Critical Phenomena

Marco Picco
LPTHE
UPMC, Sorbonne Universités and CNRS

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Plan

- Introduction
- Basic definitions : Ising model
- Mean field theory
- Landau-Ginzburg-Wilson model
- Renormalisation group
- Scaling theories
- Renormalisation and scaling theory
- Perturbative renormalisation group and ϵ expansion.

These slides correspond to a mini course on the subject of Critical Phenomena given in Ubu, Esperito Santo, Brazil, in February 2016. It corresponds to three courses at an introductory level.
This course is based on the book (and try to keep the same conventions) :
Scaling and Renormalization in Statistical Physics, John Cardy, Cambridge University Press (1996)

Introduction

Introduction

- We consider some macroscopic object (a chalk). If we cut it in two pieces, each piece will continue behaving like the original piece. Same density, compressibility, magnetization, etc.
- We continue dividing it by two. After many iterations, we will reach the microscopic scale and the properties will change. We will have reach a length which is defined as the correlation length of the considered material.
- The correlation length is the distance over which the fluctuations of the microscopic degrees of freedom are correlated. For a distance much larger than this correlation length, macroscopic laws.
- In most systems, the correlation length is very small and corresponds to few microscopic spacings.

Introduction

- By changing external parameters like temperature, pressure, etc, the behavior of a macroscopic material can change brutally. (Melting of a ferromagnet or ice are simple examples.) The changing points (in the parameters space) are defined as critical points.
- These critical points usually mark a separation between two phases : magnetized and paramagnet or ice and liquid, etc.
- Two types of transitions.
  i) Transition with coexistence of the phases (melting ice) and discontinuity in some thermodynamics quantities (latent heat) : First order phase transitions.
  ii) No coexistence of the two phases. At the transition point, a unique critical phase, with fluctuation acting on the whole system, with an infinite correlation length : continuous or second order phase transition.

Introduction

- Critical phenomena is associated with the study of physics at the critical point of second order phase transitions.
- Infinite correlation length implies no scale in the system : scale invariance.
- The fact that there is a large correlation length can make the study very complex. In fact, it will lead to many simplifications.
- One of the most important is universality : a system close to the continuous phase transition is largely independent of the microscopic underlying model. It will be in one of a small number of universality class depending on global properties such as the symmetries, the spatial dimension, etc.
- The universality will be manifest when computing the critical exponents associated to the critical transition : these exponents will depend only on the universality class, even for models which correspond to a different microscopic model.

Introduction

- Critical phenomenas are present in many places in real life. To give some definitions, we will first present some simple examples. We will present two well known examples of systems which exhibit a second order phase transition : i) Ferromagnets ii) simple fluids.
- Other examples : Binary fluids, antiferromagnets, Helium I/ Helium II transition, Conductor /superconductor transition, Baryogenesis and Electroweak Phase Transition, cosmic inflation, etc.
- Ferromagnets : a system with two external parameters, temperature T and external magnetic field H.
  Local magnetization can be in 3 dimensions (Heisenberg model), 2 dimensions (XY model) or just one dimensional (Ising model).

Introduction

- We will consider the simple case restricted along one dimension.
- Very simple phase diagram : one line of singularities for $H = 0, T < T_c$.
- In the rest of the phase diagram, all the thermodynamical quantities are regular (i.e. analytical functions of H and T).
- We will consider the magnetization M : order parameter.
- $T < T_c$, $M(H)$ has a discontinuity for $H = 0 \rightarrow$ First order phase transition.
- $lim_{H \rightarrow 0_+} M = M_0 = -lim_{H \rightarrow 0_-} M$: spontaneous symmetry breaking : Hamiltonian is invariant under local magnetic degree of freedom but the symmetry is not respected in an equilibrium thermodynamical state.

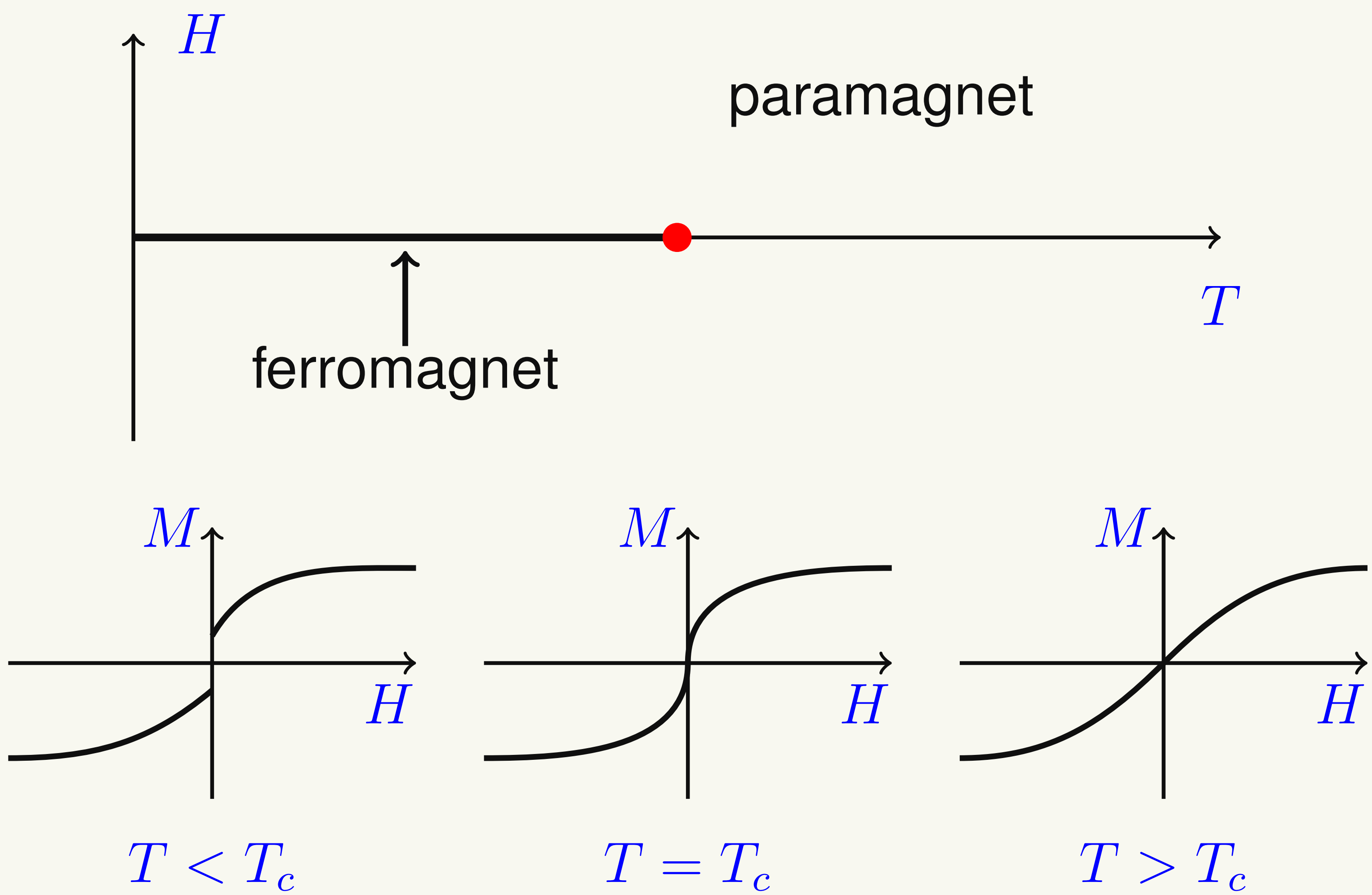


Figure 1: Top : Phase diagram of a ferromagnet. Bottom : Magnetisation as a function of the applied magnetic field

Introduction

- The discontinuity is a power of the deviation to the critical point. We defined $t = \frac{(T-T_c)}{T_c}$ the reduced temperature. This reduced temperature will be frequently used in the following as a parameter to describe the transition.
- At $T = T_c \rightarrow$ Second order phase transition. No discontinuity in the order parameter but on his first derivative. T_c is the critical temperature or Curie temperature.
- We will now define the quantities of interest, the critical exponents, at the critical point.

Introduction

- α : Specific heat in zero field : $C \simeq A|t|^{-\alpha}$. A is the critical amplitude.
- β : Spontaneous magnetization : $lim_{H\to 0_+} M \simeq (-t)^{\beta}$.
- γ : Zero field susceptibility : $\chi = \left(\frac{\partial M}{\partial H}\right)_{H=0} \simeq |t|^{-\gamma}$.
- δ : At $T = T_c, M \simeq |H|^{1/\delta}$
- ν : Correlation length exponent : $\xi \simeq |t|^{-\nu}$. ξ can be defined, for $T \neq T_c$ by

$$G(r) \simeq \frac{e^{-\frac{r}{\xi}}}{r^{(d-1)/2}} \tag{1}$$

- η : anomalous magnetic dimension : $G(r) \simeq r^{d-2+\eta}$.

Introduction

- The second example is the one of the perfect fluid with a transition between vapor and liquid. At the end of 19 century, Van der Waals showed that, by using an appropriate scaling of temperature and pressures, all fluids behave in a similar way.
- Scaling is done compared to some critical value of the temperature, pressure and density, which is the border between the two phases, gas or liquid.
- Along this border, very similar to the ferromagnetic transition with a critical point at the end. The order parameter in that case is the density of the fluid.

