Teoria quântica de campos em espaco-tempo curvo

ugomoschella@gmail.com

Università dell'Insubria – Como – Italia
IHES – Bures-sur-Yvette – France

II) (More or less) Canonical quantization of free fields (in flat space!)

Things we saw yesterday

- A quantum field is (distributional) map from a spacetime (a globally hyperbolic manifold) into a local field algebra

$$\mathcal{M} \ni x \to \phi(x)$$

$$\mathcal{C}^{\infty}(\mathcal{M}) \ni f \to \int \phi(x) f(x) dx$$

- The (Heisenberg) algebraic structure is provided by the commutator: a distribution that vanishes at spacelike separated pairs *(x,y)*

$$C(x,y) = [\phi(x), \phi(y)] = -i\hbar E(x,y)$$

Things we saw yesterday

Let *(M,g)* a globally hyperbolic manifold In the space of complex solution of the KG equation $\Box_g \phi + V(x)\phi = 0$. introduce the invariant Peierls aka KG inner product

$$(f,g) = -i \int_{\Sigma_t} \overline{f} \overset{\leftrightarrow}{\nabla}_\mu \, g \, d\sigma^\mu$$

Find a "complete" basis $\{u_i\}$ so that

$$(u_i, u_j) = \delta_{ij}, \quad (\bar{u}_i, \bar{u}_j) = -\delta_{ij}, \quad (u_i, \bar{u}_j) = 0$$

The commutator admits the following expansion

$$C(x,y) = \sum_i (u_i(x)\overline{u}_i(y) - u_i(y)\overline{u}_i(x))$$

It is basis independent (uniqueness)

Things we saw yesterday

- The second necessary step amounts to representing the quantum field and the commutation relations in a Hilbert space

$$\mathcal{M} \ni x \to \widehat{\phi}(x)$$

$$\mathcal{C}^\infty(\mathcal{M}) \ni f \to \int \widehat{\phi}(x) f(x) dx \in Op(\mathcal{H})$$

$$[\widehat{\phi}(x), \widehat{\phi}(y)] = C(x,y) \qquad (= -i\hbar E(x,y))$$

- This problem has uncountably many solutions (as opposed to the Stone - Von Neumann uniqueness). How to construct (some of) them?

Quantum field theory (1927)

Quantum Mechanics + Special Relativity =

Quantum Field Theory

- The most successful theory we have (together with GR)
- (Embarassing) Long standing question:
- **Does any nontrivial relativistic QFT in spacetime dimension 4 exist (at the nonperturbative level)?**
- i.e. are QM and SR compatible?

QM + SR requirements

- **Locality or Microcausality:**

$$[\widehat{\phi}(x), \widehat{\phi}(y)] = 0 \text{ if } (x-y)^2 < 0$$

- **Relativistic invariance:**

  A strongly continuousunitary rep. $U(a, \Lambda)$of the Poincaré group acts on $\mathcal{H}$

- **Poincaré invariance of the fields:**

$$U(a,\Lambda)\widehat{\phi}(x)U(a,\Lambda)^{-1} = \widehat{\phi}(\Lambda x + a)$$

- **Spectral condition on energy and momentum:**

$$U(a) = e^{ia^\mu P_\mu} \quad \text{spectrum}(P_\mu) \subset \overline{V^+}$$

- **Existence and uniqueness of the vacuum:**

$$U(a,\Lambda)\Psi_0 = \Psi_0$$

Quantum field theory

- Microcausality (i.e. Local commutativity) and/or Poincaré (actually translation) invariance forbid the existence of this map: ultraviolet singularities are unavoidable!

$$\mathrm{M}^4 \ni x \to \widehat{\phi}\,(x) \in Op(\mathcal{H}) \to \text{trivial theory}$$

- Fields are singular objects: only spacetime averaged fields make sense

$$\mathcal{S}(\mathrm{M}^4) \ni f \to \widehat{\phi}\,(f) = \int \widehat{\phi}\,(x)\, f(x) d^4x \in Op(\mathcal{H})$$

- Quantum fields are distributions.
- Previous properties have to be understood in the sense of distributions.
- Good news: they can be differentiated freely.
- Bad news: <u>they cannot be multiplied</u>. Even writing nonlinear field equations is meaningless:

$$\Box\phi + \lambda\phi^3 = 0$$

Vacuum Expectation Values

- The theory is completely characterized by the knowledge of a set of distributions satisfying a number of properties:

$$\langle \Psi_0, \hat{\phi}(x_1) \ldots \hat{\phi}(x_n) \Psi_0 \rangle = W_n(x_1, \ldots, x_n)$$

- **Relativistic invariance:**

$$W_n(\Lambda x_1 + a, \ldots, \Lambda x_n + a) = W_n(x_1, \ldots, x_n)$$

- **Locality or microcausality:**

$$\text{if} \quad (x_j - x_{j+1})^2 < 0$$
$$W_n(x_1, \ldots, x_j, x_{j+1}, \ldots) = W_n(x_1, \ldots, x_{j+1}, x_j, \ldots)$$

- **Nonlinear Conditions of Positive definiteness** (Necessary for the Q.M. interpretation)

- **Spectral Condition**

Positive definiteness

- Consider a terminating sequence of test functions

$$\mathbf{f} = (f_0, f_1(x_1), f_2(x_1, x_2), \ldots, 0, \ldots) \quad x_j \in \mathbf{M}^4$$

