Sheaves and D-modules on Lorentzian manifolds

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Normal cones

A manifold means a real C^∞-manifold and a morphism of manifolds $f\colon M \to N$ is a map of class C^∞. For any subset A, we denote by $\overline{A}$ its closure, by $\mathrm{Int}(A)$ its interior and we set $\partial A = \overline{A} \setminus \mathrm{Int}(A)$.

Let A, B be two subsets of M. The *Whitney cone* $C(A, B)$ is a closed conic subset of TM. In a chart, it is described as follows.

$$\begin{cases} v \in C_{x_0}(A,B) \subset T_{x_0}M \\ \text{if and only if} \\ \text{there exists a sequence } \{(x_n, y_n, \lambda_n)\}_n \subset A \times B \times \mathbb{R}_{>0} \\ \text{such that} \\ x_n \xrightarrow{n} x_0,\ y_n \xrightarrow{n} x_0,\ \lambda_n(x_n - y_n) \xrightarrow{n} v. \end{cases}$$

For N a smooth submanifold of M, one denotes by $C_N(A)$ the image of $N \times_M C(A, N)$ in $T_N M = (N \times_M TM)/TN$.

Let A be a subset of M. The *strict normal cone* of A is is an open convex cone of TM defined by

$$N(A) = TM \setminus C(M \setminus A, A).$$

In a local chart of M,

$$(x,v) \in N(A) \Leftrightarrow \begin{cases} \text{there exists an open cone } \gamma_0 \text{ with } v \in \gamma_0 \\ \text{and an open neighborhood } U \text{ of } x \text{ such that} \\ U \cap (U \cap A + \gamma_0) \subset A. \end{cases}$$

Preorders

We denote by Δ_M, or simply Δ, the diagonal of $M \times M$.
Let M_i $(i = 1, 2, 3)$ be manifolds. For short, we write $M_{ij} := M_i \times M_j$ $(1 \leq i, j \leq 3)$ and $M_{123} = M_1 \times M_2 \times M_3$. We denote by q_i the projection $M_{ij} \to M_i$ or the projection $M_{123} \to M_i$ and by q_{ij} the projection $M_{123} \to M_{ij}$. For $A_1 \subset M_{12}$ and $A_2 \subset M_{23}$, one sets

$$A_1 \underset{2}{\circ} A_2 = q_{13}(q_{12}^{-1} A_1 \cap q_{23}^{-1} A_2).$$

Consider a preorder $\preceq$ on a manifold M and its graph $\Delta_{\preceq} \subset M \times M$. Then

$$\Delta \subset \Delta_{\preceq}\,,$$
$$\Delta_{\preceq} \circ \Delta_{\preceq} = \Delta_{\preceq}\,.$$

For a subset $A \subset M$, one sets

$$\begin{cases} J^-_{\preceq}(A) = q_1(q_2^{-1}(A) \cap \Delta_{\preceq}) = \{x \in M; \exists y \in A \text{ with } x \preceq y\}, \\ J^+_{\preceq}(A) = q_2(q_1^{-1}(A) \cap \Delta_{\preceq}) = \{x \in M; \exists y \in A \text{ with } y \preceq x\}. \end{cases}$$

For $x \in M$, we write $J^+_{\preceq}(x)$ and $J^-_{\preceq}(x)$ instead of $J^+_{\preceq}(\{x\})$ and $J^-_{\preceq}(\{x\})$ respectively. One calls $J^-_{\preceq}(A)$ (resp. $J^+_{\preceq}(A)$) the past (resp. future) of A for the preorder $\preceq$.

Let $\preceq$ be a preorder on M. The next results are obvious:

- $J^-_{\preceq}(A) = \bigcup_{x \in A} J^-_{\preceq}(x)$, and similarly with $J^+_{\preceq}(A)$,
- $A \subset J^-_{\preceq}(A)$, $J^-_{\preceq}(J^-_{\preceq}(A)) = J^-_{\preceq}(A)$ and similarly with $J^+_{\preceq}(A)$,
- $A = J^+_{\preceq}(A) \Leftrightarrow M \setminus A = J^-_{\preceq}(M \setminus A)$.

Definition

(a) The preorder is *closed* if $\Delta_{\preceq}$ is closed in $M \times M$.

(b) The preorder is *proper* if q_{13} is proper on $\Delta_{\preceq} \times_M \Delta_{\preceq}$. Equivalently, for any two compact subsets A and B of M, the so-called *causal diamond* $J^+_{\preceq}(A) \cap J^-_{\preceq}(B)$ is compact.

- If $\preceq$ is closed and A is a compact subset of M, then $J^-_{\preceq}(A)$ and $J^+_{\preceq}(A)$ are closed.
- If $\preceq$ is proper, then it is closed.

Definition

(a) A *causal manifold* (M, γ) is a manifold M equipped with an open convex cone $\gamma \subset TM$ such that $\gamma_x \neq \varnothing$ for all $x \in M$.

(b) A *morphism of causal manifolds* $f : (M, \gamma_M) \to (N, \gamma_N)$ is a morphism of manifolds such that $Tf(\overline{\gamma_M}) \subset \overline{\gamma_N}$.