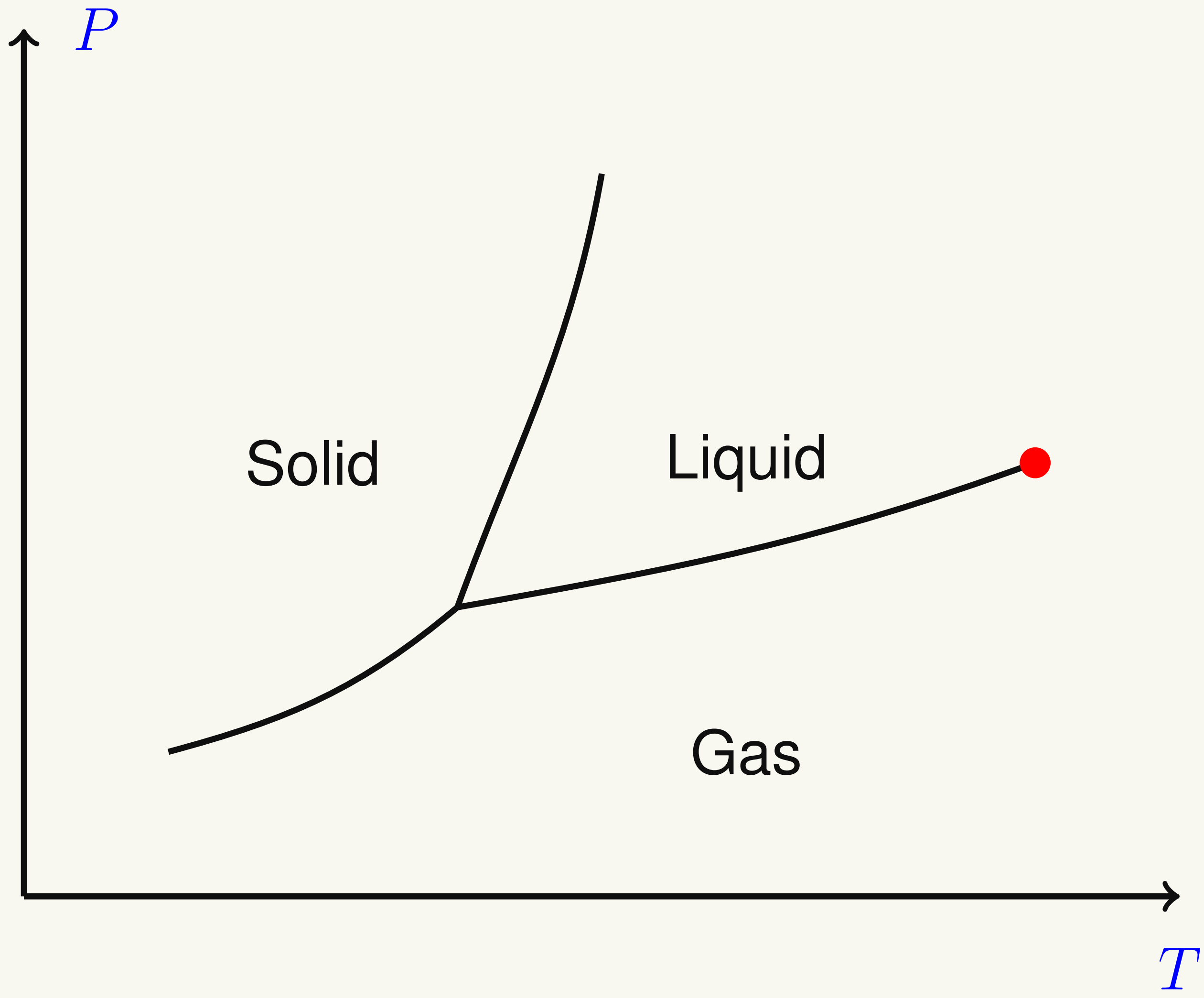


Figure 2: Phase diagram of a simple fluid

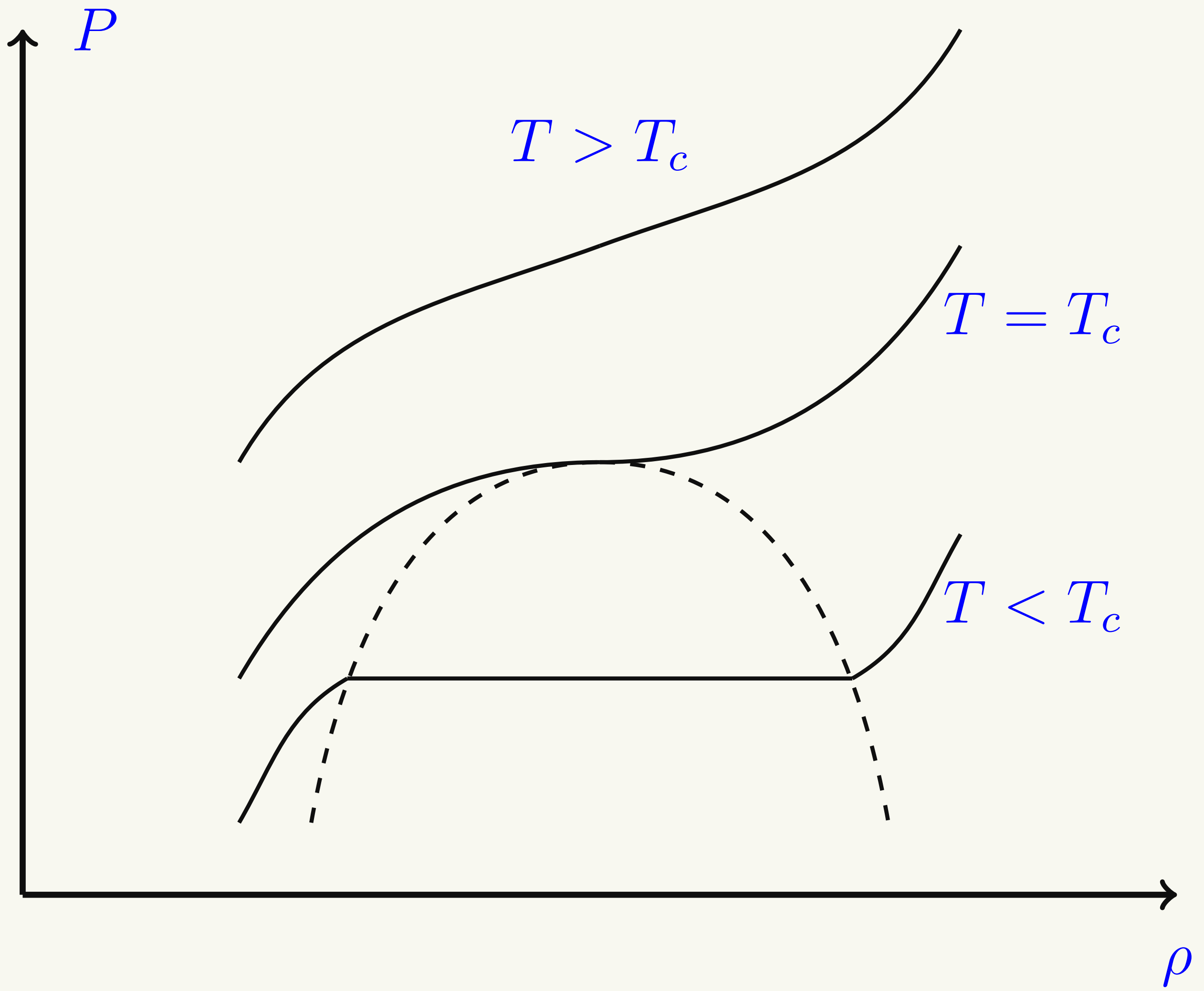


Figure 3: Liquid gas transition

Universality and comparison with experimental systems :

Transition type	Material	α	β	γ	ν
Ferro. (n=3)	Fe, Ni	-0.1	0.34	1.4	0.7
Superfluid (n=2)	He^4	0	0.3	1.2	0.7
Liquid-gas (n=1)	CO_2, Xe	0.11	0.32	1.24	0.63
Superconductors		0	1/2	1	1/2
Mean Field		0	1/2	1	1/2

Basic model : Ising model

Basic model : Ising model

- We introduce one of the simplest model that we use as a basic example in the following : the Ising model. It consist of a system with a variable, the spin S, which takes two values, $+1$ or -1 on each point of a regular lattice with nearest neighbor interactions. The associated energy is

$$E(J,h) = -\sum_{<ij>} J_{ij} S_i S_j - \sum_i h_i S_i \ . \qquad (2)$$

  The first sum is over the nearest neighbor interactions, indicated by $< ij >$. The second sum corresponds to a local magnetic field which couples to the spins S_i.

- $J_{ij} \to J$ and $h_i \to H$. Otherwise, model with disorder (spin glasses) or Random Fields Ising model, which are more difficult to treat.

- $$E(J,H) = -J \sum_{<ij>} S_i S_j - H \sum_i S_i \ . \qquad (3)$$

  The theory is then defined by the partition function

  $$\mathcal{Z}(J,H) = \sum_{S_i} e^{-\beta E(J,H)} \ , \qquad (4)$$

  with $\beta = 1/T$ the inverse temperature. We can compute the ordinary quantities from the expression of $\mathcal{Z}$.

  $$< E >= \frac{1}{\mathcal{Z}(J,H)} \frac{\partial \mathcal{Z}(J,H)}{\partial \beta} \ ; \ < M >= \frac{1}{\beta \mathcal{Z}(J,H)} \frac{\partial \mathcal{Z}(J,H)}{\partial H} \qquad (5)$$

Basic model : Ising model

- The Ising model can be considered as a simple theory of the magnetism.
- at low temperature *i.e.* at large value of β, the interaction term will be important and the spins will tend to be aligned = magnetic phase
- at high temperature *i.e.* at small value of β, the interaction term is less important. The system will be in a disordered phase = paramagnetic phase.
- $$\mathcal{Z}(J,H) = \int dE \mathcal{N}(E) e^{-\beta E} \,, \tag{6}$$

  with $\mathcal{N}(E)$ the number of configurations with energy E.

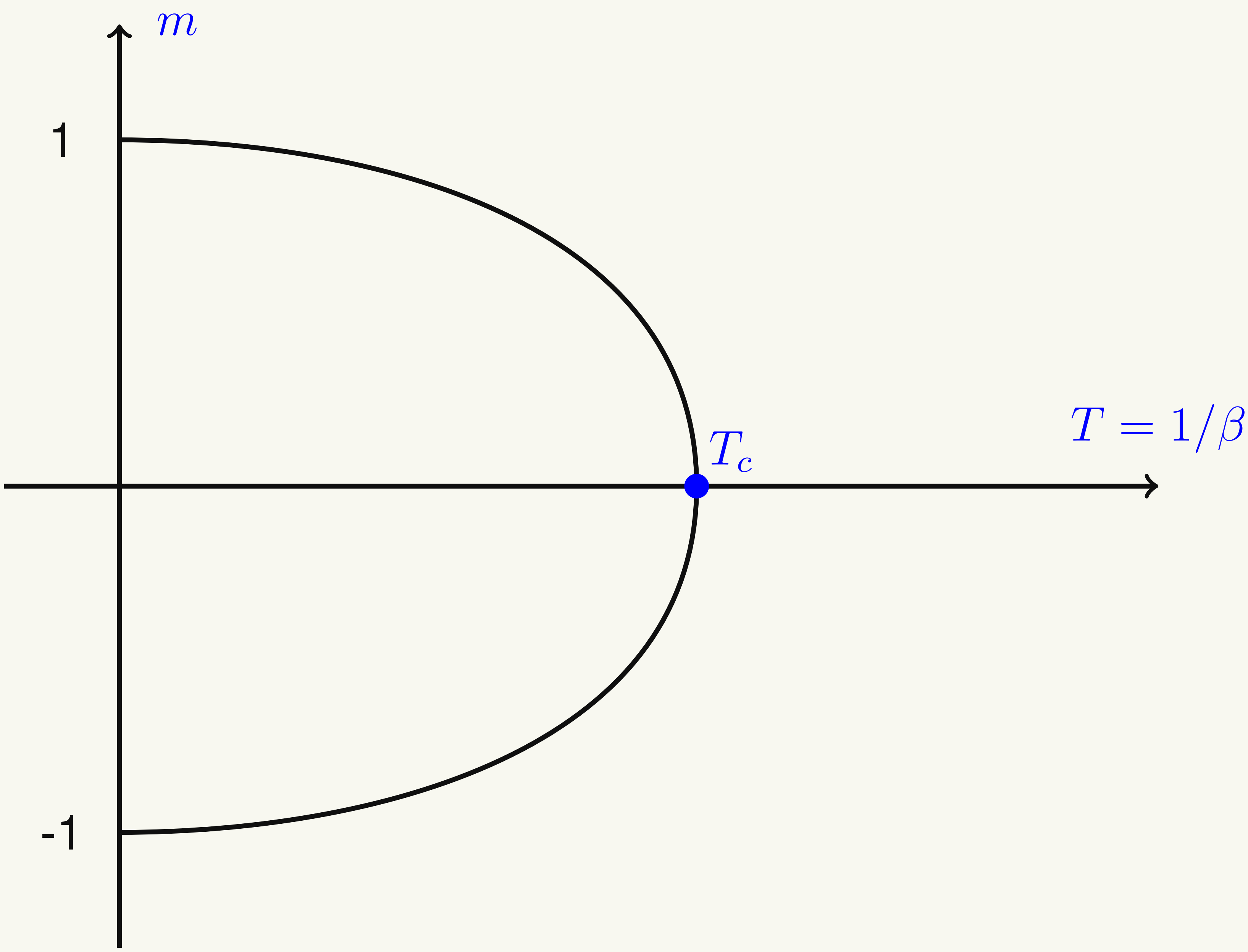
Basic model : Ising model

- If we consider a system with N spins in dimension d, then the lowest energy is $E = -d \times N$: all the spins are up or all the spins down
- A simple check shows that (having fixed $J = 1$ for simplicity)

$$\begin{aligned} \mathcal{Z}(\beta, H=0) &= 2e^{\beta d N}(1 + N e^{-4d\beta} \\ &\quad + O(N)e^{-(8d-2)\beta} + O(N^2)e^{-8d\beta} \cdots) \end{aligned} \tag{7}$$

- For $d \geq 2$, there exists a value of $\beta = \beta_c$ at which the magnetization cancels.
- For $\beta > \beta_c$, $m =< M > /N \to 1$: Energy dominates.
- For $\beta < \beta_c$, $m \to 0$: Entropy (the number of configurations) dominates.