- Construct the vector:

$$\psi = f_0\Psi_0 + \int dx_1 f_1(x_1)\phi(x_1)\Psi_0 + \\ + \int dx_1 dx_2\, f_2(x_1, x_2)\, \phi(x_1)\phi(x_2)\Psi_0 + \ldots$$

- Compute the norm of this vector (it has to be positive):

$$\langle\psi, \psi\rangle \;=\; \sum_{jk} \int dx dy \bar{f}_j(x_1, \ldots, x_j) f_k(y_1, \ldots, y_k) \\ W_{j+k}(x_j, \ldots x_1; y_1, \ldots y_k) \geq 0$$

Spectral Condition

The Fourier transform of the n-point function

$$\tilde{W}_n(p_1, p_2, \ldots, p_n) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int e^{i(p_1 x_1 + \ldots + p_n x_n)} W_n(x_1, \ldots, x_n) dx_1 \ldots dx_n$$

$$\tilde{W}(p_1, p_2, \ldots, p_n) = 0$$

unless

$$p_1 \in \overline{V^+}, \quad p_1 + p_2 \in \overline{V^+}, \ldots \quad p_1 + p_2 + \ldots p_n \in \overline{V^+}$$

Spectral Condition

$$\tilde{W}(p_1, p_2, \ldots, p_n) = 0$$

unless

$$p_1 \in \overline{V^+}, \quad p_1 + p_2 \in \overline{V^+}, \ldots \quad p_1 + p_2 + \ldots p_n \in \overline{V^+}$$

equivalent to

$$W(x_1, x_2, \ldots, x_n) = b.v. W(z_1, z_2, \ldots z_n)$$

holomorphic in the tube domain T_n^-

$$T_n^- = \{z_j \in \mathbf{M}^{4(c)} : \mathrm{Im}(z_1 - z_2) \in V^-, \ldots \mathrm{Im}(z_{n-1} - z_n) \in V^-\}$$

Reconstruction ©

- Examples have been constructed in spacetime dimension two and three (hard, hard work in constructive quantum field theory.
- Limits of the method seem to have been attained

- No example known in spacetime dimension 4, after 80 years of history of QFT!

- Are quantum mechanics and special relativity compatible?
- We don't know yet!

(Generalized) free theories

©

- Completely characterized by the two-point functions:

$$\langle \Psi_0, \widehat{\phi}(x_1)\widehat{\phi}(x_2)\Psi_0 \rangle = W(x_1, x_2)$$

- Truncated n-point functions vanish;
- N-point functions are tensor products of two-point functions;
- Example: the 4-point function:

$$W_4(x_1, x_2, x_3, x_4) =$$
$$= W(x_1, x_2)W(x_3, x_4) + W(x_1, x_3)W(x_2, x_4) + W(x_1, x_4)W(x_2, x_3)$$

- GFF have trivial S-matrix
- GFF are the essential ingredient for perturbation theory

Generalized free theories ©

- Once you have a two-point function, what you do with it?
- First thing: check locality

$$\langle\Psi_0, [\phi(x)\phi(y) - \phi(x)\phi(y)]\Psi_0\rangle = W(x,y) - W(y,x) = C(x,y)$$

- It must be $C(x,y) = 0$ when $(x-y)^2 < 0$

Generalized free theories ©

- Second check positive-definiteness: now it is simply

$$\int W(x,y)\bar{f}(x)f(y) \geq 0$$

- If you have translation invariance:

$$W(x,y) = W(x-y) = \frac{1}{(2\pi)^2}\int e^{ip\cdot(x-y)}\tilde{W}(p)d^4p$$

$$\int W(x-y)\bar{f}(x)g(y)dxdy = \frac{1}{(2\pi)^2}\int e^{ip\cdot(x-y)}\tilde{W}(p)d^4p\bar{f}(x)g(y)dxdy =$$

$$= (2\pi)^2\int \tilde{W}(p)\overline{\tilde{f}(-p)}\tilde{f}(-p)d^4p \geq 0$$

- The Fourier transform of W is a positive measure of polynomial growth (Bochner-Schwarz theorem)

Generalized free theories

- Introduce a pre-Hilbert scalar product in the test function space

$$\langle f, g \rangle = \int W(x,y)\bar{f}(x)g(y)d^4x = (2\pi)^2 \int \tilde{W}(p)\overline{\tilde{f}(-p)}\tilde{g}(-p)d^4p$$

- Completing and quotienting the test function space w.r.t. this Hilbert topology gives the 1-particle space:

$$\mathcal{H}^{(1)} = \overline{\mathcal{S}(\mathbf{M}_x^4)/\mathcal{N}} = L^2(\mathbf{M_p}^4, d\mu(p))$$
$$d\mu(p) = \tilde{W}(p)d^4p$$

- The n-particle space is the symmetric tensor product

$$\mathcal{H}^{(n)} = Sym(\mathcal{H}^{(1)} \otimes \ldots \otimes \mathcal{H}^{(1)})$$

- The full Hilbert space is the symmetric Fock space

$$\mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)} \qquad \mathcal{H}^{(0)} = \mathbf{C}$$

Generalized free theories ©

- The vacuum vector: $\Psi_0 = (1, 0, 0, \ldots)$
- A dense set of vectors: $\Psi_f = (f_0, f_1, \ldots, f_k, 0 \ldots)$
- The field may be decomposed into creation and annihilation operators on the dense set of vectors: $\widehat{\phi}^-(h) = \widehat{\phi}^-(h) + \widehat{\phi}^+(h)$

$$\left(\widehat{\phi}^-(h)\Psi_f\right)_n (x_1, \ldots, x_n) = \sqrt{n+1} \int W(x, x')h(x) f_{n+1}(x', x_1, \ldots, x_n) dx dx',$$

$$\left(\widehat{\phi}^+(h)\Psi_f\right)_n (x_1, \ldots, x_n) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} h(x_k) f_{n-1}(x_1, \ldots, \widehat{x}_k, \ldots, x_n).$$

