t\}/\sim = H^2_{dR}(M)[[t]]$ (Nest-Tsygan [1995] and others)

In particular, $H^2_{dR}(\mathbb{R}^{2n})$ is trivial, all deformations are equivalent.

Kontsevich: $\{\text{Equivalence classes of star-products}\} \equiv \{\text{equivalence classes of formal Poisson tensors } \pi_t = \pi + t\pi_1 + \cdots\}$.

Newer metamorphoses: operadic approach, algebraic varieties, etc.

Tamarkin [1998]: for any associative algebra $A$, $C^\bullet(A,A)$ and $H^\bullet(A,A)$ are algebras over the same operad (up to homotopy) $\rightarrow$ KoSo2000.

Kontsevich [2001]: noncommutative deformations over the ring of formal power series of projective and affine algebraic Poisson manifolds satisfying natural geometric conditions (e.g. Poisson-Lie groups).
10. This is Quantization

A star-product provides an autonomous quantization of a manifold $M$. BFFLS '78: Quantization is a deformation of the composition law of observables of a classical system: $(A, \cdot) \rightarrow (A[[t]], \star_t)$, $A = C^\infty(M)$.

Star-product $\star (t = \frac{i}{2}\hbar)$ on Poisson manifold $M$ and Hamiltonian $H$; introduce the star-exponential: $\text{Exp}_\star(\frac{\tau H}{i\hbar}) = \sum_{r \geq 0} \frac{1}{r!} \left(\frac{\tau}{i\hbar}\right)^r H^{*r}$.

Corresponds to the unitary evolution operator, is a singular object i.e. does not belong to the quantized algebra $(A[[t]], \star)$ but to $(A[[t, t^{-1}]], \star)$. Spectrum and states are given by a spectral (Fourier-Stieltjes in the time $\tau$) decomposition of the star-exponential.

Paradigm: Harmonic oscillator $H = \frac{1}{2}(p^2 + q^2)$, Moyal product on $\mathbb{R}^{2\ell}$.

$\text{Exp}_\star(\frac{\tau H}{i\hbar}) = (\cos(\frac{\tau}{2}))^{-1} \exp \left(\frac{2H}{i\hbar} \tan(\frac{\tau}{2})\right) = \sum_{n=0}^{\infty} \exp \left(-i(n + \frac{\ell}{2})\tau\right) \pi_n^{\ell}$.

Here ($\ell = 1$ but similar formulas for $\ell \geq 1$, $L_n$ is Laguerre polynomial of degree $n$) $\pi_n^1(q, p) = 2\exp \left(\frac{-2H(q, p)}{\hbar}\right)(-1)^n L_n \left(\frac{4H(q, p)}{\hbar}\right)$. 
Remarks. The Gaussian function $\pi_0(q, p) = 2 \exp \left( -\frac{2H(q, p)}{\hbar} \right)$ describes the vacuum state. As expected the energy levels of $H$ are $E_n = \hbar(n + \frac{\ell}{2})$: $H \star \pi_n = E_n \pi_n$; $\pi_m \star \pi_n = \delta_{mn} \pi_n$; $\sum_n \pi_n = 1$. With normal ordering, $E_n = n\hbar$: $E_0 \to \infty$ for $\ell \to \infty$ in Moyal ordering but $E_0 \equiv 0$ in normal ordering, preferred in Field Theory.

- Other standard examples of QM can be quantized in an autonomous manner by choosing adapted star-products: angular momentum with spectrum $n(n + (\ell - 2))\hbar^2$ for the Casimir element of $\mathfrak{so}(\ell)$; hydrogen atom with $H = \frac{1}{2}p^2 - |q|^{-1}$ on $M = T^*S^3$, $E = \frac{1}{2}(n + 1)^{-2}\hbar^{-2}$ for the discrete spectrum, and $E \in \mathbb{R}^+$ for the continuous spectrum; etc.

- Feynman Path Integral (PI) is, for Moyal, Fourier transform in $p$ of star-exponential; equal to it (up to multiplicative factor) for normal ordering) [Dito’90]. Cattaneo-Felder [2k]: Kontsevich star product as PI.

- Cohomological renormalization. ("Subtract infinite cocycle.")
General remarks.

• After that it is a matter of practical feasibility of calculations, when there are Weyl and Wigner maps to intertwine between both formalisms, to choose to work with operators in Hilbert spaces or with functional analysis methods (distributions etc.) Dealing e.g. with spectroscopy (where it all started; cf. also Connes) and finite dimensional Hilbert spaces where operators are matrices, the operatorial formulation is easier.

• When there are no precise Weyl and Wigner maps (e.g. very general phase spaces, maybe infinite dimensional) one does not have much choice but to work (maybe “at the physical level of rigor”) with functional analysis.

