Things we saw yesterday

- A quantum field is (distributional) map from a spacetime (a globally hyperbolic manifold) into a local field algebra

$$\mathcal{M} \ni x \to \phi(x)$$

$$\mathcal{C}^{\infty}(\mathcal{M}) \ni f \to \int \phi(x) f(x) dx$$

- The (Heisenberg) algebraic structure is provided by the commutator: a distribution that vanishes at spacelike separated pairs *(x,y)*

$$C(x,y) = [\phi(x), \phi(y)] = -i\hbar E(x,y)$$

Things we saw yesterday

Let *(M,g)* a globally hyperbolic manifold In the space of complex solution of the KG equation $\Box_g \phi + V(x)\phi = 0$. introduce the invariant Peierls aka KG inner product

$$(f,g) = -i \int_{\Sigma_t} \overline{f} \overleftrightarrow{\nabla}_\mu \, g \, d\sigma^\mu$$

Find a "complete" basis $\{u_i\}$ so that

$$(u_i, u_j) = \delta_{ij}, \quad (\bar{u}_i, \bar{u}_j) = -\delta_{ij}, \quad (u_i, \bar{u}_j) = 0$$

The commutator admits the following expansion

$$C(x,y) = \sum_i (u_i(x)\overline{u}_i(y) - u_i(y)\overline{u}_i(x))$$

It is basis independent (uniqueness)

Things we saw yesterday

- Quantizing is representing the commutation rules in a Hilbert space

$$\phi(x) \to \widehat{\phi}(x) \in Op(\mathcal{H})$$

$$[\widehat{\phi}(x), \widehat{\phi}(y)] = C(x,y)\mathbf{1}_{\mathcal{H}}$$

- Realized by finding any positive two-point functions that solves the functional relation

$$W(x,y) - W(y,x) = C(x,y)$$

- *W(x,y)* Interpreted as the VEV of the quantum field

$$W(x,y) = \langle \Psi_0, \widehat{\phi}(x)\widehat{\phi}(y)\Psi_0 \rangle$$

Things we saw yesterday

- Saw a family of inequivalent quantizations parametrized by the temperature 1/β:

$$\langle\Psi_0, , \widehat{\phi}_\beta(x)\widehat{\phi}_\beta(y)\Psi_0\rangle_\beta = W_\beta(x,y) =$$
$$\frac{1}{(2\pi)^3}\int e^{-ip^0(t-s)+i\mathbf{p}(\mathbf{x}-\mathbf{y})}\left[\frac{1}{1-e^{-\beta p^0}}\right]\epsilon(p^0)\delta(p^2-m^2)dp$$

- All these fields solve the same Klein-Gordon QFT

$$\Box\widehat{\phi}_\beta(x) + m^2\widehat{\phi}_\beta(x) = 0$$

- All these fields have the same commutation rules

$$W_\beta(x,y) - W_\beta(y,x) = C(x,y)$$

Bogoliubov Tranformations

KG fields: standard construction

Let (M,g) a globally hyperbolic manifold. Consider the KG equation

$$\Box_g \phi + V(x)\phi = 0. \qquad (f,g) = -i \int_{\Sigma_t} \overline{f} \overleftrightarrow{\nabla}_\mu \, g \, d\sigma^\mu$$

Find a basis $\{u_i\}$ so that $(u_i, u_j) = \delta_{ij}, \quad (\bar{u}_i, \bar{u}_j) = -\delta_{ij}, \quad (u_i, \bar{u}_j) = 0$

Write the unequal time commutator

$$C(x,y) = \sum [u_i(x)\overline{u_i(y)} - u_i(y)\overline{u_i(x)}]$$

If there are no infrared divergences the two point functions

$$W(x,y) = \sum u_i(x)\overline{u_i(y)} \qquad W(y,x) = \sum u_i(y)\overline{u_i(x)}$$

trivially solve the split functional relation $C(x,y) = W(x,y) - W(y,x)$ and defines a <u>pure state</u> on the field algebra (an irreducible representation).

$$\int \overline{f}(x) W(x,y) f(y) = \sum \left| \int \overline{f}(x) u_i(x) \right|^2 \geq 0$$