(c) A morphism of causal manifolds f is *strict* if $Tf(\gamma_M) \subset \gamma_N$.

Causal manifolds and their causal (resp. strictly causal) morphisms form a category.
For U a open subset of M, $(U, \gamma|_U)$ is a causal manifold and the embedding $U \hookrightarrow M$ induces a morphism of causal manifolds.

Notation

For an open interval I of $\mathbb{R}$ (which we will implicitly assume to contain $[0, 1]$) we simply denote by $(I, +)$ the causal manifold $(I, I \times \mathbb{R}_{>0})$.

Example: Lorentzian manifolds

A *Lorentzian manifold* (M, g) is a *connected* C^∞-manifold M with a C^∞ nondegenerate bilinear form g on M of signature $(+, -, \dots, -)$. Let

$$g_{>0} = \{(x; v) \in TM; g_x(v, v) > 0\}.$$

Then $g_{>0}$ has at most two connected components. The Lorentzian manifold (M, g) is *time-orientable* if the cone $g_{>0}$ has two connected components. It is *time-oriented* if furthermore one connected component has been chosen. In this case, it defines a causal manifold denoted by (M, γ_g), or simply (M, γ).

Definition

A *Lorentzian spacetime* is a connected time-oriented Lorentzian manifold.

In our study, we shall simply ask that γ is an open convex cone, non empty at each $x \in M$. We don't ask any regularity on γ.

γ-sets and γ-topology

Let (M, γ) be a causal manifold.
(i) A *constant cone* in γ is a triple (φ, U, θ) where $\varphi\colon U \to \mathbb{R}^d$ is a chart and $\theta \subset \mathbb{R}^d$ is an open convex cone, such that in this chart, $U \times \theta \subset \gamma$ (that is, $\varphi(U) \times \theta \subset T\varphi(\gamma|_U)$). A constant cone (φ, U, θ) will often be denoted simply by $U \times \theta$.
(ii) A *basis of constant cones* contained in γ is a family of constant cones whose union is γ.
(iii) A subset $A \subset M$ is a γ*-set* if $\gamma \subset N(A)$. Equivalently, there exists a basis of constant cones $U \times \theta$ contained in γ such that $U \cap (U \cap A + \theta) \subset A$.

- The family of γ-sets is closed under arbitrary unions and intersections and under taking closure and interior.
- If A is a γ-set, then $\overline{\mathrm{Int} A} = \overline{A}$ and $\mathrm{Int}\overline{A} = \mathrm{Int} A$.
- If A is a γ-set and $\mathrm{Int} A \subset B \subset \overline{A}$, then B is a γ-set.

The preceding results allows us to generalize the notion of γ-topology of [KS90] in which M was affine and the cone was constant.

Definition

Let (M, γ) be a causal manifold. The γ-*topology* on M is the topology for which the open sets are the open sets of M which are γ-sets.

A subset $A \subset M$ is called γ-open if it is open for the γ-topology. In other words, if it is open in the usual topology and is a γ-set.

Remark

A set which is closed for the γ-topology is not in general a γ-set, but is a γ^a-set.

The chronological preorder

Definition

For $A \subset M$, we denote by $I_\gamma^+(A)$ the intersection of all the γ-sets which contain A and call it the *chronological future* of A.

Note that a set A is a γ-set if and only if $I_\gamma^+(A) = A$.
One easily checks that

The relation $y \in I_\gamma^+(\{x\})$ is a preorder.

Definition

We denote by $\preceq_\gamma$ the preorder given by $x \preceq_\gamma y$ if $y \in I_\gamma^+(x)$ and we denote by Δ_γ its graph. Hence, $I_\gamma^+(x) = J_{\preceq_\gamma}^+(x)$ and $\Delta_\gamma = \Delta_{\preceq_\gamma}$.
We call $\preceq_\gamma$ the *chronological preorder*.

On $(I, +)$ the chronological preorder $\preceq_\gamma$ is the usual order $\leq$.

Causal paths

Definition

A path $c\colon I \to M$ is a piecewise smooth map. A path c is *causal* if $c_l'(t), c_r'(t) \in (\overline{\gamma})_{c(t)}$ for any $t \in I$ and it is *strictly causal* if $c_l'(t), c_r'(t) \in \gamma_{c(t)}$ for any $t \in I$.

- if c_1 and c_2 are two (strictly) causal paths with $c_1(1) = c_2(0)$, the concatenation $c = c_1 \cup c_2$ is (strictly) causal.
- Let $f\colon (M, \gamma_M) \to (N, \gamma_N)$ be a morphism of causal manifolds and let $c\colon I \to M$ be a causal path. Then $f \circ c\colon I \to N$ is a causal path and similarly with strictly causal.
- The piecewise smooth preorder, ps-preorder for short, is defined by $x \preceq_{\mathrm{ps}} y$ if there is a causal path c with $c(0) = x$, $c(1) = y$.

Causal preorders

Let (M, γ) be a causal manifold and let $\preceq$ be a preorder on M. The following assertions are equivalent:

(i) One has $\Delta_\gamma \subset \Delta_\preceq$.