• Digression. In atomic physics we really know the forces. The more indirect physical measurements become, the more one has to be careful. “Curse” of experimental sciences. Mathematical logic: if $A$ and $A \rightarrow B$, then $B$. But in real life, not so. Imagine model or theory $A$. If $A \rightarrow B$ and “$B$ is nice” (e.g. verified), then $A$! (It ain’t necessarily so.)
“... One should examine closely even the elementary and the satisfactory features of our Quantum Mechanics and criticize them and try to modify them, because there may still be faults in them. The only way in which one can hope to proceed on those lines is by looking at the basic features of our present Quantum Theory from all possible points of view. Two points of view may be mathematically equivalent and you may think for that reason if you understand one of them you need not bother about the other and can neglect it. But it may be that one point of view may suggest a future development which another point does not suggest, and although in their present state the two points of view are equivalent they may lead to different possibilities for the future. Therefore, I think that we cannot afford to neglect any possible point of view for looking at Quantum Mechanics and in particular its relation to Classical Mechanics. Any point of view which gives us any interesting feature and any novel idea should be closely examined to see whether they suggest any modification or any way of developing the theory along new lines. A point of view which naturally suggests itself is to examine just how close we can make the connection between Classical and Quantum Mechanics. That is essentially a purely mathematical problem – how close can we make the connection between an algebra of non-commutative variables and the ordinary algebra of commutative variables? In both cases we can do addition, multiplication, division...” Dirac, The relation of Classical to Quantum Mechanics (2nd Can. Math. Congress, Vancouver 1949). U.Toronto Press (1951) pp 10-31.
11. Other aspects & avatars of Deformation Quantization.

Index theorems. Pseudodifferential operators (standard ordering), composition of symbols. Closed star products [CFS’92] (trace $\simeq \int_M$), cyclic cohomology, algebraic index theorems. Fractional analytic index.

Star-representation theory of Lie groups and algebras. $G$ Lie group acts on symplectic $(M, P)$, Lie algebra $\mathfrak{g} \ni x$ realized by $u_x \in C^\infty(M)$, with $P(u_x, u_y) = [u_x, u_y] \equiv \frac{1}{2t} (u_x \ast u_y - u_y \ast u_x)$ (preferred observables)

Define (group element) $E(e^x) = \text{Exp}(x) \equiv \sum_{n=0}^{\infty} (n!)^{-1} (u_x/2t)^{*n}$.

Star Representation: $\text{Im}E$-valued distribution on $M$

(test functions on $M$) $D \ni f \mapsto \mathcal{E}(f) = \int_G f(g)E(g^{-1})dg$.

Further physical applications.

- Quantization of constrained systems (second class constraints reduce to symplectic submanifold, e.g. $\mathbb{R}^8$ to $T^*S^3$, first class to Poisson).
- Quantum field theory (infinite dimensional Poisson manifolds).
- Statistical mechanics (e.g. KMS), $\beta$ deformation.
(Topological) Quantum Groups. Deform Hopf algebras of functions (differentiable vectors) on Poisson-Lie group, and/or their topological duals (as nuclear t.v.s., Fréchet or dual thereof). Preferred deformations (deform either product or coproduct) e.g. $G$ semi-simple compact: $A = C^\infty(G)$ (gets differential star product) or its dual (compactly supported distributions on $G$, completion of $\mathcal{H}g$, deform coproduct with Drinfeld twist); or $A = \mathcal{H}(G)$, coefficient functions of finite dimensional representations of $G$, or its dual.

“Noncommutative Gelfand duality theorem.” Commutative topological algebra $A \simeq$ “functions on its spectrum.” What about $(A[[t]], \star_t)$? Woronowicz’s matrix $C^*$ pseudogroups. Gelfand’s NC polynomials.

Noncommutative geometry vs. deformation quantization.

Strategy: formulate usual differential geometry in an unusual manner, using in particular algebras and related concepts, so as to be able to “plug in” noncommutativity in a natural way (cf. Dirac quote).
Noncommutative (quantized) manifolds. E.g. quantum 3- and 4-spheres (Connes–Dubois-Violette; similar strategy; also $q$AdS$_n$, possibly simpler).


“Science fiction”. Deform Minkowski to AdS$_4$ gives massless particles composed of singletons: Massless particles are (as UIR of the AdS$_4$ group $SO(3,2)$) composites of Dirac’s singletons $D_i$ and $Rac$, massless particles in a 2+1 dimensional flat space-time “where they live” (a manifestation of ’t Hooft’s holographic principle).

Dynamically, photons as 2-Rac states is consistent with QED. The 9 leptons (3 flavors) can be obtained as 2-singleton states massified by interaction with 5 pairs of Higgs. Quarks? Then, or simultaneously, quantize it (at even root of unity?): “partially granular” structure of space-time with $q$AdS$_4$ black holes from where matter emerges, singletons interacting with Higgs (or dark matter or energy). [Black holes à la ’t Hooft, can communicate with information loss: “Nature’s book keeping system: the data can be written onto a surface, and the pen with which the data are written has a finite size”. Hidden variables?].