Bogoliubov transformation of the fields

- Can write the field expansion

$$\phi(x) = \sum_i \left(u_i(x) a_i + u_i^*(x) a_i^+ \right)$$

The pure state $W(x, y) = \sum u_i(x) \overline{u_i(y)}$

Can be seen as the expectation value of the field represented in the vacuum of the annihilation operators a's

$$\widehat{\phi}(x) = \sum_i \left(u_i(x) \widehat{a}_i + u_i^*(x) \widehat{a}_i^+ \right) \qquad \widehat{a}_i |\Psi_a >= 0$$

Example (again)

$$W(x,y) = \langle \Psi_0, \phi(x)\phi(0)\Psi_0 \rangle = \frac{1}{(2\pi)^3} \int e^{-ip(x)} \theta(p^0) \delta(p^2 - m^2) d^4 p$$

$$W(x,y) = \frac{1}{(2\pi)^3} \int e^{-i\omega(x^0 - y^0) + i\mathbf{p}(\mathbf{x}-\mathbf{y})} \frac{d^3\mathbf{p}}{2\omega}$$

$$= \int \frac{e^{-i\omega x^0 + i\mathbf{p}\mathbf{x}}}{\sqrt{(2\pi)^3 2\omega}} \frac{e^{i\omega y^0 - i\mathbf{p}\mathbf{y}}}{\sqrt{(2\pi)^3 2\omega}} d^3\mathbf{p} \quad = \sum u_i(x)\overline{u_i(y)}$$

Bogoliubov transformations of the modes

Consider two complete sets of modes

$\{u_i,\ i \in A\}$ $\{u'_j,\ j \in B\}$

that diagonalize the commutator and suppose that we can write

$$u'_i = \alpha_{ij} u_j + \beta_{ij} u_j^*$$

$$u'^*_i = \alpha^*_{ij} u^*_j + \beta^*_{ij} u_j$$

Bogoliubov transformation of the fields

$$\phi(x) = \sum_i \left(u_i(x)a_i + u_i^*(x)a_i^+\right)$$

$$\phi(x) = \sum_i \left(u_i'(x)a_i' + {u_i'}^*(x){a_i'}^+\right)$$

These expressions are perfectly equivalent at the algebraic level because they yield the same commutator.

Quantization: pure states

- Choose the corresponding Fock vacua (both are pure states)

$$W(x,y) = \langle \Psi_a | \phi(x)\phi(y) | \Psi_a \rangle = \sum u_i(x) u_i(y)$$

$$W_b(x,y) = \langle \Psi_b | \phi(x)\phi(y) | \Psi_b \rangle = \sum u'_i(x) u'_i(y)$$

- **Bogoliubov transformations are a tool to generate infinitely many pure states from a given one**
- **Are they equivalent?**

The Klein-Gordon normalization reads

$$(u'_i, u'_j) = (\alpha_{ik} u_k + \beta_{ik} u_k^*, \alpha_{jl} u_l + \beta_{jl} u_l^*)$$
$$= \alpha_{ik}^* \alpha_{jk} - \beta_{ik}^* \beta_{jk} = \delta_{ij}$$

$$(u'^*_i, u'_j) = (\alpha_{ik}^* u_k^* + \beta_{ik}^* u_k, \alpha_{jl} u_l + \beta_{jl} u_l^*)$$
$$= \beta_{ik} \alpha_{jk} - \alpha_{ik} \beta_{jk} = 0$$

In matrix notation

$$\alpha\alpha^+ - \beta\beta^+ = \mathbb{I} \qquad \alpha\beta^T - \beta\alpha^T = 0$$
$$\alpha^*\alpha^T - \beta^*\beta^T = \mathbb{I} \qquad \alpha^*\beta^+ - \beta^*\alpha^+ = 0$$

$$\alpha\alpha^{+} - \beta\beta^{+} = \mathbb{I} \qquad \alpha\beta^{T} - \beta\alpha^{T} = 0$$

$$\alpha^{*}\beta^{+} - \beta^{*}\alpha^{+} = 0 \qquad \alpha^{*}\alpha^{T} - \beta^{*}\beta^{T} = \mathbb{I}$$