(ii) $\Delta_\preceq$ is a $(\gamma^a \times \gamma)$-set,

(iii) For any $x \in M$, $J^+_\preceq(x)$ is a γ-set.

(iv) For any $y \in M$, $J^-_\preceq(y)$ is a γ^a-set.

(v) For any $x \in M$, $I^+_\gamma(x) \subset J^+_\preceq(x)$.

Definition

A preorder $\preceq$ is *causal* if the equivalent conditions above are satisfied.

The cc and the ps preorders

Graphs of transitive relations, closed sets, and γ-sets in a causal manifold, are all closed under intersections.

Definition

(i) The canonical closed causal preorder, the cc-preorder for short, is defined as follows. Its graph Δ_{cc} is the intersection of all graphs of closed causal preorders. One denotes by $J^+_{\mathrm{cc}}(A)$ and $J^-_{\mathrm{cc}}(A)$ the future and past sets of A for the cc-preorder.
(ii) The piecewise smooth preorder, the ps-preorder for short, is given by $x \preceq_{\mathrm{ps}} y$ if there exists a causal path c with $c(0) = x$ and $c(1) = y$. One denotes by Δ_{ps} its graph and one denotes by $J^+_{\mathrm{ps}}(A)$ and $J^-_{\mathrm{ps}}(A)$ the future and past sets of A for the ps-preorder.

The cc-preorder and the ps-prorder are causal:

$$\Delta_\gamma \subset \Delta_{\mathrm{cc}} \text{ and } \Delta_\gamma \subset \Delta_{\mathrm{ps}}.$$

Globally hyperbolic spacetimes

Let (M, g) be a Lorentzian spacetime and let (M, γ) be the associated causal manifold.

(a) One has

$$\Delta_\gamma \subset \Delta_{\mathrm{ps}} \subset \overline{\Delta_\gamma} \subset \Delta_{\mathrm{cc}}.$$

(b) The preoreder Δ_{ps} is a proper order if and only if the preorder Δ_{cc} is a proper order and in this case, one has $\overline{\Delta_\gamma} = \Delta_{\mathrm{ps}} = \Delta_{\mathrm{cc}}$.

One shall be aware that the inclusion $\overline{\Delta_\gamma} \subset \Delta_{\mathrm{cc}}$ may be strict since $\overline{\Delta_\gamma}$ is not necessarily transitive, even in Lorentzian spacetimes.
We now extend the classical definition of global hyperbolicity of Lorentzian spacetimes to general causal manifolds as follows:

Definition

A causal manifold (M, γ) is *globally hyperbolic* if Δ_{cc} is a proper order.

Example

Let $M = \mathbb{R}^2 \setminus \{(1,0)\}$ and $\gamma = M \times (\mathbb{R}_{>0})^2$. Then (M, γ) is a causal manifold. One easily checks that

$$\begin{aligned} I_\gamma^+((0,0)) &= \{(0,0)\} \cup (\mathbb{R}_{>0})^2, \\ J_{\mathrm{ps}}^+((0,0)) &= (\mathbb{R}_{\geq 0})^2 \setminus \big([1,+\infty) \times \{0\}\big), \\ J_{\mathrm{cc}}^+((0,0)) &= \overline{I_\gamma^+((0,0))} = (\mathbb{R}_{\geq 0})^2 \setminus \{(1,0)\}. \end{aligned}$$

In particular, $J_{\mathrm{ps}}^+((0,0))$ is neither closed nor open.

Cauchy time functions and G-causal manifolds

The terminology G-causal below is not inspired by gravitation but by the name of Geroch.

Definition

(a) A *Cauchy time function* on a causal manifold (M, γ) is a submersive causal morphism $q\colon (M, \gamma) \to (\mathbb{R}, +)$ which is proper on the sets $J^+_{\mathrm{cc}}(K)$ and $J^-_{\mathrm{cc}}(K)$ for any compact set $K \subset M$. (One proves that is is enough to assume that q is proper on the sets $J^+_{\mathrm{cc}}(x)$ and $J^-_{\mathrm{cc}}(x)$ for any $x \in M$.)
(b) A *G-causal manifold* (M, γ, q) is the data of a causal manifold (M, γ) together with a Cauchy time function q.

- A Cauchy time function on a causal manifold (M,γ) is strictly causal and is increasing as a function from $(M,\preceq_{\mathrm{cc}})$ to $(\mathbb{R},\leq)$. In particular, it has no strictly causal loops.
- A Cauchy time function is strictly increasing on strictly causal paths.
- If a causal manifold admits a Cauchy time function, then its cc-preorder is proper.
- Let q be a Cauchy time function on (M,γ) and let $x \in M$. Then $q(I_\gamma^+(x)) = q(J_{\mathrm{cc}}^+(x)) = [q(x), +\infty)$. In particular, G-causal manifolds cannot be compact and Cauchy time functions are surjective.