- <u>The commutator is a c-number i.e. proportional to the identity operator</u>:

$$[\widehat{\phi}(x_1), \widehat{\phi}(x_2)] = \langle \Psi_0, [\widehat{\phi}(x_1), \widehat{\phi}(x_2)]\Psi_0 \rangle =$$
$$= W(x_1, x_2) - W(x_2, x_1) = C(x_1, x_2)$$

The other way

$$\mathcal{S}(\mathrm{M}^4) \ni f \to \phi(f) = \int \phi(x) f(x) d^4x \in \mathcal{F}$$

$$[\phi(x), \phi(y)] = C(x,y)\mathbf{1}$$

- Notice that the hat has again disappeared: now the field algebra is abstract
- Problem: find (all) positive definite two-point distribution such that $W(x,y) - W(y,x) = C(x,y)$
- Fock–space representation is obtained by the previous construction: $\phi(f) \to \widehat{\phi}(f) : F_W \to F_W$
- There are infinitely many inequivalent answers even in flat spacetime!

Curved spacetime ©

$$\mathbf{M}^4 \longrightarrow \mathcal{M}$$

$$\mathcal{S}(\mathbf{M}^4) \ni f \to \widehat{\phi}(f) = \int \widehat{\phi}(x)\, f(x) d^4x \in Op(\mathcal{H})$$

replaced by

$$\mathcal{C}_0^\infty(\mathcal{M}) \ni f \to \widehat{\phi}(f) = \int \widehat{\phi}(x)\, f(x) d^4x \in Op(\mathcal{H})$$

Curved spacetime ©

- The only property we can in general retain is local commutativity

$$[\phi(x), \phi(y)] = C(x,y)\mathbf{1}$$

$C(x,y) = 0$ for "spacelike separated" events x and y

The problem of constructing a linear quantum field theory amounts to find a positive definite two-point distribution such that the splitting holds

$$W(x,y) - W(y,x) = C(x,y)$$

Curved spacetime ©

➤ The construction then goes as in flat spacetime (but no symmetry group implemented in general)

$$\langle f, g \rangle = \int_{\mathcal{M}} W(x,y)\bar{f}(x)g(y)d^4x$$

$$\mathcal{H}^{(1)} = \overline{\mathcal{C}_0^{\infty}(\mathcal{M})/\mathcal{N}}, \qquad \mathcal{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}$$

$$\left(\hat{\phi}^-(h)\Psi_f\right)_n(x_1,\ldots,x_n) = \sqrt{n+1}\int_{\mathcal{M}} W(x,x')h(x)f_{n+1}(x',x_1,\ldots,x_n)dxdx',$$

$$\left(\hat{\phi}^+(h)\Psi_f\right)_n(x_1,\ldots,x_n) = \frac{1}{\sqrt{n}}\sum_{k=1}^{n} h(x_k)f_{n-1}(x_1,\ldots,\hat{x}_k,\ldots,x_n).$$

$$\langle \Psi_0, \hat{\phi}(x)\hat{\phi}(y)\Psi_0\rangle = W(x,y), \quad [\hat{\phi}(x),\hat{\phi}(y)] = C(x,y)\mathbf{1}$$

Summary

- **How find a positive definite two-point distribution that solves the equation**

$$W(x,y) - W(y,x) = C(x,y) \ ?$$

- **How many solutions has this problem?**

- **Which solution is physically relevant?**

KG fields: vacuum representation

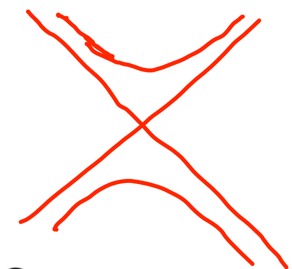
$$\Box\phi(x) + m^2\phi(x) = 0$$
$$C(x,y) = \frac{1}{(2\pi)^3}\int e^{-ip(x-y)}[\theta(p^0) - \theta(-p^0)]\delta(p^2 - m^2)dp$$

$$\phi \longrightarrow \hat{\phi}, \qquad W(x,y) = \langle \Psi_0, \hat{\phi}(x)\hat{\phi}(y)\Psi_0\rangle$$

$$\left(\Box_x + m^2\right) W(x,y) = \left(\Box_y + m^2\right) W(x,y) = 0$$

$$W(x,y) = W(x-y) = \int e^{-ip(x-y)}\tilde{W}(p)\frac{dp}{(2\pi)^3}, \quad px = p^0 t - \mathbf{px}$$

$$(p^2 - m^2)\tilde{W}(p) = 0 \;\rightarrow\; \tilde{W}(p) = \delta(p^2 - m^2)$$



$$\delta(p^2 - m^2) = \frac{\delta(p^0 - \sqrt{\mathbf{p}^2 - m^2})}{2\sqrt{\mathbf{p}^2 + m^2}} + \frac{\delta(p^0 + \sqrt{\mathbf{p}^2 - m^2})}{2\sqrt{\mathbf{p}^2 + m^2}}$$

$$\tilde{W}(p) \simeq \alpha\theta(p^0)\delta(p^2 - m^2) + \beta\theta(-p^0)\delta(p^2 - m^2)$$