Basic model : Ising model



Basic model : Ising model

- In one dimension, the Ising model is rather trivial. For $T > 0$ it is always in the paramagnetic phase : see later.
- In two dimensions can be solved exactly (Onsager): equivalent to a problem of free fermions (QFT) or one of the simple Conformal Field Theories (with central charge $c = 1/2$).
- In $d > 2$ no solution. Only approximate methods.
- Main problem is that it is too difficult to compute the partition function $\mathcal{Z}$.

Mean Field Theory

Mean Field Theory

- Mean Field theory is a rather general way of describing phases transition which uses general arguments to obtain a qualitative description of the phase diagram of simple models.
- In large dimensions it can give exact results for the critical exponents
- Mean Field theory dates back to Van der Waals who derived the first mean field theory for transition between liquid and vapor in 1873. Next, in 1895, Pierre Curie noticed the analogy with ferromagnets. This was developed further by Pierre Weiss in 1907. General theory is associated to Lev Landau (1937).
- We will first consider the simple case of the Ising model

$$\mathcal{Z}(J,H) = \sum_{S_i} e^{\beta \frac{J}{2} \sum_{<ij>} S_i S_j + \beta H \sum_i S_i} \, . \tag{8}$$

- The first step of the Mean Field approach is to replace the spin variable S_i by an average magnetization plus some fluctuation

$$S_i = M + (S_i - M) = M + dS_i \tag{9}$$

$$\begin{aligned} S_i S_j &= (M + (S_i - M))(M + (S_i - M)) \\ &= M^2 + M(S_i - M) + M(S_j - M) + O(dS^2) \\ &= M(S_i + S_j) - M^2 + O(dS^2) \end{aligned} \tag{10}$$

- We end with the simplified model

$$\mathcal{Z}(J,H) = \sum_{S_i} e^{-N\beta \frac{J}{2} M^2 + \beta(JM+H)\sum_i S_i} \ . \tag{11}$$

- What we have done is to neglect the correlation between the spins. Later on, we will give a criterium for the validity of this approach.

Mean Field Theory

- The summation on the spin is now trivial since there is no more interaction :

$$\begin{aligned}\mathcal{Z}(J,H) &= e^{-N\beta\frac{J}{2}M^2}\prod_i\sum_{S=\pm 1}e^{\beta(JM+H)S}\\ &= e^{-N\beta\frac{J}{2}M^2}\left[2cosh\beta(JM+H)\right]^N\\ &= e^{-N(\beta\frac{J}{2}M^2-\log(cosh\beta(JM+H)))}\\ &= e^{-N\beta f_{MF}(M)}\ , \end{aligned} \tag{12}$$

with $f_{MF}(M)$ the free energy per site. From the previous expression, we can easily obtain the magnetisation :

$$M = \frac{1}{N\beta\mathcal{Z}}\frac{\partial\mathcal{Z}}{\partial H} = tanh\beta(JM+H) \tag{13}$$

- A simple assumption is that the partition function is dominated by the minimum of the free energy (which is multiplied by N)

Mean Field Theory

- We then expend to the first orders in M the free energy (for $H=0$)

$$f_{MF}(M) = \frac{J}{2}M^2 - \frac{1}{\beta}\ln\left(cosh\beta(JM)\right) \tag{14}$$
$$= \frac{J}{2}M^2 - \frac{1}{\beta}\ln\left(1+\frac{1}{2}(\beta JM)^2+\frac{1}{4!}(\beta JM)^4+\cdots\right)$$
$$= \frac{J}{2}M^2 - \frac{1}{\beta}(\frac{1}{2}(\beta JM)^2+\frac{1}{4!}(\beta JM)^4$$
$$-\frac{1}{2}(\frac{1}{2}(\beta JM)^2)^2)+\cdots$$

We end with the expression

$$f_{MF}(M) = \frac{J}{2}(1-\beta J)M^2 + \frac{1}{12}\beta^3J^4M^4 + O(M^6) \tag{15}$$

Mean Field Theory

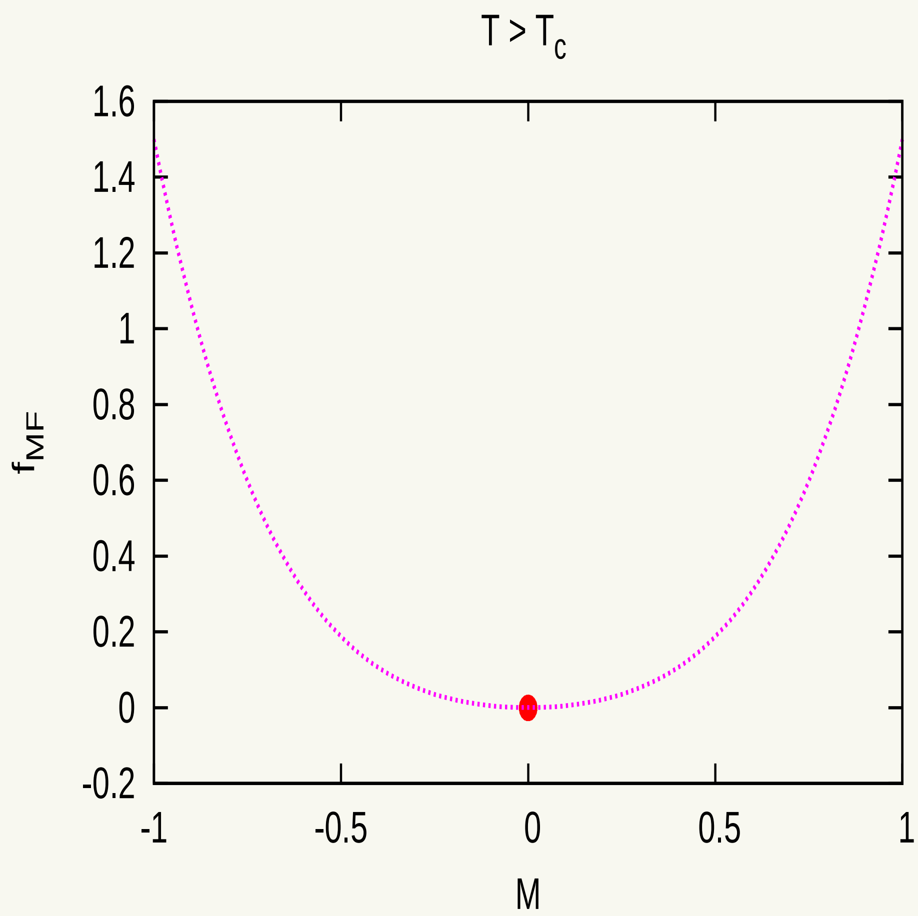
- Using $\beta = 1/T$ we can rewrite

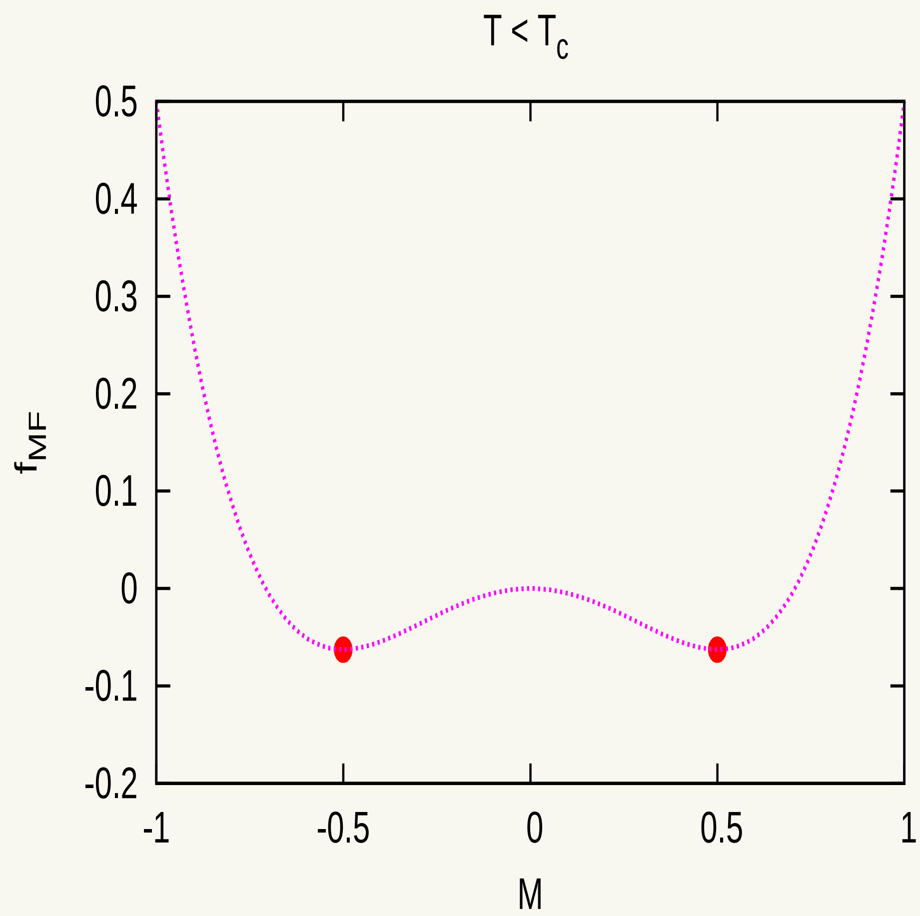
$$f_{MF}(M) = a(T - T_c)M^2 + bM^4 + O(M^6) \ , \tag{16}$$

  with a and $b > 0$ and $T_c = J$.

- If $T > T_c$, there is a unique minimum of the free energy for $M = 0$ and the free energy is symmetric under $M \to -M$.
- If $T < T_c$, there is two minimums at $M \simeq \pm\sqrt{T_c - T} \simeq \pm\sqrt{-t}$ and the symmetry is broken.
- The critical exponent associated to the magnetisation $M \simeq (-t)^{\beta}$ is thus $\beta = 1/2$ for the mean field theory. (β in that case is the critical exponent associated to the magnetisation, NOT the inverse temperature !!!)