Example

Let $M = \mathbb{S}^1 \times \mathbb{R}$ and $\gamma = T\mathbb{S}^1 \times \{(t; v) \in T\mathbb{R}; v > 0\}$. The map $q\colon M \to \mathbb{R}, (x, t) \mapsto t$, is a Cauchy time function on (M, γ). Denote by x a coordinate on $\mathbb{S}^1$ (hence, $x + 2\pi = x$). The path $[0, 2\pi] \ni s \mapsto (s, 0) \in M$ is a causal loop.

Theorem

If a Lorentzian spacetime is globally hyperbolic, then it admits a Cauchy time function.

This follows from the results of R. Geroch (1970) and Minguzzi-Sánchez (2008). See also Fathi-Siconolfi (2011) for a more general version.

We denote by $\mathbf{k}$ a field and let X be a topological space.
A presheaf F on X associates to each open subset $U \subset X$ a $\mathbf{k}$-module $F(U)$, and to an open inclusion $V \subset U$, a linear map, called the restriction map, $\rho_{VU}\colon F(U) \to F(V)$, such that for each open inclusions $W \subset V \subset U$, one has:

$$\rho_{UU} = \mathrm{id}_U, \qquad \rho_{WU} = \rho_{WV} \circ \rho_{VU}.$$

A morphism of presheaves $\varphi\colon F \to G$ is the data for any open set U of a linear map $\varphi(U)\colon F(U) \to G(U)$ such that for any open inclusion $V \subset U$, the diagram below commutes:

$$\begin{tikzcd} F(U) \arrow[r, "\varphi(U)"] \arrow[d] & G(U) \arrow[d] \\ F(V) \arrow[r, "\varphi(V)"] & G(V) \end{tikzcd}$$

For $s \in F(U)$ one writes $s|_V$ instead of $\rho_{VU}(s)$.

A presheaf is a sheaf if it satisfies the condition

for any open subset $U \subset X$, any open covering $U = \bigcup_i U_i$, any family $\{s_i \in F(U_i), i \in I\}$ satisfying $s_i|_{U_{ij}} = s_j|_{U_{ij}}$ for all i, j, there exists a unique $s \in F(U)$ with $s|_{U_i} = s_i$ for all i.

Example

(i) The presheaf $\mathscr{C}^0_X$ is a sheaf.
(ii) The presheaf $\mathbf{k}_X$ of locally constant $\mathbf{k}$-valued functions on X is a sheaf, called the constant sheaf, and denoted $\mathbf{k}_X$.
(iii) Let M be a real manifold. We have the classical sheaves $\mathscr{C}^\infty_X$, $\mathscr{D}b_X$ and on a complex manifold X, the sheaf $\mathscr{O}_X$ of holomorphic functions.
(iv) On a topological space X, the presheaf $U \mapsto \mathscr{C}^{0,b}_X(U)$ of continuous bounded functions is not a sheaf in general. To be bounded is not a local property.

(v) For a locally closed subset Z of M, we denote by $\mathbf{k}_Z$ the constant sheaf with stalk $\mathbf{k}$ on Z, extended by 0 on $M \setminus Z$.

The category of sheaves $\mathrm{Mod}(\mathbf{k}_X)$ is an abelian categories and admits a bounded derived category $\mathrm{D^b}(\mathbf{k}_M)$. Essentialy, an object of $\mathrm{D^b}(\mathbf{k}_M)$ is a bounded complex of sheaves and a complex quasi-isomorphic to zero (that is, an exact complex) is 0.

Example

The de Rham complex

$$0 \to \mathbb{C}_M \to \Omega^0_M \xrightarrow{d} \cdots \to \Omega^n_M \to 0$$

is exact. It is isomorphic to 0 in $\mathrm{D^b}(\mathbf{k}_M)$. Equivalently, in $\mathrm{D^b}(\mathbf{k}_M)$. the sheaf $\mathbb{C}_M$ is isomorphic to the complex

$$0 \to \Omega^0_M \xrightarrow{d} \cdots \to \Omega^n_M \to 0.$$

Microlocal theory

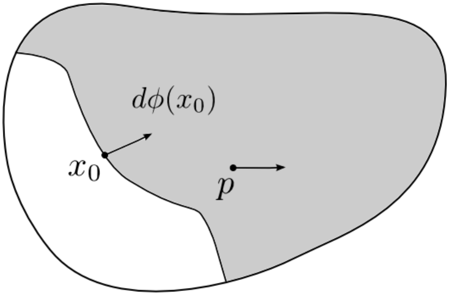
We shall recall some notions and results of [KS90].
Let M be a real manifold, $\pi_M\colon T^*M \to M$ its cotangent bundle.

Definition

Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$. The singular support, or micro-support, $\mathrm{SS}(F)$ is the closed conic subset of T^*M defined as follows. An open subset W of T^*M does not intersect $\mathrm{SS}(F)$ if for any C^1-function $\varphi\colon M \to \mathbb{R}$ and any $x_0 \in M$ such that $(x_0; d\varphi(x_0)) \in W$, setting $U = \{x; \varphi(x) < \varphi(x_0)\}$, one has for all $j \in \mathbb{Z}$

$$\varinjlim_{V \ni x_0} H^j(U \cup V; F) \simeq H^j(U; F).$$

Therefore, if $(x_0; d\varphi(x_0)) \notin \mathrm{SS}(F)$, then any cohomology class defined on an open subset U as above extends through the boundary in a neighborhood of x_0.