Positivity of the energy spectrum: β = 0

Klein-Gordon fields ©

$$W(x,y) = \frac{\alpha}{(2\pi)^3} \int e^{-ip(x-y)} \theta(p^0) \delta(p^2 - m^2) dp$$

$$W'(x,y) = W(y,x) = \frac{\alpha}{(2\pi)^3} \int e^{+ip(x-y)} \theta(p^0) \delta(p^2 - m^2) dp$$
$$= \frac{\alpha}{(2\pi)^3} \int e^{-ip(x-y)} \theta(-p^0) \delta(p^2 - m^2) dp$$

$$C(x,y) = W(x,y) - W(y,x) = \frac{\alpha}{(2\pi)^3} \int e^{-ip(x-y)} \epsilon(p^0) \delta(p^2 - m^2) dp$$

$$\epsilon(p^0) = \theta(p^0) - \theta(-p^0)$$

Consequences of the spectral condition

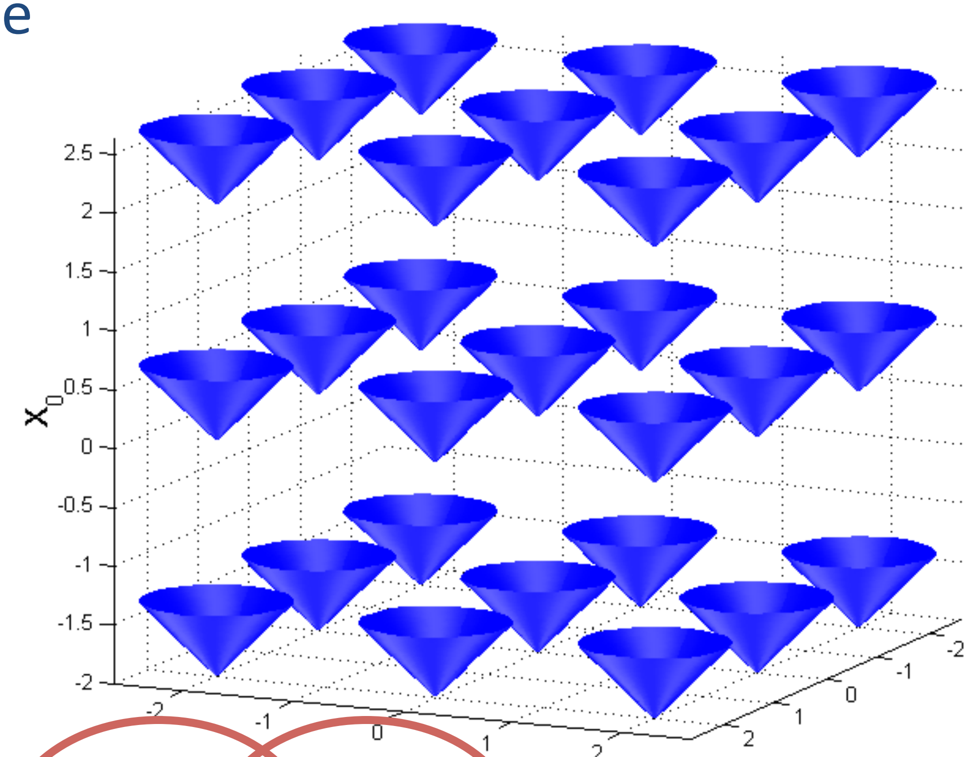
2) The spectral condition implies that the integral representation makes sense in a larger complex domain of the complex Minkowski space-time

$z \in T^- \ (\mathrm{Im}\, z \in V^-)$

$z' \in T^+ \ (\mathrm{Im}\, z' \in V^+)$

$z' - z \in T^+$

$(\mathrm{Im}(z' - z) \in V^+)$

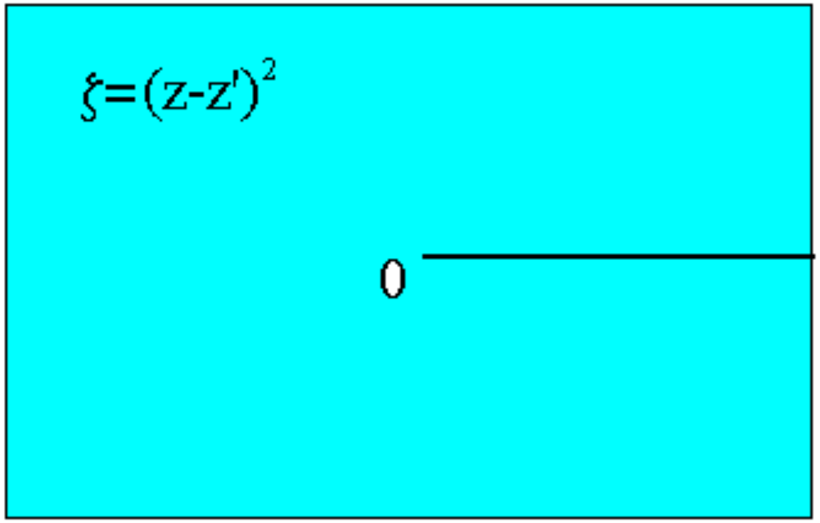


$$W(z - z') = \int e^{-ip\cdot z} e^{ip\cdot z'} \theta(p^0)\delta(p^2 - m^2)$$

Maximal analyticity

Spectral Property + Lorentz invariance = maximal analyticity

$W(z,z') = W(z-z') = W(\zeta)$ is maximally analytic in the complex Lorentz invariant variable $\zeta = (z-z')^2$:



The cut reflects causality and QM

Klein-Gordon fields II

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$$\Box\phi(x) + m^2\phi(x) = 0$$
$$C(x,y) = \frac{1}{(2\pi)^3}\int e^{-ip(x-y)}[\theta(p^0) - \theta(-p^0)]\delta(p^2 - m^2)dp$$

The fundamental split equation

$$C(x,y) = W(x,y) - W(y,x)$$
$$\tilde{C}(p) = \tilde{W}(p) - \tilde{W}(-p)$$

Has been solved according with spectral condition

$$\tilde{W}(p) = \theta(p^0)\delta(p^2 - m^2) \quad \tilde{W}(-p) = \theta(-p^0)\delta(p^2 - m^2)$$
$$\tilde{C}(p) = \tilde{W}(p) - \tilde{W}(-p) = \epsilon(p^0)\delta(p^2 - m^2)$$

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$$\Box\phi(x) + m^2\phi(x) = 0$$
$$C(x,y) = \frac{1}{(2\pi)^3}\int e^{-ip(x-y)}[\theta(p^0) - \theta(-p^0)]\delta(p^2 - m^2)dp$$

$$C(x,y) = W(x,y) - W(y,x) \qquad \tilde{C}(p) = \tilde{W}(p) - \tilde{W}(-p)$$

$$\tilde{W}(p) = (\alpha(p)\theta(p^0) + \gamma(p)\theta(-p^0))\delta(p^2 - m^2)$$
$$\tilde{W}(-p) = (\alpha(-p)\theta(-p^0) + \gamma(-p)\theta(p^0))\delta(p^2 - m^2)$$

$$\alpha(p) - \gamma(-p) = 1, \qquad \alpha(-p) - \gamma(p) = 1$$

- Immediate (trivial) solution: α and γ constant such that $\alpha - \gamma = 1$

$$\tilde{W}_\gamma(p) = [(1+\gamma)\theta(p^0) + \gamma\theta(-p^0)]\delta(p^2 - m^2)$$

- It defines an inequivalent local and covariant quantization of the KG field; negative energy states are present. The representation is not irreducible.

Klein-Gordon fields II ©

$$\tilde{W}_\gamma(p) = [(1+\gamma)\theta(p^0) + \gamma\theta(-p^0)]\delta(p^2 - m^2)$$

$$\gamma = \frac{1}{e^\beta - 1} \qquad \alpha = 1 + \gamma = \frac{e^\beta}{e^\beta - 1} = \frac{1}{1 - e^{-\beta}}$$

$$\gamma(p) = \frac{1}{e^{-\beta p^0} - 1} \qquad \alpha(p) = \frac{1}{1 - e^{-\beta p^0}}$$

$$\alpha(p) - \gamma(-p) = \frac{1}{1 - e^{-\beta p^0}} - \frac{1}{e^{\beta p^0} - 1} = 1$$

$$\alpha(-p) - \gamma(p) = \frac{1}{1 - e^{\beta p^0}} - \frac{1}{e^{-\beta p^0} - 1} = 1$$

- Non trivial solution to the split equation:

$$\tilde{W}_\beta(p) = \left[\frac{\theta(p^0)}{1 - e^{-\beta p^0}} + \frac{\theta(-p^0)}{e^{-\beta p^0} - 1}\right]\delta(p^2 - m^2)$$

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$$\Box\phi(x) + m^2\phi(x) = 0$$
$$C(x,y) = \frac{1}{(2\pi)^3}\int e^{-ip(x-y)}[\theta(p^0) - \theta(-p^0)]\delta(p^2 - m^2)dp$$

$$\tilde{W}_\beta(p) = \left[\frac{\theta(p^0)}{1 - e^{-\beta p^0}} + \frac{\theta(-p^0)}{e^{-\beta p^0} - 1}\right]\delta(p^2 - m^2) = \frac{\epsilon(p^0)\delta(p^2 - m^2)}{1 - e^{-\beta p^0}}$$

$$W_\beta(x,y) = \frac{1}{(2\pi)^3}\int e^{-ip(x-y)}\left[\frac{1}{1 - e^{-\beta p^0}}\right]\epsilon(p^0)\delta(p^2 - m^2)dp$$

$$W_\beta(x,y) - W_\beta(y,x) = C(x,y)$$

- Positive definiteness holds.
- Stationary local quantization of the KG field depending on a positive parameter β.
- Lorentz invariance is broken. There exists a preferred class of referentials.

Klein-Gordon fields II ©

- Two crucial properties: analyticity

$$W_\beta(x,y) = \frac{1}{(2\pi)^3}\int e^{-ip^0(t-s)+i\mathbf{p}(\mathbf{x}-\mathbf{y})}\left[\frac{1}{1-e^{-\beta p^0}}\right]\epsilon(p^0)\delta(p^2-m^2)dp$$

analytic in the strip $-\beta < \mathrm{Im}(t-s) < 0$

$$W_\beta(y,x) = \frac{1}{(2\pi)^3}\int e^{+ip^0(t-s)-i\mathbf{p}(\mathbf{x}-\mathbf{y})}\left[\frac{1}{1-e^{-\beta p^0}}\right]\epsilon(p^0)\delta(p^2-m^2)dp$$