Mean Field Theory





Mean Field Theory

- Other critical exponents can be computed in a similar way. For instance, starting from $M = tanh\beta(JM + H)$ and using $\tanh x = x(1 - x^2/3) + O(x^4)$, we get, at $T_c = J$

$$M \simeq M + H/J - M^3/3 \,, \tag{17}$$

which gives the behaviour of the magnetisation in function of the external magnetic field at the critical point as

$$M \simeq B^{\frac{1}{3}} = B^{\frac{1}{\delta}} \tag{18}$$

with $\delta = 3$ the corresponding critical exponent.

Mean Field Theory

General approach of the Mean Field (Landau Theory)

- Determine the order parameter (M)
- Consider the symmetry of the problem
- Construct the more general free energy in powers of the order parameter compatible with the symmetry
  For the ferromagnetism, with invariance under $M \to -M$

$$F(M) = a_2 M^2 + a_4 M^4 + a_6 M^6 + \cdots \tag{19}$$

- We minimize (saddle point) the corresponding partition function

$$Z = \int dM e^{-\beta F(M)} \tag{20}$$

Mean Field Theory

- For the ferromagnetic system (or the simple Ising model), we had $a_2 \simeq T - T_c$ and $a_4 > 0$.
- If we consider $a_4 < 0$ then we get a first order phase transition.
- If we consider $a_4 = 0$ then we get a tricritical point corresponding to the separation between a line of second order phase transition and a line of first order phase transition. Example : Magnetic system with vacancies.

Landau-Ginzburg-Wilson model

Landau-Ginzburg-Wilson model

- One way of obtaining or derive the mean field Hamiltonian is by starting from a continuous spin variable $S(\vec{r})$.
  (in the following, we will drop the vector for $\vec{r}$ and replace it by a single parameter r. The generalisation to a vector is rather trivial).
- We will impose that this spin variable is peaked around the values ± 1

$$\mathcal{H} = -\frac{1}{2}\sum_{r,r'} J(r-r')S(r)S(r') - H\sum_{r} S(r) + \lambda \sum (S(r)^2 - 1)^2 \quad (21)$$

  with the last term to impose the condition $S(r) \simeq \pm 1$.
- $J(r-r')$ is a coupling between the spin $S(r)$ at some distance $r-r'$. More details on this later.

Landau-Ginzburg-Wilson model

- The partition function is simply

$$\mathcal{Z} = \int \prod_r dS(r) e^{-\mathcal{H}} \tag{22}$$

- Now we use

$$\begin{aligned} \sum_{r,r'} J(r-r')S(r)S(r') &= \sum_{r,r'} J(r-r')S(r) \times \\ &\times \left(S(r) + (r-r')\nabla S(r) + \frac{1}{2}(r-r')^2\nabla^2 S(r) + \cdots\right) \\ &= J\sum_r \left(S(r)^2 - R^2a^2(\nabla S(r))^2 + \cdots\right) \end{aligned} \tag{23}$$

with

$$J = \sum_r J(r) \;\; ; R^2 J = \sum_r r^2 J(r) \; , \tag{24}$$

and a a unit of length.

Landau-Ginzburg-Wilson model

- Putting all the terms together, one obtain

$$\mathcal{H} = \int \frac{d^d r}{a^d}[\frac{1}{2}Ja^2R^2(\nabla S(r))^2 - (2\lambda + J)S^2(r) + \lambda S^4(r) - H(r)S(r)] \tag{25}$$

- Next we will rescale the field $S(r)$ such that

$$S^2(r) \to (a^{d-2}/JR^2)S^2(r)\ . \tag{26}$$

- We end up with

$$\mathcal{H} = \int d^d r[\frac{1}{2}(\nabla S(r))^2 + ta^{-2}S^2(r) + ua^{d-4}S^4 + ha^{-d/2-1}S] \tag{27}$$

- The new parameters t, u and h are dimensionless.

Landau-Ginzburg-Wilson model

- If we want to impose invariance under rescaling (why ? see later....) of this Hamiltonian under a rescaling $a \to ba$, then we need also to rescale the parameters such that

$$\begin{aligned} t' &= b^2 t \\ h' &= b^{d/2+1} h \\ u' &= b^{4-d} u \end{aligned} \tag{28}$$

- If $d > 4$, it means that the S^4 term becomes less and less important.
- A term S^6 would have got a contribution $u_6' = b^{6-2d} u_6$. Even less relevant.
- S^{2n} parameter u_{2n} is rescaled as $u_{2n}' = a^{(n-1)d-2n}$. This justifies to ignore largest powers in the field. The same is true for additional derivatives, etc.

Landau-Ginzburg-Wilson model

We will return to the Landau-Ginzburg-Wilson model later after having understood the importance of rescaling. Before, some comments :

- If we ignore the kinetic term (with derivatives), we ignore the local fluctuation of the spin variable $S(r)$. We recover the mean field Hamiltonian from the Landau theory.
- The result does not depend much on the type of interaction $J(r - r')$ as far as J and $R^2 J$ are finite numbers. The simplest choice is with nearest neighbour interactions (like for the Ising model on the lattice) :

$$|r| = 1 \rightarrow J(r) = 1$$
$$|r| > 1 \rightarrow J(r) = 0$$

- a more general choice is $J(r) \simeq \frac{1}{r^{d+\sigma}}$.

Landau-Ginzburg-Wilson model

- We then have the condition

$$J = \sum_r J(r) \simeq \int_a^\Lambda r^{d-1} dr \frac{1}{r^{d+\sigma}} = r^{-\sigma}|_a^\Lambda \,, \tag{29}$$

which will be finite for any $\sigma > 0$.

- A second condition is

$$JR^2 = \sum_r r^2 J(r) \simeq \int_a^\Lambda r^{d+1} dr \frac{1}{r^{d+\sigma}} = r^{2-\sigma}|_a^\Lambda \,, \tag{30}$$

which will give a finite result for any $\sigma > 2$ which is the condition for having short range interactions, equivalent to the nearest neighbour interactions.
We then expect that any interaction with this condition will lead to the same result : Universality of interactions.

Renormalisation group

Renormalisation group

- We will present now some very basic version of the renormalisation group.
- One of the main characteristics of a critical phenomena is the property of scale invariance.
- We can rescale a system and observe again the same thing (in average !!!) : coarse graining.
- This can be visualized on simple systems simulated numerically.
- We will first show some examples for the $2d$ Ising model, at T_c and close to T_c.
- Next we try to see the consequences of the scale invariance for a simple model in one dimension.

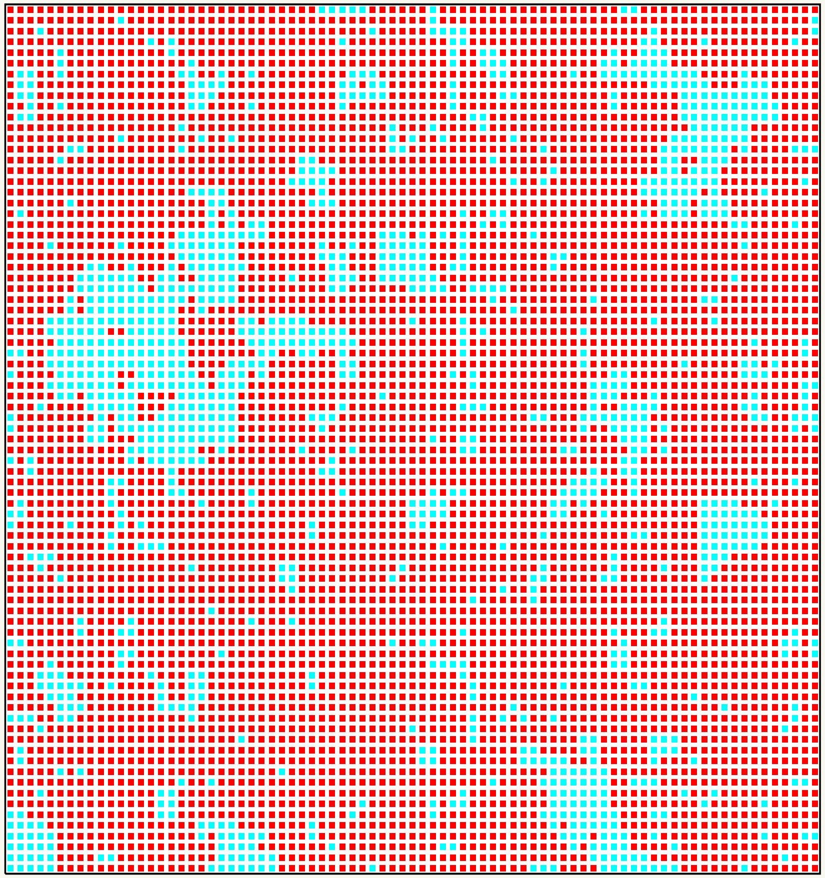
Renormalisation group, $2d$ Ising model

- (i) We will start from a configuration of an equilibrated $2d$ Ising model on a large square lattice. We will show only 81×81 spins $S_{ix,iy}$.
- (ii) The next step is to transform this configuration in 27×27 new spins $NS_{ix,iy}$, such that each of the $NS_{ix,iy}$ is obtained by summing over 3×3 spins : block spin transformation

$$NS_{ix,iy} = S_{3ix-2,3iy-2} + S_{3ix-1,3iy-2} + S_{3ix,3iy-2} + S_{3ix-2,3iy-1} + S_{3ix-1,3iy-1} + S_{3ix,3iy-1} + S_{3ix-2,3iy} + S_{3ix-1,3iy} + S_{3ix,3iy}$$

  If $NS_{ix,iy} > 0$, the new spin is $+1$, otherwise it is -1.
- (iii) Next we rescale by a factor 3 to consider again a system of 81×81 spins
- We go again to step (ii)

Renormalisation group, $T_c, L = 6561$



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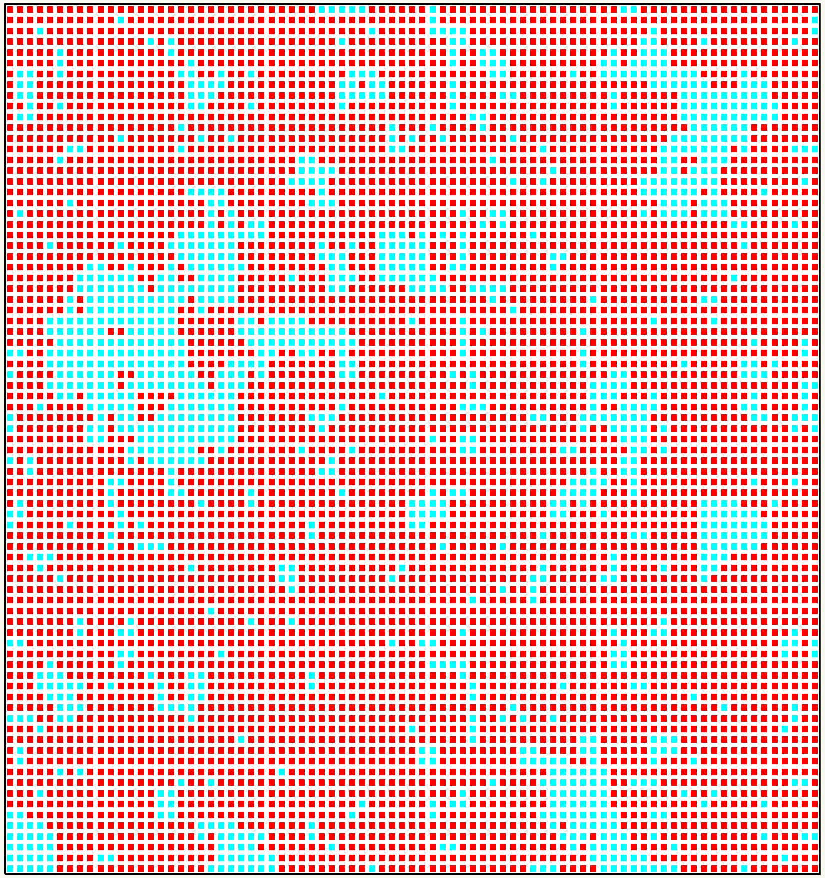