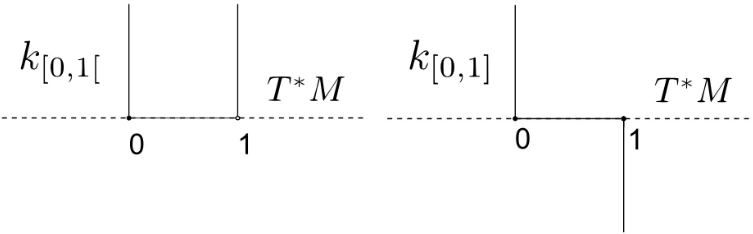


- The microsupport is closed and is $\mathbb{R}^+$-conic,
- $\mathrm{SS}(F) \cap T^*_M M = \pi_M(\mathrm{SS}(F)) = \mathrm{Supp}(F)$,
- if $F_1 \to F_2 \to F_3 \xrightarrow{+1}$ is a distinguished triangle in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$, then $\mathrm{SS}(F_i) \subset \mathrm{SS}(F_j) \cup \mathrm{SS}(F_k)$ for all $i, j, k \in \{1, 2, 3\}$ with $j \neq k$.
- The microsupport is involutive (one also says, co-isotropic). (No precise definition here.)

Examples

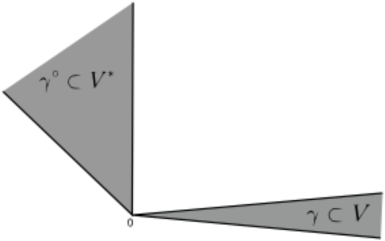
(i) If F is a non-zero local system on M and M is connected, then $\mathrm{SS}(F) = T^*_M M$, the zero-section.
(ii) If N is a closed submanifold of M and $F = \mathbf{k}_N$, then $\mathrm{SS}(F) = T^*_N M$, the conormal bundle to N in M.
(iii) Let φ be a C^1-function with $d\varphi(x) \neq 0$ for $\varphi(x) = 0$. Let $U = \{x \in M; \varphi(x) > 0\}$, $Z = \{x \in M; \varphi(x) \geq 0\}$. Then

$$\mathrm{SS}(\mathbf{k}_U) = U \times_M T^*_M M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \leq 0\},$$
$$\mathrm{SS}(\mathbf{k}_Z) = Z \times_M T^*_M M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \geq 0\}.$$



(iv) Assume $M = V$ is a vector space and let γ be a cone with vertex at 0. The dual cone γ° is a convex closed cone.

$$\gamma^\circ = \{(x;\xi) \in E^*; \langle \xi, v\rangle \geq 0 \text{ for all } v \in \gamma_x\}.$$



If γ is a closed convex cone, then

$$\mathrm{SS}(\mathbf{k}_\gamma) \cap \pi^{-1}(0) = \gamma^\circ.$$

Note that, the smallest γ is, the biggest γ° is. (A variant of the uncertaintly principle.)

(v) Let $(X, \mathscr{O}_X)$ be a complex manifold and let $\mathscr{M}$ be a coherent module over the ring $\mathscr{D}_X$ of holomorphic differential operators. (Hence, $\mathscr{M}$ represents a system of linear partial differential equations on X.) Denote by $F = \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)$ the complex of holomorphic solutions of $\mathscr{M}$. Then $\mathrm{SS}(F) = \mathrm{char}(\mathscr{M})$, the characteristic variety of $\mathscr{M}$.

Theorem

Let $Z, U \subset M$. Assume that Z is closed and U is open. Then $\mathrm{SS}(\mathbf{k}_Z) \subset N(Z)^\circ$ and $\mathrm{SS}(\mathbf{k}_U) \subset N(U)^{\circ a}$.

Operations

Let $f\colon M \to N$ be a morphism of real manifolds. Denote by p_1 and p_2 the projections from $T^*(M\times N)$ to T^*M and T^*N and set $\Lambda_f = T^*_{\Gamma_f}(M\times N)$. Then $\Lambda_f \xrightarrow{\sim} M\times_N T^*N$ by the map $p_1 \times p_2^a$.

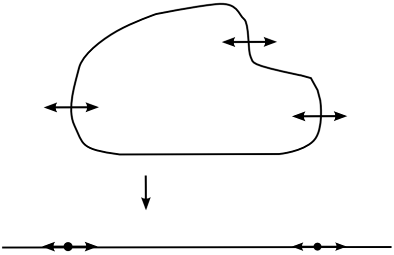
$$
\begin{tikzcd}
 & T^*_{\Gamma_f}(M\times N) = \Lambda_f \arrow[dl, "p_1"'] \arrow[d, "\sim"] \arrow[dr, "p_2^a"] & \\
T^*M & M\times_N T^*N \arrow[l, "f_d"'] \arrow[r, "f_\pi"] & T^*N.
\end{tikzcd}
$$

Theorem

Let $F \in \mathrm{D^b}(\mathbf{k}_M)$ and let $G \in \mathrm{D^b}(\mathbf{k}_N)$.