analytic in the strip $0 < \mathrm{Im}(t-s) < \beta$

- Periodicity in imaginary time

$$\begin{aligned}
W_\beta(s,\mathbf{y},t+i\beta,\mathbf{x}) &= \frac{1}{(2\pi)^3}\int e^{+ip^0(t+i\beta-s)-i\mathbf{p}(\mathbf{x}-\mathbf{y})}\left[\frac{1}{1-e^{-\beta p^0}}\right]\epsilon(p^0)\delta(p^2-m^2)dp \\
&= \frac{1}{(2\pi)^3}\int e^{+ip^0(t-s)-i\mathbf{p}(\mathbf{x}-\mathbf{y})}\left[\frac{e^{-\beta p^0}}{1-e^{-\beta p^0}}\right]\epsilon(p^0)\delta(p^2-m^2)dp \\
&= \frac{1}{(2\pi)^3}\int e^{+ip^0(t-s)-i\mathbf{p}(\mathbf{x}-\mathbf{y})}\left[\frac{1}{e^{\beta p^0}-1}\right]\epsilon(p^0)\delta(p^2-m^2)dp \\
(p^0\to -p^0) \quad &= \frac{1}{(2\pi)^3}\int e^{-ip^0(t-s)+i\mathbf{p}(\mathbf{x}-\mathbf{y})}\left[\frac{1}{1-e^{-\beta p^0}}\right]\epsilon(p^0)\delta(p^2-m^2)dp \\
&= W_\beta(t,\mathbf{x},s,\mathbf{y})
\end{aligned}$$

KMS condition ©

- Consider a quantum system confined to a compact subset of space.

Its time-evolution is generated by a self-adjoint Hamiltonian H on a Hilbert space $\mathcal{H}$.
The energy spectrum of H is discrete.

- $Q_I,\ldots,Q_N$ self-adjoint operators on $\mathcal{H}$ representing conserved quantities and commuting with all ``observables''.

- $\mu_I,\ldots,\mu_N$ denote the chemical potentials conjugate to the conserved quantities.

- The state describing thermal equilibrium at inverse temperature β and chemical potentials $\mu_I,\ldots,\mu_N$ is given by the density matrix (Gibbs, Landau, von Neumann)

$$\rho_{\beta,\mu} := Z_{\beta,\mu}^{-1} \exp[-\beta H_\mu], \qquad \langle A \rangle_{\beta,\underline{\mu}} := \mathrm{tr}_{\mathcal{H}}[\rho_{\beta,\mu} A]$$

$$H_\mu := H - \sum_{i=1}^{N} \mu_i Q_i, \qquad Z_{\beta,\mu} = \mathrm{tr}_{\mathcal{H}}[e^{-\beta H_\mu}]$$

KMS condition ©

$$\rho_{\beta,\mu} := Z_{\beta,\mu}^{-1}\exp[-\beta H_\mu], \quad \langle A\rangle_{\beta,\underline{\mu}} := \mathrm{tr}_{\mathcal{H}}[\rho_{\beta,\mu}A]$$

- Time evolution in the Heisenberg representation

$$\alpha_t(A) := e^{itH}Ae^{-itH} = e^{itH_\mu}Ae^{-itH_\mu}$$

- $<\alpha_t(A)\ B>_{\beta,\mu}$ analytic in the strip $-\beta < \mathrm{Im}\ t < 0$

- $<B\alpha_t(A)>_{\beta,\mu}$ analytic in the strip $0 < \mathrm{Im}\ t < \beta$

- Cyclicity of the trace implies the famous KMS periodicity condition

$$\langle\alpha_t(A)B\rangle_{\beta,\mu} = \langle B\alpha_{t+i\beta}(A)\rangle_{\beta,\mu}$$

Klein-Gordon fields II ©

- KMS Thermal equilibrium quantization at inverse temperature β=1/T:

$$W_\beta(x,y) = \frac{1}{(2\pi)^3}\int e^{-ip^0(t-s)+i\mathbf{p}(\mathbf{x}-\mathbf{y})}\left[\frac{1}{1-e^{-\beta p^0}}\right]\epsilon(p^0)\delta(p^2-m^2)dp$$

$$W_\beta(x,y) \text{ analytic in the strip } -\beta < \mathrm{Im}(t-s) < 0$$

$$W_\beta(y,x) \text{ analytic in the strip } \quad 0 < \mathrm{Im}(t-s) < \beta$$

$$W_\beta(s,\mathbf{y},t+i\beta,\mathbf{x}) = W_\beta(t,\mathbf{x},s,\mathbf{y})$$

- For every β=1/T we have an inequivalent canonical quantization:

$$\Box\widehat{\phi}_\beta(x) + m^2\widehat{\phi}_\beta(x) = 0$$

$$[\widehat{\phi}_\beta(x),\widehat{\phi}_\beta(y)] = \frac{1}{(2\pi)^3}\int(\theta(p^0)-\theta(-p^0))\delta(p^2-m^2)d^4p$$

$$[\widehat{\phi}_\beta(t_0,\mathbf{x}),\, \widehat{\pi}_\beta(t_0,\mathbf{y})] = i\hbar\delta(\mathbf{x}-\mathbf{y})$$

$$[\widehat{\phi}_\beta(t_0,\mathbf{x}),\, \widehat{\phi}_\beta(t_0,\mathbf{y})] = [\widehat{\pi}_\beta(t_0,\mathbf{x}),\, \widehat{\pi}_\beta(t_0,\mathbf{y})] = 0.$$