The previous relations are condensed as follows

$$\begin{pmatrix} \alpha & \beta \\ \beta^{*} & \alpha^{*} \end{pmatrix} \begin{pmatrix} \alpha^{+} & -\beta^{T} \\ -\beta^{+} & \alpha^{T} \end{pmatrix} = \mathbb{I}$$

Uniqueness of the inverse provides two more relations (and their complex conjugates):

$$\begin{pmatrix} \alpha^{+} & -\beta^{T} \\ -\beta^{+} & \alpha^{T} \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta^{*} & \alpha^{*} \end{pmatrix} = \mathbb{I}$$

$$\alpha^{+}\alpha - \beta^{T}\beta^{*} = \mathbb{I} \qquad \alpha^{+}\beta - \beta^{T}\alpha^{*} = 0$$

$$\alpha^{T}\beta^{*} - \beta^{+}\alpha = 0 \qquad \alpha^{T}\alpha^{*} - \beta^{+}\beta = \mathbb{I}$$

Summary

$$\alpha\alpha^{+} - \beta\beta^{+} = \mathbb{I} \qquad \alpha\beta^{T} - \beta\alpha^{T} = 0$$

$$\alpha^{*}\alpha^{T} - \beta^{*}\beta^{T} = \mathbb{I} \qquad \alpha^{*}\beta^{+} - \beta^{*}\alpha^{+} = 0$$

$$\alpha^{+}\alpha - \beta^{T}\beta^{*} = \mathbb{I} \qquad \alpha^{+}\beta - \beta^{T}\alpha^{*} = 0$$

$$\alpha^{T}\alpha^{*} - \beta^{+}\beta = \mathbb{I} \qquad \alpha^{T}\beta^{*} - \beta^{+}\alpha = 0$$

$$\begin{pmatrix} \alpha^+ & -\beta^T \\ -\beta^+ & \alpha^T \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} = \mathbb{I}$$

In matrix form the Bogoliubov transformation reads

$$\begin{pmatrix} u' \\ u'^* \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} u \\ u^* \end{pmatrix}$$

The inverse transform is therefore

$$\begin{pmatrix} u \\ u^* \end{pmatrix} = \begin{pmatrix} \alpha^+ & -\beta^T \\ -\beta^+ & \alpha^T \end{pmatrix} \begin{pmatrix} u' \\ u'^* \end{pmatrix}$$

Inverse transform

$$\begin{pmatrix} u \\ u^* \end{pmatrix} = \begin{pmatrix} \alpha^+ & -\beta^T \\ -\beta^+ & \alpha^T \end{pmatrix} \begin{pmatrix} u' \\ u'^* \end{pmatrix}$$

$$u_i = (\alpha^+)_{ij} u'_j - (\beta^T)_{ij} {u'_j}^*$$

$$u_i^* = -(\beta^+)_{ij} u'_j + (\alpha^T)_{ij} {u'_j}^*$$

Bogoliubov transformation of the fields

$$\phi(x) = \sum_i \left(u_i(x)a_i + u_i^*(x)a_i^+\right)$$

$$\phi(x) = \sum_i \left(u_i'(x)a_i' + u_i'^*(x)a_i'^+\right)$$

These expressions are perfectly equivalent at the algebraic level because they yield the same commutator.

$$\phi(x) = \sum_i \left(u_i(x)a_i + u_i^*(x)a_i^+\right)$$

$$= \sum_{ij} \left((\alpha^+)_{ij}u'_j - (\beta^T)_{ij}{u'_j}^*\right) a_i + \sum_{ij} \left(-(\beta^+)_{ij}u'_j + (\alpha^T)_{ij}{u'_j}^*\right) a_i^+$$

$$= \sum_{ij} \left(\alpha^*_{ji}a_i - \beta^*_{ji}a_i^+\right) u'_j \ + \sum_{ij} \left(-\beta_{ji}a_i + \alpha_{ji}a_i^+\right) {u'_j}^*$$

$$a'_j = \sum_i \left(\alpha^*_{ji}a_i - \beta^*_{ji}a_i^+\right)$$

$${a'_j}^+ = \sum_i \left(-\beta_{ji}a_i + \alpha_{ji}a_i^+\right)$$

One important point

- In matrix form

$$\begin{pmatrix} a' \\ a'^{+} \end{pmatrix} = \begin{pmatrix} \alpha^* & -\beta^* \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} a \\ a^{+} \end{pmatrix}$$