(i) *Assume that f is proper on* $\mathrm{Supp}(F)$. *Then* $\mathrm{SS}(\mathrm{R}f_*F) \subset f_\pi f_d^{-1}\,\mathrm{SS}(F) = \mathrm{SS}(F)\overset{a}{\circ}\Lambda_f$.

(ii) *Assume that f_d is proper on $f_\pi^{-1}\,\mathrm{SS}(G)$. Then* $\mathrm{SS}(f^{-1}G) \subset f_d f_\pi^{-1}\,\mathrm{SS}(G) = \Lambda_f\overset{a}{\circ}\mathrm{SS}(G)$.



Theorem

Let $F_1, F_2 \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$.

(i) *Assume that* $\mathrm{SS}(F_1) \cap \mathrm{SS}(F_2)^a \subset T^*_M M$*. Then* $\mathrm{SS}(F_1 \otimes F_2) \subset \mathrm{SS}(F_1) + \mathrm{SS}(F_2)$*.*

(ii) *Assume that* $\mathrm{SS}(F_1) \cap \mathrm{SS}(F_2) \subset T^*_M M$*. Then* $\mathrm{SS}(\mathrm{R}\mathscr{H}om\,(F_1, F_2)) \subset \mathrm{SS}(F_1)^a + \mathrm{SS}(F_2)$*.*

Hyperbolicity for sheaves

Consider a vector bundle $\tau\colon E \to M$. It gives rise to the maps $T^*E \leftarrow E \times_M T^*M \to T^*M$. By restricting to the zero-section of E, we get the map $T^*M \hookrightarrow T^*E$. Now assume that M is a closed submanifold of a manifold X. Applying this construction to the bundle $T^*_M X$ above M, and using the Hamiltonian isomorphism we get the maps

$$T^*M \hookrightarrow T^*T^*_M X \simeq T_{T^*_M X} T^*X.$$

Theorem

Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_X)$. Then

$$\begin{aligned} \mathrm{SS}(\mathrm{R}\Gamma_M F) &\subset T^*M \cap C_{T^*_M X}(\mathrm{SS}(F)), \\ \mathrm{SS}(F|_M) &\subset T^*M \cap C_{T^*_M X}(\mathrm{SS}(F)). \end{aligned}$$

Direct images for Causal manifolds

Notation

Here, for a causal manifold (M,γ) we set

$$\lambda := \gamma^{\circ}.$$

Note that λ is a closed convex proper cone of T^*M, $\lambda \supset T^*_M M$ and $\gamma = \mathrm{Int}(\lambda^{\circ})$.

Let (M,γ) be a causal manifold and $\preceq$ a closed causal preorder on M. Let $Z, U \subset M$ with U open and Z closed.

- Assume $U = J^{+}_{\preceq}(U)$. Then $\mathrm{SS}(\mathbf{k}_U) \subset \lambda^{a}$.
- Assume that $Z = J^{-}_{\preceq}(Z)$. Then $\mathrm{SS}(\mathbf{k}_Z) \subset \lambda^{a}$.

By using the theorem on the microsupport for proper direct images, one proves:

Theorem

Let $f\colon (M,\gamma_M) \to (N,\gamma_N)$ *be a morphism of causal manifolds, let* $\preceq$ *be a closed causal preorder on* M *and let* $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$*. Assume that*

(a) $f\colon M \to N$ *is submersive,*

(b) *for any compact* $K \subset M$*, the map* f *is proper on the closed set* $J^{-}_{\preceq}(K)$*,*

(c) $\mathrm{SS}(F) \cap \lambda_M \subset T^*_M M$*.*

Then $\mathrm{SS}(\mathrm{R}f_*F) \cap \mathrm{Int}(\lambda_N) = \varnothing$*.*

Theorem

Let (M, γ, q) be a G-causal manifold and let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$.

(i) *Assume that $\mathrm{SS}(F) \cap \lambda^a \subset T^*_M M$ and let B be a closed subset satisfying $B = J^-_{\preceq}(B)$ and $B \subset q^{-1}((-\infty, a])$ for some $a \in \mathbb{R}$. Then*

$$\mathrm{R}\Gamma_B(M; F) \simeq 0.$$

(ii) *Assume that $\mathrm{SS}(F) \cap (\lambda \cup \lambda^a) \subset T^*_M M$. Then, setting $M_0 = q^{-1}(0)$, the natural restriction morphism below is an isomorphism:*

$$\mathrm{R}\Gamma(M; F) \xrightarrow{\sim} \mathrm{R}\Gamma(M_0; F|_{M_0}).$$

Characteristic variety

Let $(X, \mathscr{O}_X)$ be a complex manifold and let $\mathscr{D}_X$ be the sheaf of rings of holomorphic (finite order) differential operators.
A left coherent $\mathscr{D}_X$-module $\mathscr{M}$ may be locally represented as the cokernel of a matrix $\cdot P_0$ of differential operators acting on the right:

$$\mathscr{M} \simeq \mathscr{D}_X^{N_0} / \mathscr{D}_X^{N_1} \cdot P_0.$$

By classical arguments, $\mathscr{M}$ is locally isomorphic to the cohomology of a bounded complex

$$\mathscr{M}^\bullet := 0 \to \mathscr{D}_X^{N_r} \to \cdots \to \mathscr{D}_X^{N_1} \xrightarrow{\cdot P_0} \mathscr{D}_X^{N_0} \to 0.$$