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$$\Box\phi(x) + m^2\phi(x) = 0$$
$$[\phi(t_0, \mathbf{x}), \pi(t_0, \mathbf{y})] = i\hbar\delta(\mathbf{x} - \mathbf{y})$$
$$[\phi(t_0, \mathbf{x}), \phi(t_0, \mathbf{y})] = [\pi(t_0, \mathbf{x}), \pi(t_0, \mathbf{y})] = 0.$$

$$W(x, y) = \frac{1}{(2\pi)^3}\int e^{-i\omega(x^0 - y^0) + i\mathbf{p}(\mathbf{x}-\mathbf{y})}\frac{d^3\mathbf{p}}{2\omega}$$

$$= \int \frac{e^{-i\omega x^0 + i\mathbf{p}\mathbf{x}}}{\sqrt{(2\pi)^3 2\omega}}\ \frac{e^{i\omega y^0 - i\mathbf{p}\mathbf{y}}}{\sqrt{(2\pi)^3 2\omega}}\ d^3\mathbf{p} \qquad \omega = \sqrt{\mathbf{p}^2 + m^2}$$

The function $u_{\mathbf{p}}(t, \mathbf{x}) = \frac{e^{-i\omega x^0 + i\mathbf{p}\mathbf{x}}}{\sqrt{(2\pi)^3 2\omega}}$ is a complex classical solution of the KG equation:

$$\Box u_{\mathbf{p}}(t, \mathbf{x}) = (-\omega^2 + \mathbf{p}^2)u_{\mathbf{p}}(t, \mathbf{x}) = -m^2 u_{\mathbf{p}}(t, \mathbf{x})$$

$$W(x, y) = \int u_{\mathbf{p}}(t, x)u^*_{\mathbf{p}}(s, y)d^3\mathbf{p}$$

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$$(\phi_1, \phi_2) = -i\Omega(\overline{\phi}_1, \phi_2) = i\int_{t=const} [\overline{\phi}_1(t,\mathbf{x})\pi_2(t,\mathbf{x}) - \overline{\pi}_1(t,\mathbf{x})\phi_2(t,\mathbf{x})] d^3\mathbf{x}$$

$$= i\int_{t=const} \overline{\phi}_1(t,\mathbf{x})\; i\overleftrightarrow{\partial_t}\; \phi_2(t,\mathbf{x}) d^3\mathbf{x}$$

$$(u_{\mathbf{p}}, u_{\mathbf{p}'}) = i\int \left[\frac{e^{i\omega t - i\mathbf{p}\mathbf{x}}}{\sqrt{(2\pi)^3 2\omega}}\,(-i\omega')\frac{e^{-i\omega' t + i\mathbf{p}'\mathbf{x}}}{\sqrt{(2\pi)^3 2\omega'}} - i\omega\frac{e^{i\omega t - i\mathbf{p}\mathbf{x}}}{\sqrt{(2\pi)^3 2\omega}}\;\frac{e^{-i\omega' t + i\mathbf{p}'\mathbf{x}}}{\sqrt{(2\pi)^3 2\omega'}}\right] d^3\mathbf{x} = \delta(\mathbf{p} - \mathbf{p}')$$

$$(\overline{u}_{\mathbf{p}}, \overline{u}_{\mathbf{p}'}) = i\int \left[\frac{e^{-i\omega t + i\mathbf{p}\mathbf{x}}}{\sqrt{(2\pi)^3 2\omega}}\,(i\omega')\frac{e^{i\omega' t - i\mathbf{p}'\mathbf{x}}}{\sqrt{(2\pi)^3 2\omega'}} + i\omega\frac{e^{-i\omega t + i\mathbf{p}\mathbf{x}}}{\sqrt{(2\pi)^3 2\omega}}\;\frac{e^{i\omega' t - i\mathbf{p}'\mathbf{x}}}{\sqrt{(2\pi)^3 2\omega'}}\right] d^3\mathbf{x} = -\delta(\mathbf{p} - \mathbf{p}')$$

$$(u_{\mathbf{p}}, \overline{u}_{\mathbf{p}'}) = i\int \left[\frac{e^{i\omega t - i\mathbf{p}\mathbf{x}}}{\sqrt{(2\pi)^3 2\omega}}\,(i\omega')\frac{e^{i\omega' t - i\mathbf{p}'\mathbf{x}}}{\sqrt{(2\pi)^3 2\omega'}} - i\omega\frac{e^{i\omega t - i\mathbf{p}\mathbf{x}}}{\sqrt{(2\pi)^3 2\omega}}\;\frac{e^{i\omega' t - i\mathbf{p}'\mathbf{x}}}{\sqrt{(2\pi)^3 2\omega'}}\right] d^3\mathbf{x} = 0$$

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Consider the space $S^{\mathbf{C}}$ of complex classical solution of the KG equation. Introduce a sesquilinear (symplectic) form

$$(f,g) = i\int_{t=const} \bar{f}(t,\mathbf{x}) \overleftrightarrow{\partial_t} g(t,\mathbf{x}) d^3\mathbf{x}$$

The complex solutions $u_{\mathbf{p}}(x) = \frac{e^{-i\omega t + i\mathbf{p}\mathbf{x}}}{\sqrt{(2\pi)^3 2\omega}}$ are a "basis" for $S^{\mathbf{C}}$ in the following sense:

$(u_{\mathbf{p}}, u_{\mathbf{p}'}) = \delta(\mathbf{p}-\mathbf{p}'), \quad (\bar{u}_{\mathbf{p}}, \bar{u}_{\mathbf{p}'}) = -\delta(\mathbf{p}-\mathbf{p}'), \quad (u_{\mathbf{p}}, \bar{u}_{\mathbf{p}'}) = 0$