- The question now is: is the above transformation unitarily impementable?
- If YES the corresponding Fock representations are physically and mathematically equivalent.
- If NOT they are inequivaent and have distinct physical interpretations

The unitary implementer

$\mathcal{H} =$ Fock space of the operators $a\,,\quad |\Psi\rangle \in \mathcal{H}$

$\mathcal{H}' =$ Fock space of the operators $a'\,,\quad |\Psi'\rangle \in \mathcal{H}'$

We need a unitary operator $U : \mathcal{H}' \to \mathcal{H}$ such that

$$U^+ a\, U = a' \qquad U^+ a^+ U = a'^+$$

$$a\, U = U a' = U(\alpha^* a - \beta^* a^+)$$

$$a^+ U = U a'^+ = U(\alpha a^+ - \beta a)$$

Fock states

Consider a pair of canonical operators and the corresponding Fock vacuum and Fock basis

$$[a, a^+] = 1 \qquad a|0\rangle = 0$$

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^+)^n|0\rangle \qquad \langle n|m\rangle = \delta_{n,m}$$

Completeness of the basis $\displaystyle\sum_{n=0}^{\infty} |n\rangle\langle n| = \mathbb{I}$

Coherent states

$$|z\rangle = \exp(za^{+})|0\rangle$$

$$= \sum_n \frac{z^n}{n!}(a^{+})^n|0\rangle \;= \sum_n \frac{z^n}{\sqrt{n!}}|n\rangle$$

Coherent states are eigenstates of the destruction operator

$$a|z\rangle = z|z\rangle$$

BCH formula: $e^{X}e^{Y} = e^{X+Y+\frac{1}{2}[X,Y]}$

$$e^{za^{+}}e^{ua} = e^{ua+za^{+}-\frac{1}{2}uz}$$
$$e^{ua}e^{za^{+}} = e^{ua+za^{+}+\frac{1}{2}uz}$$
$$e^{ua}e^{za^{+}} = e^{uz}e^{za^{+}}e^{ua}$$

Applying this formula to the vacuum

$$\exp(ua)|z\rangle = \exp(uz)|z\rangle$$

In general for an holomorphic function

$$f(a)|z\rangle = f(z)|z\rangle \qquad \langle z|f(a^{+}) = f(z^{*})\langle z|$$

$$\langle z|z'\rangle = \langle 0|e^{z^{*}a}e^{z'a^{+}}|0\rangle = e^{z^{*}z'}\langle 0|e^{z'a^{+}}e^{z^{*}a}|0\rangle =$$
$$= e^{z^{*}z'}\langle 0|0\rangle = e^{z^{*}z'}$$

As for creation operators to the right

$$a^+|z\rangle = a^+ e^{za^+}|0\rangle = \partial_z e^{za^+}|0\rangle = \partial_z|z\rangle$$

Similarly for destruction operators to the left

$$\langle z|a = \langle 0|e^{z^* a} a = \partial_{z^*}\langle 0|e^{z^* a} = \partial_{z^*}\langle z|$$

Resolution of the identity

- Bargmann measure

$$d\mu(z) = \frac{e^{-|z|^2} dz^* \wedge dz}{2\pi i} = \frac{e^{-(x^2+y^2)} dx \wedge dy}{\pi}$$

$$\int d\mu(z) |z\rangle\langle z| =$$

$$= \sum_{n,m} \frac{|n\rangle\langle m|}{\sqrt{n!m!}} \int \frac{re^{-r^2} dr}{\pi} r^{n+m} \int e^{i(n-m)\theta} d\theta$$

$$= \sum_n \frac{|n\rangle\langle n|}{n!} \int_0^\infty 2r^{2n+1} e^{-r^2} dr \;= \sum_n |n\rangle\langle n| = \mathbb{I}$$