For a coherent $\mathscr{D}_X$-module $\mathscr{M}$, one sets for short

$$\begin{aligned} \mathrm{Sol}(\mathscr{M}) &:= \mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X) \simeq \mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}^\bullet, \mathscr{O}_X) \\ &\simeq 0 \to \mathscr{O}_X^{N_0} \xrightarrow{P_0 \cdot} \mathscr{O}_X^{N_1} \to \cdots \mathscr{O}_X^{N_r} \to 0. \end{aligned}$$

One denotes by $\mathrm{char}(\mathscr{M})$ the characteristic variety of $\mathscr{M}$. If $\mathscr{M} = \mathscr{D}_X/\mathscr{I}$ for a locally finitely generated left ideal of $\mathscr{D}_X$, then

$$\mathrm{char}(\mathscr{M}) = \{(z;\zeta) \in T^*X; \sigma(P)(z;\zeta) = 0 \text{ for all } P \in \mathscr{I}\},$$

where $\sigma(P)$ denotes the principal symbol of P.

Theorem

Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then $\mathrm{char}(\mathscr{M})$ *is a closed conic complex analytic* involutive (i.e., *co-isotropic*) *subset of T^*X. Moreover,*

$$\mathrm{char}(\mathscr{M}) = \mathrm{SS}(\mathrm{Sol}(\mathscr{M})).$$

The involutivity result was first proved by Sato-Kashiwara-Kawai in 1973 using differential operators of infinite order. Then Gabber (1981) gave a purely algebraic proof. The last formula due to [KS90] gives another totally different proof of the involutivity.

Cauchy problem

Let Y be a complex submanifold of the complex manifold X. One says that Y is non-characteristic for $\mathscr{M}$ if

$$\mathrm{char}(\mathscr{M}) \cap T^*_Y X \subset T^*_X X.$$

With this hypothesis, the induced system $\mathscr{M}_Y$ by $\mathscr{M}$ on Y is a coherent $\mathscr{D}_Y$-module and one has the Cauchy–Kowalesky–Kashiwara theorem(1970):

Theorem

Assume Y is non-characteristic for $\mathscr{M}$. Then $\mathscr{M}_Y$ is a coherent $\mathscr{D}_Y$-module and the morphism

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{O}_X)|_Y \to \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{O}_Y).$$

is an isomorphism.

Hyperbolic systems

Let M be a real analytic manifold and let X be a complexification of M. Recall the natural maps

$$T^*M \hookrightarrow T^*T^*_MX \simeq T_{T^*_MX}T^*X.$$

For a coherent left $\mathscr{D}_X$-module $\mathscr{M}$, we set

$$\mathrm{hypchar}_M(\mathscr{M}) = T^*M \cap C_{T^*_MX}(\mathrm{char}(\mathscr{M})).$$

A vector $\theta \in T^*M \setminus \mathrm{hypchar}_M(\mathscr{M})$ is called hyperbolic for $\mathscr{M}$. A submanifold N of M is called *hyperbolic* for $\mathscr{M}$ if $T^*_NM \cap \mathrm{hypchar}_M(\mathscr{M}) \subset T^*_MM$.

Assume we have the local coordinate system $(x + \sqrt{-1}y)$ on X, $(x + \sqrt{-1}y; \xi + \sqrt{-1}\eta)$ on T^*X and let $M = \{y = 0\}$ so that $T^*_M X = \{y = \xi = 0\}$.
Let $(x_0; \theta_0) \in T^*M$ with $\theta_0 \neq 0$. Let $P \in \mathscr{D}_X$. We find that $(x_0; \theta_0)$ is hyperbolic for P (that is, for $\mathscr{D}_X/\mathscr{D}_X \cdot P$) if and only if

$$\begin{cases} \text{there exist an open neighborhood } U \text{ of } x_0 \text{ in } M \text{ and} \\ \text{an open conic neighborhood } \gamma \text{ of } \theta_0 \in \mathbb{R}^n \text{ such that} \\ \sigma(P)(x; \theta + \sqrt{-1}\eta) \neq 0 \text{ for all } \eta \in \mathbb{R}^n,\ x \in U,\ \theta \in \gamma. \end{cases}$$

By he local Bochner's tube theorem that this is equivalent to

$$\begin{cases} \text{there exist an open neighborhood } U \text{ of } x_0 \text{ in } M \text{ such} \\ \text{that } \sigma(P)(x; \theta_0 + \sqrt{-1}\eta) \neq 0 \text{ for all } \eta \in \mathbb{R}^n, \text{ and} \\ x \in U. \end{cases}$$

One recovers the classical notion of a (weakly) hyperbolic operator.