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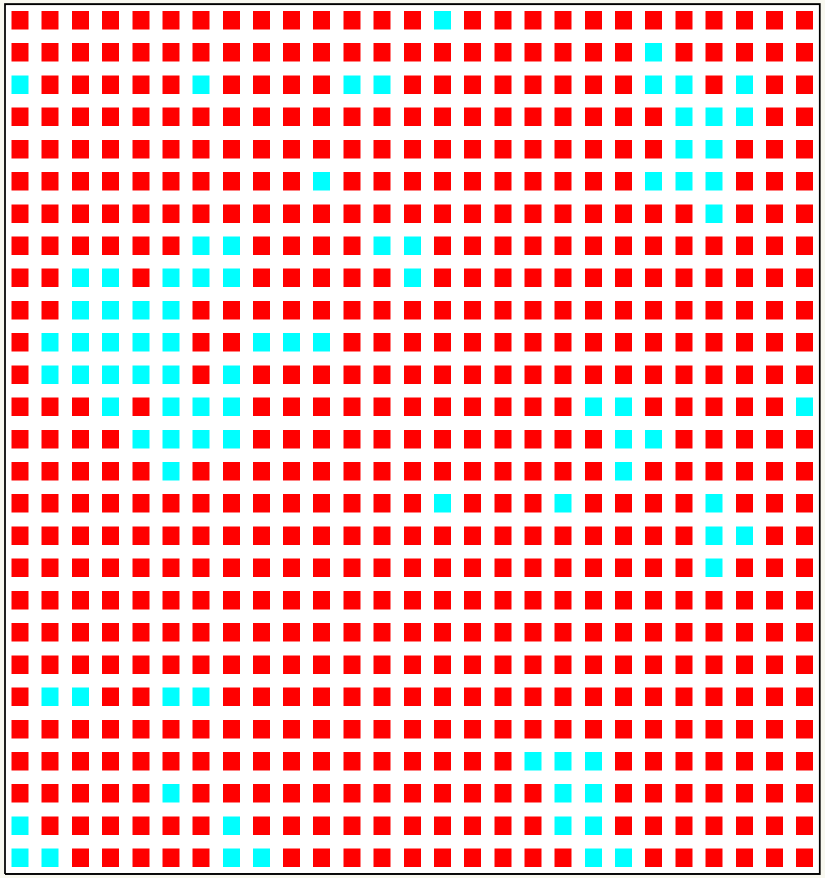

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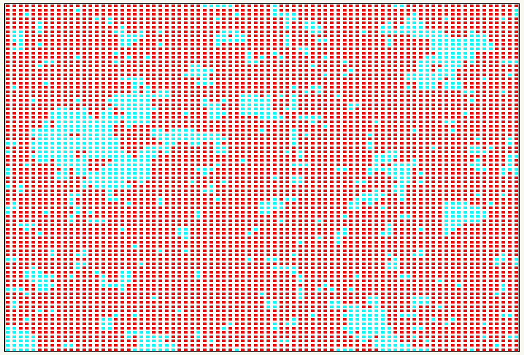

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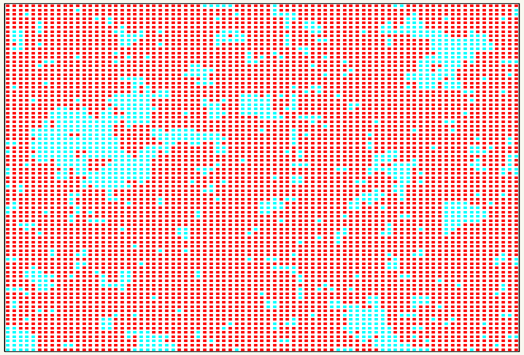

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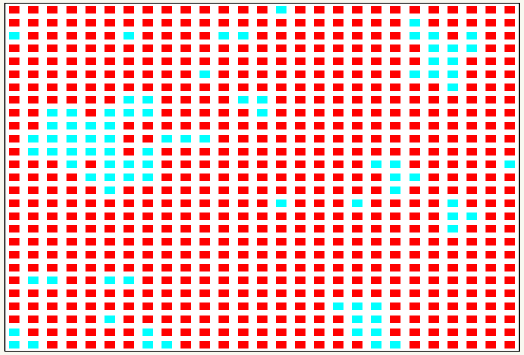

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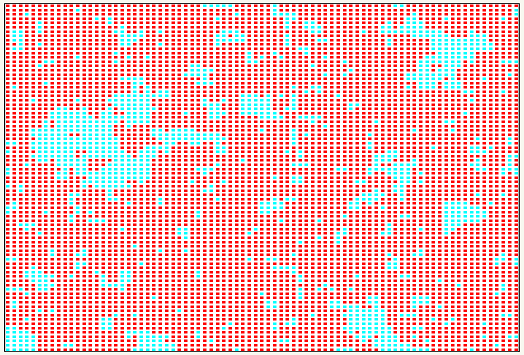
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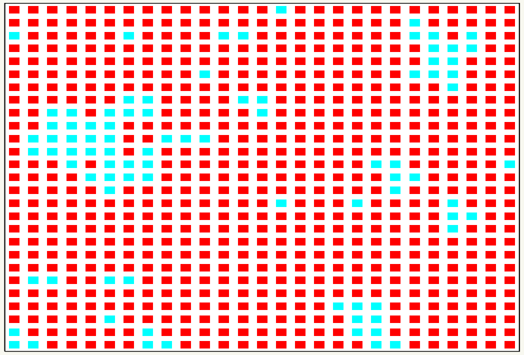
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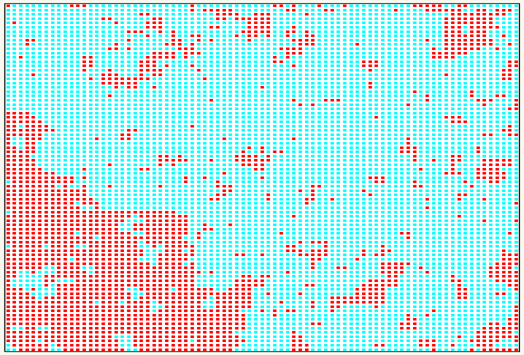
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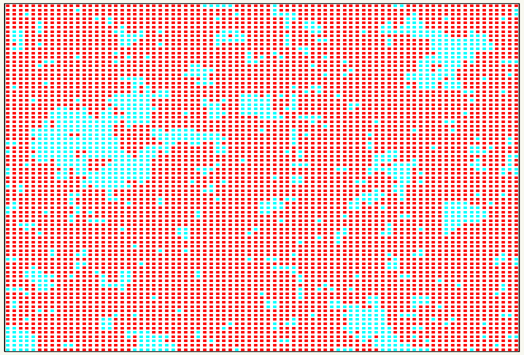

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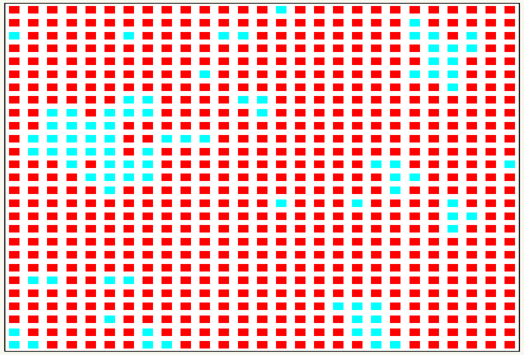

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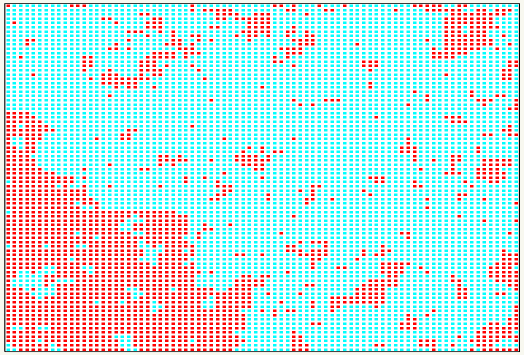

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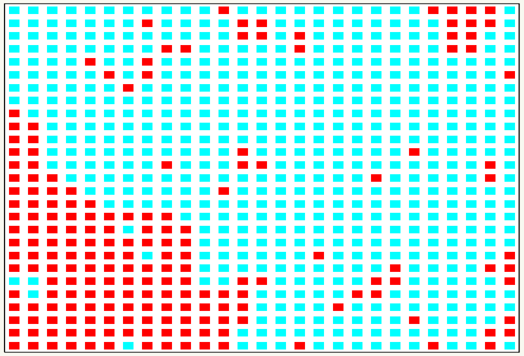
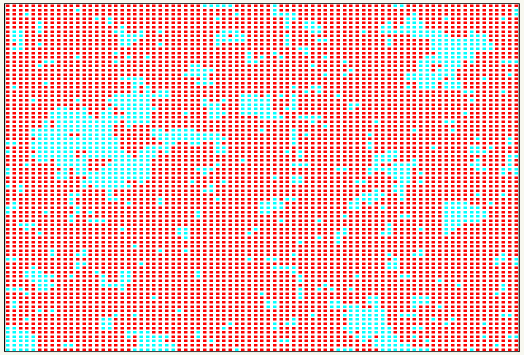


N=27

Renormalisation group, $T_c, L = 6561$



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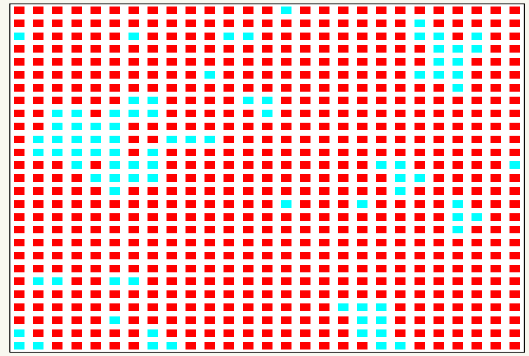

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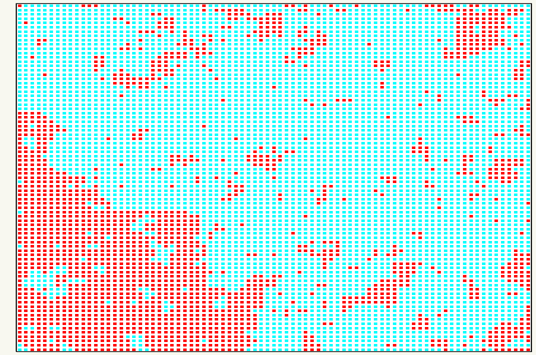