The unitary implementer (continued)

$$\langle t|aU|z\rangle = \langle t|U(\alpha^* a - \beta^* a^+)|z\rangle$$

$$\langle t|a^+ U|z\rangle = \langle t|U(\alpha a^+ - \beta a)|z\rangle$$

$$\begin{cases} \partial_{t^*}\langle t|U|z\rangle = \alpha^* z\langle t|U|z\rangle - \beta^* \partial_z \langle t|U|z\rangle \\ t^*\langle t|U|z\rangle = \alpha\partial_z\langle t|U|z\rangle - \beta z\langle t|U|z\rangle \end{cases}$$

$$\begin{cases} \partial_{t_i^*}\langle t|U|z\rangle = \alpha_{ij}^* z_j\langle t|U|z\rangle - \beta_{ij}^* \partial_{z_j} \langle t|U|z\rangle \\ t_i^*\langle t|U|z\rangle = \alpha_{ij}\partial_{z_j}\langle t|U|z\rangle - \beta_{ij} z_j\langle t|U|z\rangle \end{cases}$$

Property 1

- The matrix α^{+} is invertible.

Summary

$$\alpha\alpha^{+} - \beta\beta^{+} = \mathbb{I}$$

$$\alpha\beta^{T} - \beta\alpha^{T} = 0$$

$$\alpha^{*}\alpha^{T} - \beta^{*}\beta^{T} = \mathbb{I}$$

$$\alpha^{*}\beta^{+} - \beta^{*}\alpha^{+} = 0$$

$$\alpha^{+}\alpha - \beta^{T}\beta^{*} = \mathbb{I}$$

$$\alpha^{+}\beta - \beta^{T}\alpha^{*} = 0$$

$$\alpha^{T}\alpha^{*} - \beta^{+}\beta = \mathbb{I}$$

$$\alpha^{T}\beta^{*} - \beta^{+}\alpha = 0$$

Property 1

- The matrix α^+ is invertible. This follows from

$$\alpha\alpha^+ - \beta\beta^+ = \mathbb{I}$$

which implies that

$$\langle\Psi|\alpha\alpha^+|\Psi\rangle = \langle\Psi|\Psi\rangle + \langle\Psi|\beta\beta^+|\Psi\rangle$$

$$||\alpha^+\Psi||^2 = ||\Psi||^2 + ||\beta^+\Psi||^2$$

and therefore the kernel of α^+ is trivial.

Summary

$$\alpha\alpha^{+} - \beta\beta^{+} = \mathbb{I} \qquad \alpha\beta^{T} - \beta\alpha^{T} = 0$$

$$\alpha^{*}\alpha^{T} - \beta^{*}\beta^{T} = \mathbb{I} \qquad \alpha^{*}\beta^{+} - \beta^{*}\alpha^{+} = 0$$

$$\alpha^{+}\alpha - \beta^{T}\beta^{*} = \mathbb{I} \qquad \alpha^{+}\beta - \beta^{T}\alpha^{*} = 0$$

$$\boxed{\alpha^{T}\alpha^{*} - \beta^{+}\beta = \mathbb{I}} \qquad \alpha^{T}\beta^{*} - \beta^{+}\alpha = 0$$

$$\alpha^T \alpha^* = (\alpha^+ \alpha)^* = \mathbb{I} + \beta^+ \beta$$

implies that also α^* and therefore α has a trivial kernel. The second equation

$$t^* \langle t | a\, U | z \rangle = \alpha \partial_z \langle t | U | z \rangle - \beta z \langle t | U | z \rangle$$

is rewritten as follows.

$$\partial_z \langle t | U | z \rangle = \alpha^{-1} t^* \langle t | U | z \rangle + \alpha^{-1} \beta z \langle t | U | z \rangle$$

$$\alpha^{-1} \beta$$

Summary

$$\alpha\alpha^{+} - \beta\beta^{+} = \mathbb{I} \qquad \boxed{\alpha\beta^{T} - \beta\alpha^{T} = 0}$$

$$\alpha^{*}\alpha^{T} - \beta^{*}\beta^{T} = \mathbb{I} \qquad \alpha^{*}\beta^{+} - \beta^{*}\alpha^{+} = 0$$

$$\alpha^{+}\alpha - \beta^{T}\beta^{*} = \mathbb{I} \qquad \alpha^{+}\beta - \beta^{T}\alpha^{*} = 0$$

$$\alpha^{T}\alpha^{*} - \beta^{+}\beta = \mathbb{I} \qquad \alpha^{T}\beta^{*} - \beta^{+}\alpha = 0$$