The functions $u_{\mathbf{p}}(x)$ are "positive frequency" in the following sense: $i\partial_t u_{\mathbf{p}} = \omega u_{\mathbf{p}}$

The two-point function is a superposition of all the "positive frequency" solutions and their complex conjugates:

$$W(x,y) = \int u_{\mathbf{p}}(x)\overline{u_{\mathbf{p}}(y)} d^3\mathbf{p}$$

Klein-Gordon fields III ©

- The structure of the 2-point function guarantees canonicity:

$$W(x,y) = \int u_{\mathbf{p}}(x)\overline{u_{\mathbf{p}}(y)}d^3\mathbf{p}$$

$$(u_{\mathbf{p}}, u_{\mathbf{p}'}) = \delta(\mathbf{p}-\mathbf{p}'),$$
$$(\bar{u}_{\mathbf{p}}, \bar{u}_{\mathbf{p}'}) = -\delta(\mathbf{p}-\mathbf{p}'),$$
$$(u_{\mathbf{p}}, \bar{u}_{\mathbf{p}'}) = 0$$

$$C(x,y) = W(x,y) - W(y,x) = \int [u_{\mathbf{p}}(x)\overline{u_{\mathbf{p}}(y)} - u_{\mathbf{p}}(y)\overline{u_{\mathbf{p}}(x)}]d^3\mathbf{p}$$

$$\frac{\partial C}{\partial y^0}(x,y)\Big|_{x^0=y^0} = \int \left[u_{\mathbf{p}}(x)\overline{\frac{\partial u_{\mathbf{p}}(y)}{\partial y^0}} - \frac{\partial u_{\mathbf{p}}(y)}{\partial y^0}\overline{u_{\mathbf{p}}(x)}\right]_{x^0=y^0} d^3\mathbf{p} = i\delta^3(\mathbf{x}-\mathbf{y})$$

- The 2-point function is a map into the positive frequency subspace of $S^{\mathbf{C}}$

$$f \in C_0^\infty(\mathbf{M}^4) \to W{*}f = \int W(x,y)f(y)d^4y = \int u_{\mathbf{p}}(x)\overline{u_{\mathbf{p}}(y)}f(y)d^4y d^3\mathbf{p}$$

- The permuted 2-point function is a map into the negative frequency subspace of $S^{\mathbf{C}}$

$$f \in C_0^\infty(\mathbf{M}^4) \to W'{*}f = \int W(y,x)f(y)d^4y = \int \overline{u_{\mathbf{p}}(x)}u_{\mathbf{p}}(y)f(y)d^4y d^3\mathbf{p}$$

Bogoliubov Transformations ©

- A basis in the space of complex solutions of the KG equation on a manifold M

$$(u_i, u_j) = \delta_{ij}, \quad (\bar{u}_i, \bar{u}_j) = -\delta_{ij}, \quad (u_i, \bar{u}_j) = 0$$

- Another basis $(v_i, v_j) = \delta_{ij}, \quad (\bar{v}_i, \bar{v}_j) = -\delta_{ij}, \quad (v_i, \bar{v}_j) = 0$

- Completeness formally gives

$$v_i(x) = \sum [a_{ij} u_j(x) + b_{ij}\, \bar{u}_j(x)] \qquad u_j(x) = \sum [\bar{a}_{ij} v_i(x) - b_{ij}\, \bar{v}_i(x)]$$

$$(v_i, u_k) = a_{ik}, \quad (v_i, \bar{u}_k) = -b_{ik},$$

$$v_i(x) = \sum [(a_{ij}\bar{a}_{lj} - b_{ij}\bar{b}_{lj}) v_l(x) - (a_{ij} b_{lj} - b_{ij} a_{lj})\, \bar{v}_l(x)]$$

$$(a_{ij}\bar{a}_{lj} - b_{ij}\bar{b}_{lj}) = \delta_{ij}, \quad (a_{ij} b_{lj} - b_{ij} a_{lj}) = 0$$

Bogoliubov Transformations

- The abstract quantum field in terms of ladder operators (formally)

$$\phi(x) = \sum [u_i \, b_i + \bar{u}_i \, b_i^\dagger] = \sum [v_i \, a_i + \bar{v}_i \, a_i^\dagger]$$

$$b_i = \sum \, [a_{ji} \, a_j + \bar{b}_{ji} \, a_j^\dagger] \qquad\qquad a_i = \sum \, [\bar{a}_{ij} \, b_j - \bar{b}_{ji} \, b_j^\dagger]$$

- Choose the corresponding Fock vacua

$$W(x,y) = \langle \Psi_0, \widehat{\phi}(x)\widehat{\phi}(y)\Psi_0 \rangle_b = \sum_i u_i(x)\overline{u_i(y)}$$

$$W_{a,b}(x,y) = \langle \Psi_0, \widehat{\phi}(x)\widehat{\phi}(y)\Psi_0 \rangle_a = \sum v_i(x)\bar{v}_j(y) =$$
$$\sum \, [a_{ij} u_j(x) + b_{ij} \, \bar{u}_j(x)][\bar{a}_{il}\bar{u}_l(y) + \bar{b}_{il} \, u_l(y)],$$

- If the quantizations are unitarily equivalent the Bogoliubov transformation is implementable. The matrices a and b must be Hilbert-Schmidt.
- Otherwise the quantizations are inequivalent.