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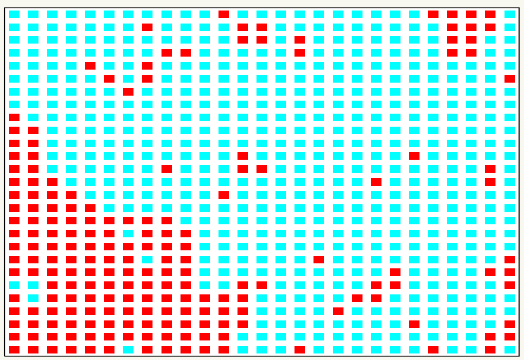

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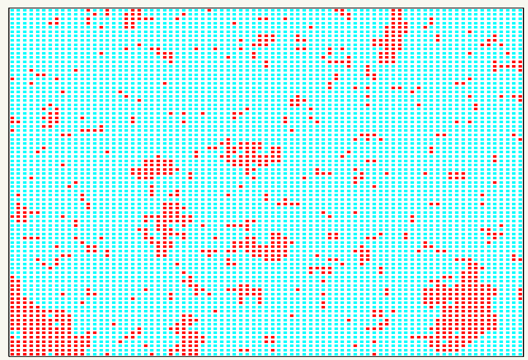

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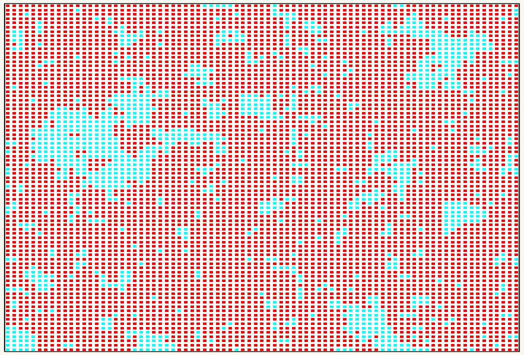

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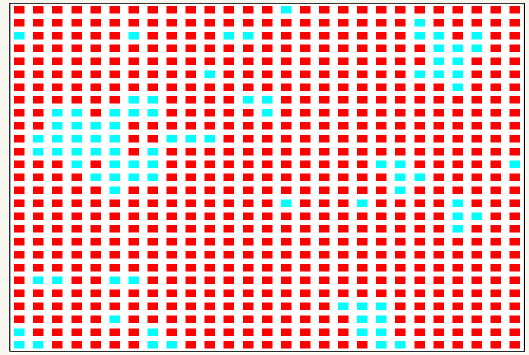

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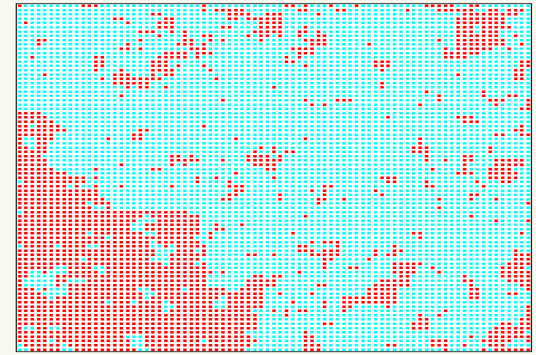

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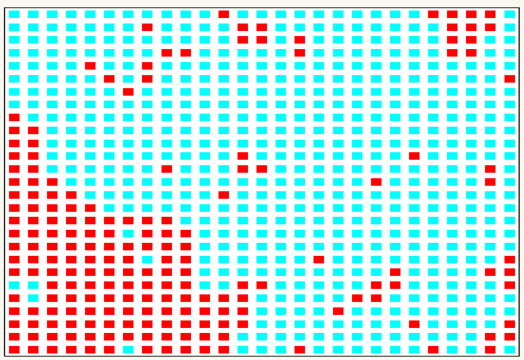

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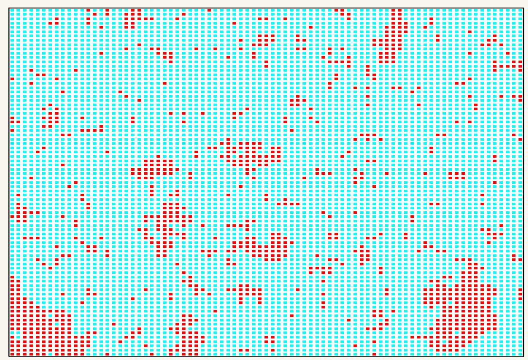

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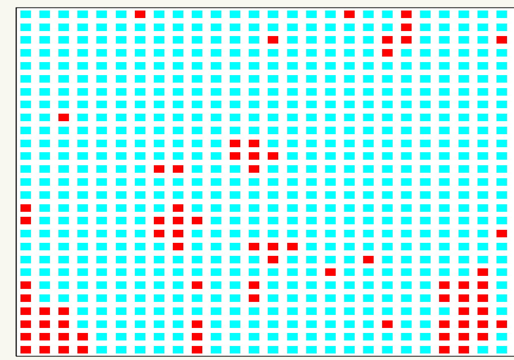

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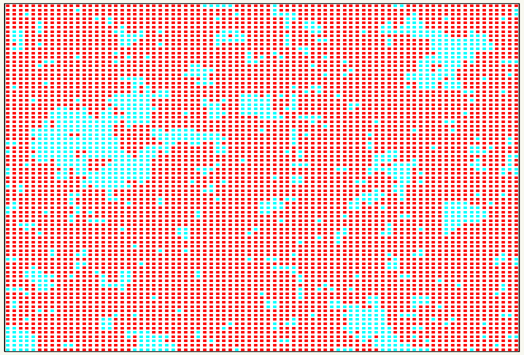

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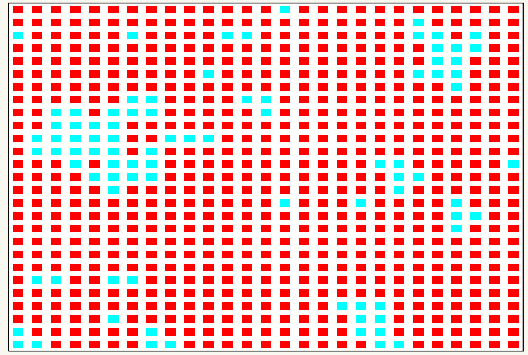

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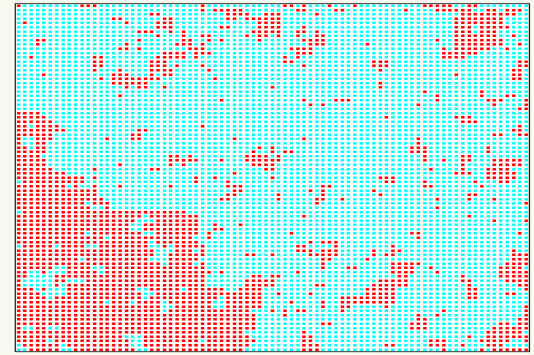

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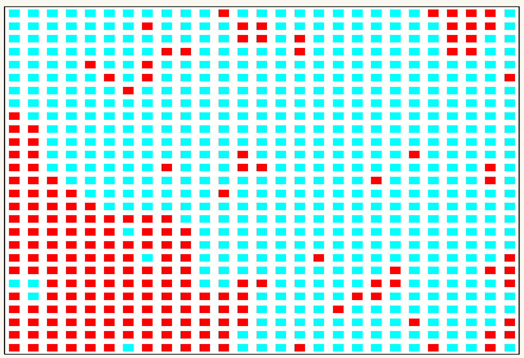

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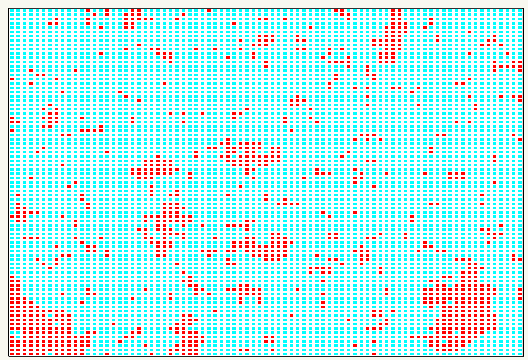

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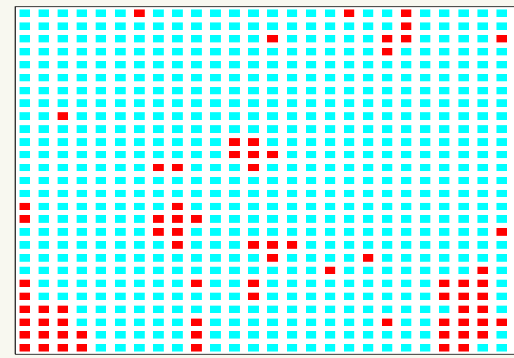

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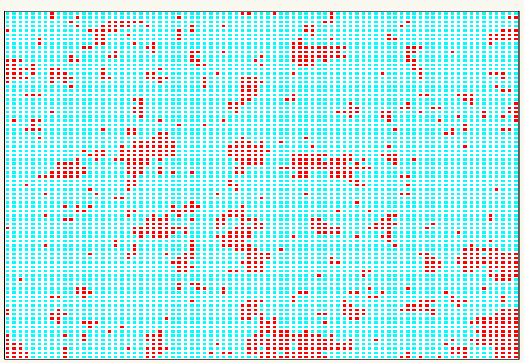

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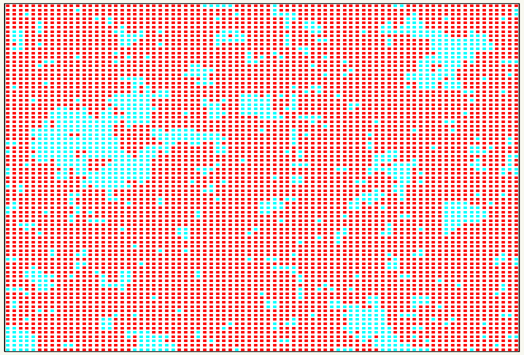

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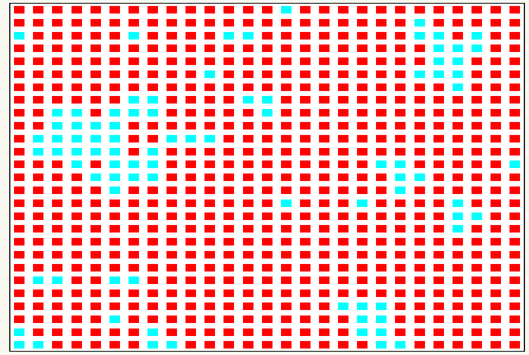

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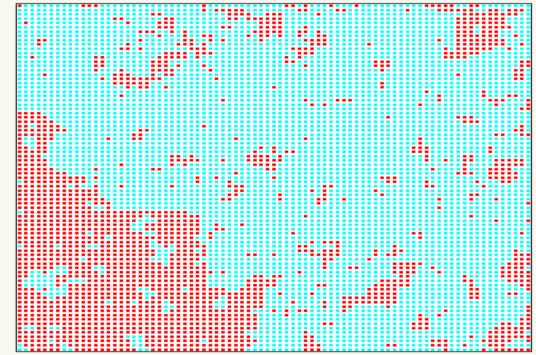