Property 2

The matrix $\alpha^{-1}\beta$ is symmetric

Since $\alpha\beta^T - \beta\alpha^T = 0$ and α is invertible

$$\beta^T = \alpha^{-1}\beta\alpha^T$$

$$(\alpha^{-1}\beta)^T = \beta^T(\alpha^{-1})^T = \alpha^{-1}\beta\alpha^T(\alpha^{-1})^T$$

$$= \alpha^{-1}\beta$$

Integration

$$\partial_{t^*}\langle t|U|z\rangle = \alpha^* z\langle t|U|z\rangle - \beta^*\partial_z\langle t|U|z\rangle$$

$$\partial_z\langle t|U|z\rangle = \alpha^{-1}t^*\langle t|U|z\rangle + \alpha^{-1}\beta z\langle t|U|z\rangle$$

The second equation can now be integrated easily

$$\langle t|U|z\rangle = \exp\left(z\alpha^{-1}t^* + \frac{1}{2}z\alpha^{-1}\beta z\right)f(t^*)$$

Inserting in the first equation we get

$$(z\alpha^{-1} + (\partial_{t^*}f)/f)\langle t|U|z\rangle =$$

$$= (\alpha^* z - \beta^*\alpha^{-1}t^* - \beta^*\alpha^{-1}\beta z)\langle t|U|z\rangle$$

$$(z\alpha^{-1} + (\partial_{t^*} f)/f)\langle t|U|z\rangle =$$

$$= (\alpha^* z - \beta^* \alpha^{-1} t^* - \beta^* \alpha^{-1} \beta z)\langle t|U|z\rangle$$

We get one equation $\partial_{t^*} f = -\beta^* \alpha^{-1} t^* f$
promptly integrated

$$f = \exp\left(-\frac{1}{2} t^* \beta^* \alpha^{-1} t^*\right)$$

and a compatibility condition

$$(\alpha^{-1})^T = \alpha^* - \beta^* \alpha^{-1} \beta$$

Summary

$$\alpha\alpha^{+} - \beta\beta^{+} = \mathbb{I}$$

$$\alpha\beta^{T} - \beta\alpha^{T} = 0$$

$$\alpha^{*}\alpha^{T} - \beta^{*}\beta^{T} = \mathbb{I}$$

$$\alpha^{*}\beta^{+} - \beta^{*}\alpha^{+} = 0$$

$$\alpha^{+}\alpha - \beta^{T}\beta^{*} = \mathbb{I}$$

$$\alpha^{+}\beta - \beta^{T}\alpha^{*} = 0$$

$$\boxed{\alpha^{T}\alpha^{*} - \beta^{+}\beta = \mathbb{I}}$$

$$\boxed{\alpha^{T}\beta^{*} - \beta^{+}\alpha = 0}$$

$$\alpha^T \beta^* - \beta^+ \alpha = 0 \quad \Rightarrow \quad \beta^+ = \alpha^T \beta^* \alpha^{-1}$$

$$\alpha^T \alpha^* - \beta^+ \beta = \mathbb{I}$$

$$\alpha^T \alpha^* - \alpha^T \beta^* \alpha^{-1} \beta = \mathbb{I}$$

and the compatibility condition follows

$$(\alpha^{-1})^T = \alpha^* - \beta^* \alpha^{-1} \beta$$

All in all

$$\langle t|U|z\rangle = C\exp\left(z\alpha^{-1}t^* + \frac{1}{2}z\alpha^{-1}\beta z - \frac{1}{2}t^*\beta^*\alpha^{-1}t^*\right)$$

The overall normalization has to be determined yet.