Now, consider the sheaves

$$\mathscr{A}_M = \mathscr{O}_X|_M, \quad \mathscr{B}_M = H^n_M(\mathscr{O}_X) \otimes \mathrm{or}_M \simeq \mathrm{R}\Gamma_M(\mathscr{O}_X) \otimes \mathrm{or}_M\,[n].$$

Here, or_M is the orientation sheaf on M and $n = \dim M$. The sheaf $\mathscr{A}_M$ is the sheaf of (complex valued) real analytic functions on M and the sheaf $\mathscr{B}_M$ is the sheaf of Sato's hyperfunctions on M. Applying the theorem which gives a bound to the microsupport of $\mathrm{R}\Gamma_M F$ and $F|_M$, we get:

Theorem (see KS90)

Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module. Then

$$\mathrm{SS}(\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)) \subset \mathrm{hypchar}_M(\mathscr{M}),$$
$$\mathrm{SS}(\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{A}_M)) \subset \mathrm{hypchar}_M(\mathscr{M}).$$

In other words, hyperfunction (as well as real analytic) solutions of the system $\mathscr{M}$ propagate in the hyperbolic directions.

The following result is easily deduced from the preceding one.

Theorem

Let M be a real analytic manifold, X a complexification of M, $\mathscr{M}$ a coherent $\mathscr{D}_X$-module. Let $N \hookrightarrow M$ be a real analytic smooth closed submanifold of M and $Y \hookrightarrow X$ is a complexification of N in X. We assume

$$T_N^*M \cap \mathrm{hypchar}_M(\mathscr{M}) \subset T_M^*M,$$

that is, N is hyperbolic for $\mathscr{M}$. Then Y is non-characteristic for $\mathscr{M}$ in a neighborhood of N and we have the isomorphism

$$\mathrm{R}\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M)|_N \xrightarrow{\sim} \mathrm{R}\mathscr{H}om_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{B}_N).$$

In other words, the Cauchy problem in a neighborhood of N for hyperfunctions on M is well-posed for hyperbolic systems.

Theorem

Let (M, γ, q) be a G-causal manifold and assume that M is real analytic. Let $\mathscr{M}$ be a coherent $\mathscr{D}_X$-module satisfying $\mathrm{hypchar}(\mathscr{M}) \cap \lambda \subset T^*_M M$.

(a) *Let A be a closed subset satisfying either $A = J^+_{\mathrm{cc}}(A)$ and $A \subset q^{-1}([a, +\infty))$ or $A = J^-_{\mathrm{cc}}(A)$ and $A \subset q^{-1}((-\infty, a])$ for some $a \in \mathbb{R}$. Then $\mathrm{RHom}_{\mathscr{D}_X}(\mathscr{M}, \Gamma_A \mathscr{B}_M) \simeq 0$. In particular, hyperfunction solutions of the system $\mathscr{M}$ defined on $M \setminus A$ extend uniquely to the whole of M as hyperfunction solutions of the system.*

(b) *Let $N = q^{-1}(0)$ and assume that N is real analytic. Let Y be a complexification of N in X. Then we have the isomorphism*

$$\mathrm{RHom}_{\mathscr{D}_X}(\mathscr{M}, \mathscr{B}_M) \to \mathrm{RHom}_{\mathscr{D}_Y}(\mathscr{M}_Y, \mathscr{B}_N).$$

In other words, the Cauchy problem for hyperfunctions with initial data on N is globally well-posed.

Examples

Let N be a real analytic manifold, $M = N \times \mathbb{R}$. We denote by $(t; w)$ the coordinates on $T\mathbb{R}$ and by $(t; \tau)$ the coordinates on $T^*\mathbb{R}$.
Let $P = \partial_t^2 - R$ be a differential operator of order 2 such that R does not depend on ∂_t and $\sigma_2(R)|_{T^*_M X} \leq 0$. We assume

> there exist a smooth function $f\colon \mathbb{R} \to \mathbb{R}_{>0}$ and a smooth complete Riemannian metric g on N such that $\sigma_2(R)(x, t; \xi) \leq f(t)|\xi|^2_{g_x}$.

Note that this condition is automatically satisfied if N is compact.
We set

$$\begin{aligned} \gamma &= \{(x, t; v, w) \in TM; w > 1/(2f(t))|v|_g\}, \\ \gamma^\circ &= \{(x, t; \xi, \tau) \in T^*M; \tau \geq 2f(t)|\xi|_g\}. \end{aligned}$$

Then

- $\mathrm{hypchar}(P) \cap \gamma^\circ \subset T^*_M M$,
- γ is the future cone of the Lorentzian spacetime $(M, dt^2 - (1/2f(t))g)$, which is globally hyperbolic.
- (M, γ, q) is a G-causal manifold,
- the Cauchy problem for hyperfunctions (and for real analytic functions) is globally well-posed.

As a particular case, if $(g_t)_{t\in\mathbb{R}}$ is an analytic family of *complete* Riemannian metrics on N and $(\Delta_t)_{t\in\mathbb{R}}$ are the associated Laplace–Beltrami operators, then the operator $P = \partial_t^2 - \Delta_t$ is such an example.