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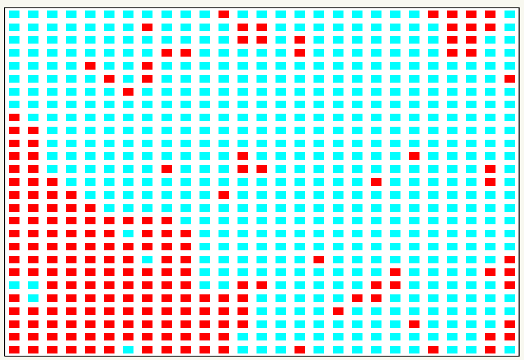

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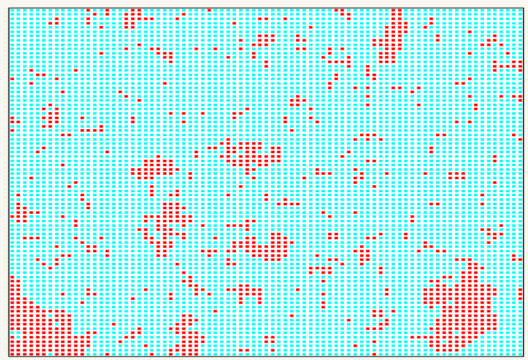

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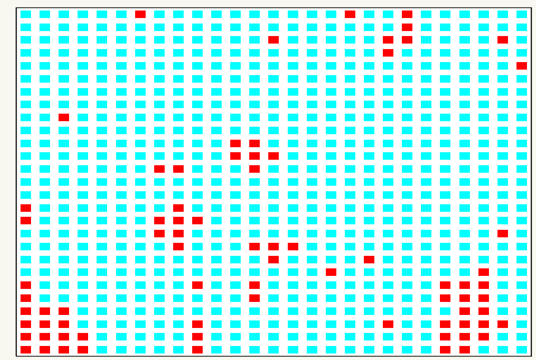

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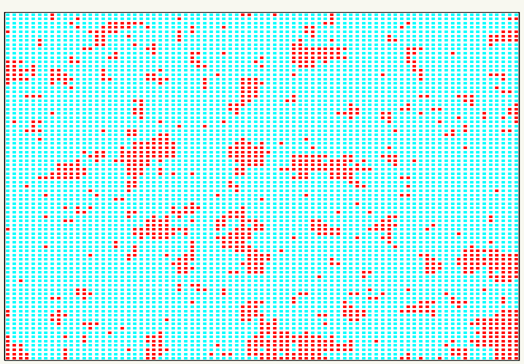

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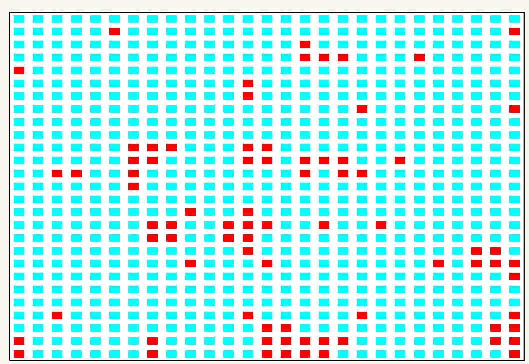

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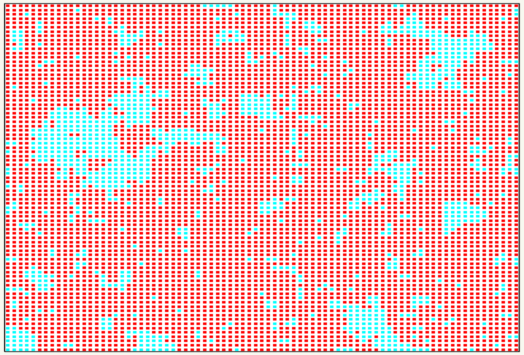

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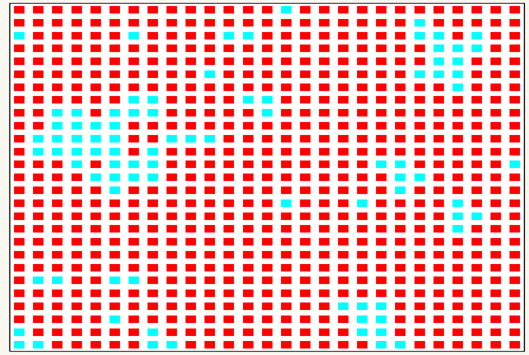

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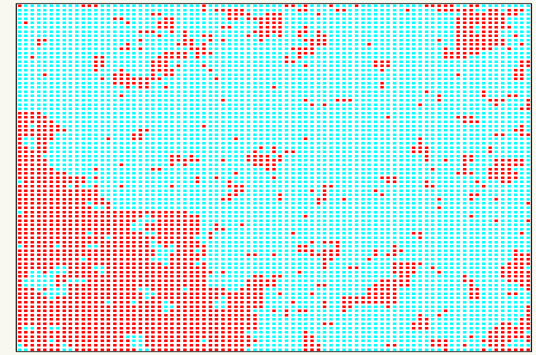

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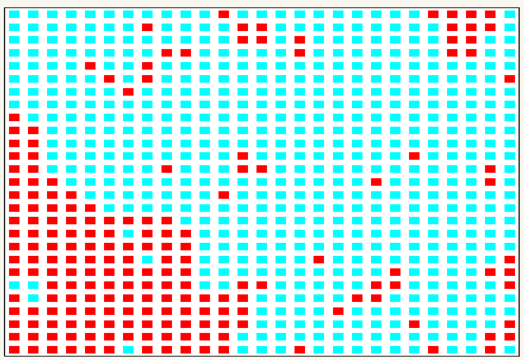

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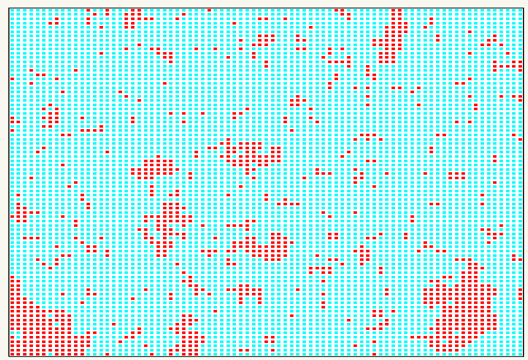

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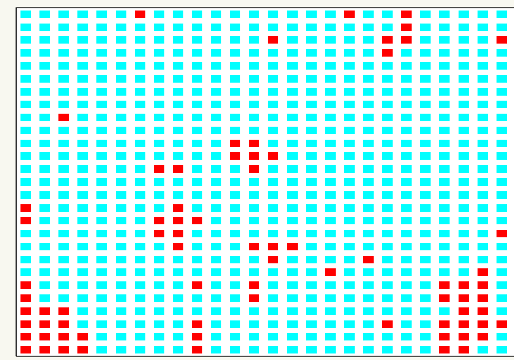

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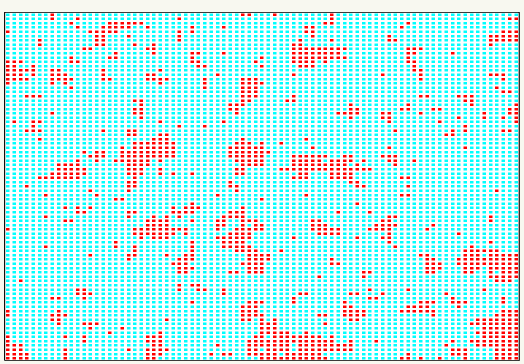

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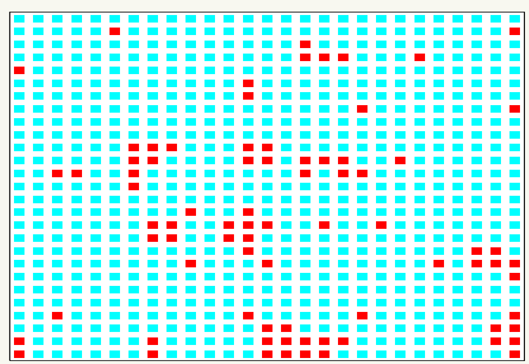

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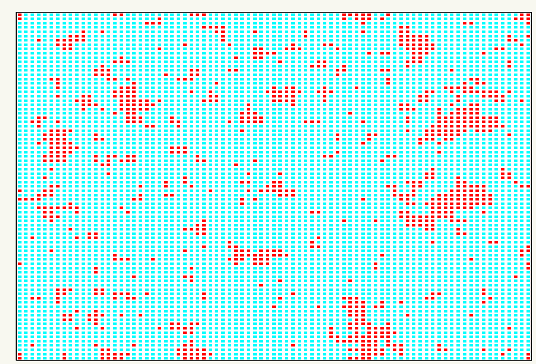

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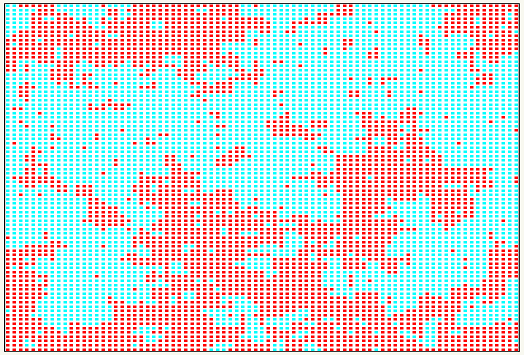

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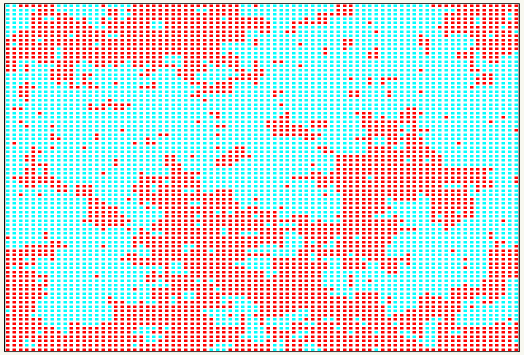

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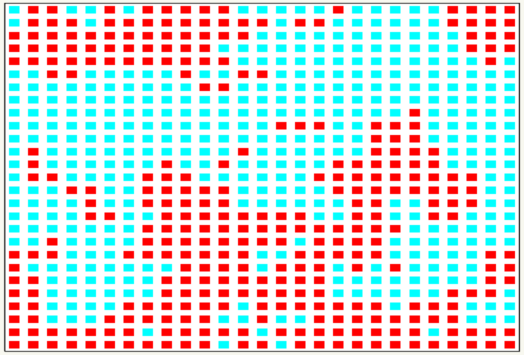

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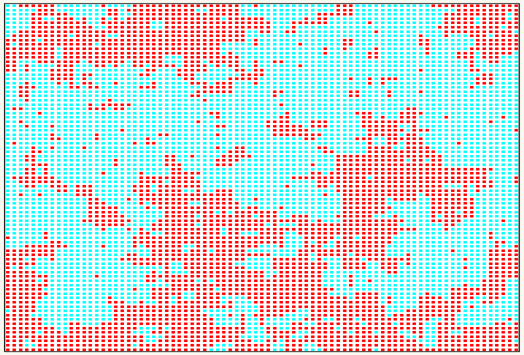