Now note that

$$\langle t| : F(a,a^+) : |z\rangle = F(z,t^*)\langle t|z\rangle$$

Example

$$\langle t| : aa^+ : |z\rangle = \langle t|a^+a|z\rangle = t^*z\langle t|z\rangle$$

$$U = C : \exp\left(a(\alpha^{-1}-1)a^+ + \frac{1}{2}a\alpha^{-1}\beta a - \frac{1}{2}a^+\beta^*\alpha^{-1}a^+\right) :$$

The normalization

$$U = C : \exp\left(a(\alpha^{-1} - 1)a^{+} + \frac{1}{2}a\alpha^{-1}\beta a - \frac{1}{2}a^{+}\beta^{*}\alpha^{-1}a^{+}\right) :$$

$$\langle 0|UU^{+}|0\rangle = 1 =$$

$$= C^{2}\langle 0|\exp\left(\frac{1}{2}a(\alpha^{-1}\beta)a\right)\exp\left(\frac{1}{2}a^{+}(\alpha^{-1}\beta)^{*}a^{+}\right)|0\rangle$$

Technical interlude

Let $M = M^T$

$$\mathcal{A} = \langle 0| \exp\left(\frac{1}{2} aMa\right) \exp\left(\frac{1}{2} a^+ M^* a^+\right) |0\rangle$$

$$= \int d\mu(z) \langle 0| \exp\left(\frac{1}{2} aMa\right) |z\rangle\langle z| \exp\left(\frac{1}{2} a^+ M^* a^+\right) |0\rangle$$

$$= \int \frac{dz^* \wedge dz}{2\pi i} e^{-zz^* + \frac{1}{2} zMz + \frac{1}{2} z^* M^* z^*}$$

$$= \int \frac{dz^* \wedge dz}{2\pi i} \exp\left[-\frac{1}{2} \begin{pmatrix} z & z* \end{pmatrix} \begin{pmatrix} -M & 1 \\ 1 & -M^* \end{pmatrix} \begin{pmatrix} z \\ z* \end{pmatrix}\right]$$

Technical interlude

Let $M = M^T$

$$\mathcal{A} = \int \frac{dz^* \wedge dz}{2\pi i} \exp\left[-\frac{1}{2} \begin{pmatrix} z & z* \end{pmatrix} \begin{pmatrix} -M & 1 \\ 1 & -M^* \end{pmatrix} \begin{pmatrix} z \\ z* \end{pmatrix} \right]$$

$$\begin{pmatrix} z & z^* \end{pmatrix} = \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$$\begin{pmatrix} z \\ z^* \end{pmatrix} = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} -M & 1 \\ 1 & -M^* \end{pmatrix} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} = \begin{pmatrix} 2 - M - M^* & -i(M - M^*) \\ -i(M - M^*) & 2 + M + M^* \end{pmatrix} = 2\mathcal{M}$$

$$\mathcal{A} = \int \frac{dxdy}{\pi} \exp\left[- \begin{pmatrix} x & y \end{pmatrix} \mathcal{M} \begin{pmatrix} x \\ y \end{pmatrix} \right] = \sqrt{\frac{1}{\det \mathcal{M}}}$$

Technical interlude 2

$$\text{Let } M = M^T$$

$$\det\left[\frac{1}{2}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}\begin{pmatrix} -M & 1 \\ 1 & -M^* \end{pmatrix}\begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}\right] = -\det\begin{pmatrix} -M & 1 \\ 1 & -M^* \end{pmatrix}$$

A theorem of linear algebra says that when C and D commute

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(AD - BC)$$

$$\det \mathcal{M} = \det(1 - MM^*)$$

$$\langle 0| \exp\left(\frac{1}{2}aMa\right) \exp\left(\frac{1}{2}a^+ M^* a^+\right)|0\rangle = \frac{1}{\sqrt{\det(1 - MM^*)}}$$

The normalization

$$\langle 0| \exp\left(\frac{1}{2} a(\alpha^{-1}\beta) a\right) \exp\left(\frac{1}{2} a^{+}(\alpha^{-1}\beta)^{*} a^{+}\right) |0\rangle = \frac{1}{\sqrt{\det(1-\alpha^{-1}\beta(\alpha^{-1}\beta)^{*})}}$$

Summary

$$\alpha\alpha^{+} - \beta\beta^{+} = \mathbb{I}$$

$$\alpha\beta^{T} - \beta\alpha^{T} = 0$$

$$\alpha^{*}\alpha^{T} - \beta^{*}\beta^{T} = \mathbb{I}$$

$$\alpha^{*}\beta^{+} - \beta^{*}\alpha^{+} = 0$$

$$\boxed{\alpha^{+}\alpha - \beta^{T}\beta^{*} = \mathbb{I}}$$

$$\boxed{\alpha^{+}\beta - \beta^{T}\alpha^{*} = 0}$$

$$\alpha^{T}\alpha^{*} - \beta^{+}\beta = \mathbb{I}$$

$$\alpha^{T}\beta^{*} - \beta^{+}\alpha = 0$$

The normalization

$$\langle 0|\exp\left(\frac{1}{2}a(\alpha^{-1}\beta)a\right)\exp\left(\frac{1}{2}a^{+}(\alpha^{-1}\beta)^{*}a^{+}\right)|0\rangle = \frac{1}{\sqrt{\det(1-\alpha^{-1}\beta(\alpha^{-1}\beta)^{*})}}$$

$$\boxed{\alpha^{+}\beta - \beta^{T}\alpha^{*} = 0} \qquad \beta^{T} = \alpha^{+}\beta(\alpha^{*})^{-1}$$

$$\boxed{\alpha^{+}\alpha - \beta^{T}\beta^{*} = \mathbb{I}} \qquad \alpha^{+}\alpha - \alpha^{+}\beta(\alpha^{*})^{-1}\beta^{*} = \mathbb{I}$$

$$\mathbb{I} - \alpha^{-1}\beta(\alpha^{*})^{-1}\beta^{*} = (\alpha^{+}\alpha)^{-1}$$

$$\frac{1}{\sqrt{\det(1-\alpha^{-1}\beta(\alpha^{-1}\beta)^{*})}} = \sqrt{\det(\alpha^{+}\alpha)}$$

$$C = (\det(\alpha^{+}\alpha))^{-\frac{1}{4}}$$

The S Matrix

$$U = \frac{1}{(\det(\alpha^{+}\alpha))^{\frac{1}{4}}} : \exp\left(a(\alpha^{-1} - 1)a^{+} + \frac{1}{2} a\alpha^{-1}\beta a - \frac{1}{2} a^{+}\beta^{*}\alpha^{-1}a^{+} \right) :$$

Other states

$$W(x,y) = \langle \Psi_0, \widehat{\phi}(x)\widehat{\phi}(y)\Psi_0 \rangle = \sum u_i(x)\overline{u_i(y)}$$

Many other quantization can be obtained starting from the family $\{u_i\}$

Mixed states (here not the most general ones)

$$W_\gamma(x,y) = \langle \Psi_{0,\gamma}, \widehat{\phi}(x)\widehat{\phi}(y)\Psi_{0,\gamma} \rangle =$$

$$= \sum \cosh^2(\gamma_i)\ u_i(x)\overline{u_i(y)} + \sinh^2(\gamma_i)\ \overline{u_i(x)}u_i(y)$$

$$\gamma_i = 0, \quad \to W_\gamma(x,y) = W(x,y)$$

Other states

$$W_\gamma(x,y) = \langle \Psi_0, \widehat{\phi}(x)\widehat{\phi}(y)\Psi_0 \rangle =$$

$$= \sum \cosh^2(\gamma_i)\ u_i(x)\overline{u_i(y)} + \sum \sinh^2(\gamma_i)\ \overline{u_i(x)}u_i(y)$$

again and again

$$C(x,y) = W_\gamma(x,y) - W_\gamma(y,x) =$$

$$= \sum \cosh^2(\gamma_i)\ u_i(x)\overline{u_i(y)} + \sum \sinh^2(\gamma_i)\ \overline{u_i(x)}u_i(y)$$

$$- \sum \cosh^2(\gamma_i)\ u_i(y)\overline{u_i(x)} - \sum \sinh^2(\gamma_i)\ \overline{u_i(y)}u_i(x)$$

$$= \sum \left(\cosh^2(\gamma_i) - \sinh^2(\gamma_i)\right)\ [u_i(x)\overline{u_i(y)} - \overline{u_i(x)}u_i(y)]$$

$$= \sum [u_i(x)\overline{u_i(y)} - \overline{u_i(x)}u_i(y)]$$