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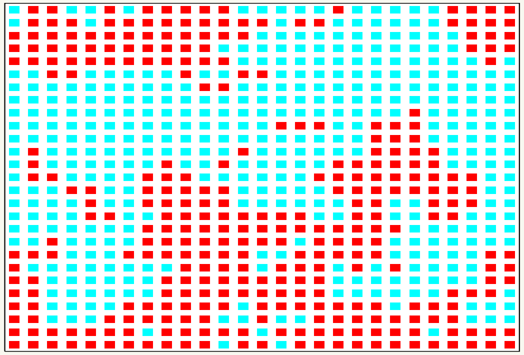

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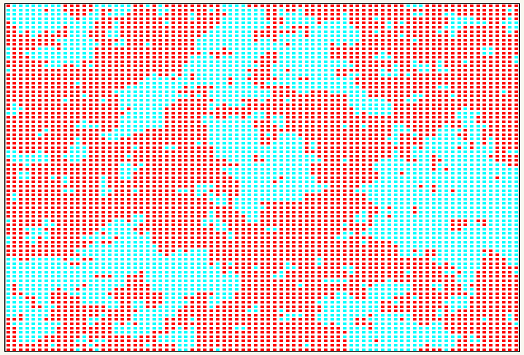

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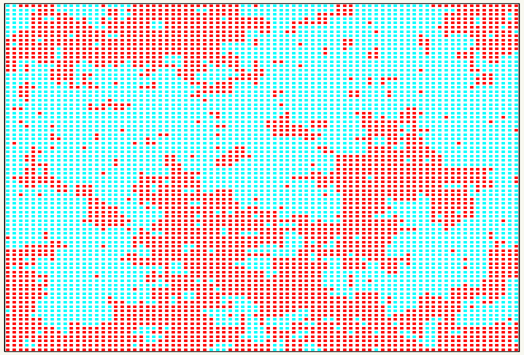

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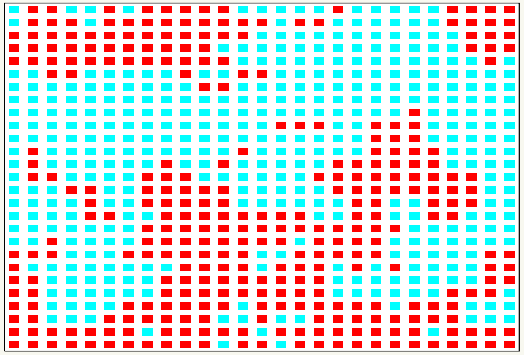

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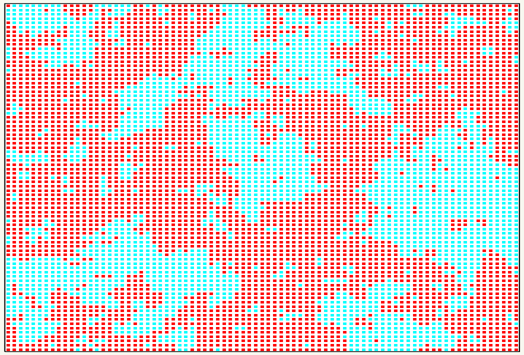

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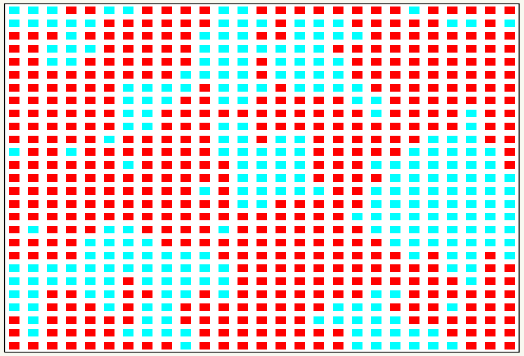

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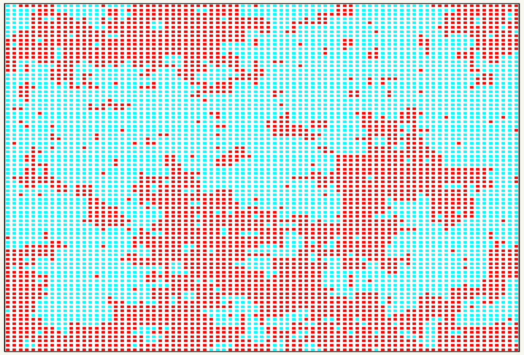

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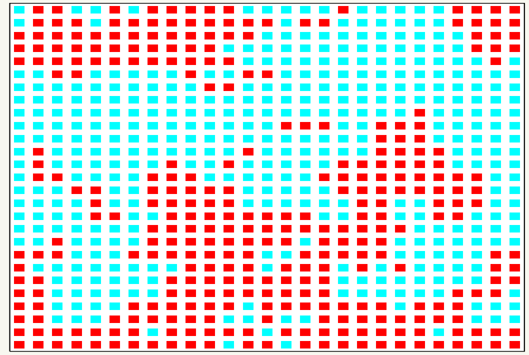

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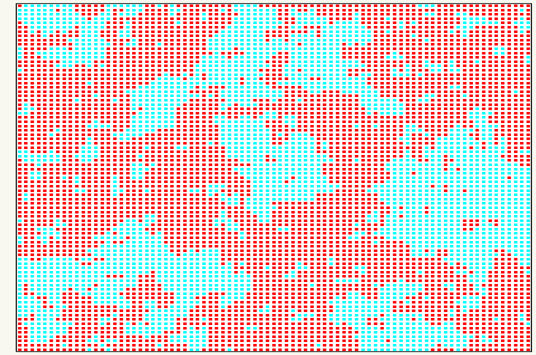

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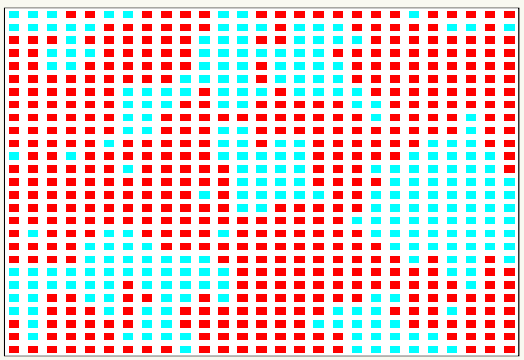

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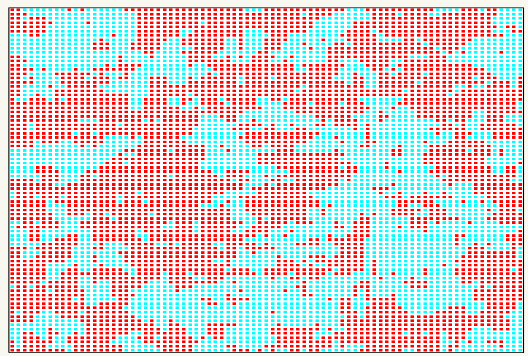

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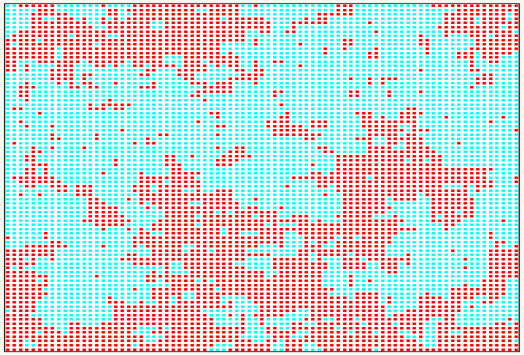

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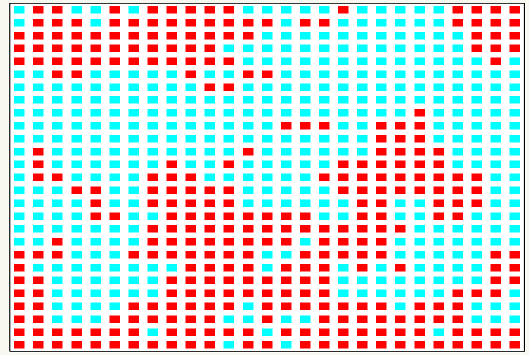

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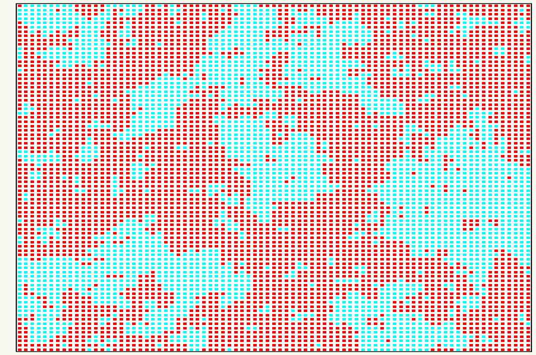

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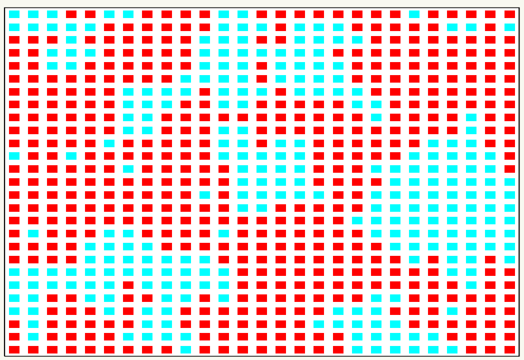

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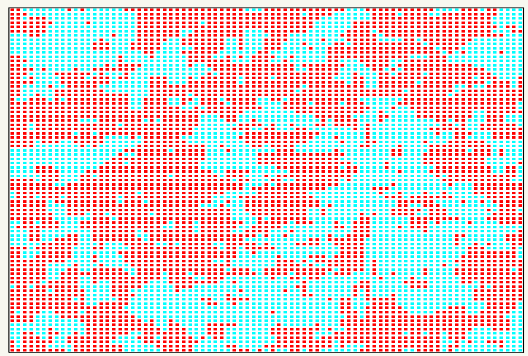

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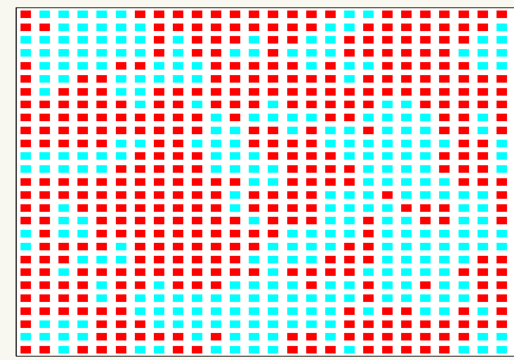

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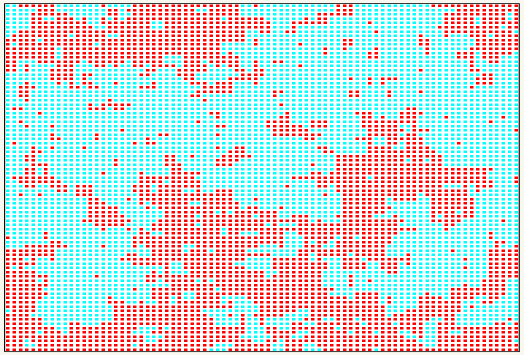

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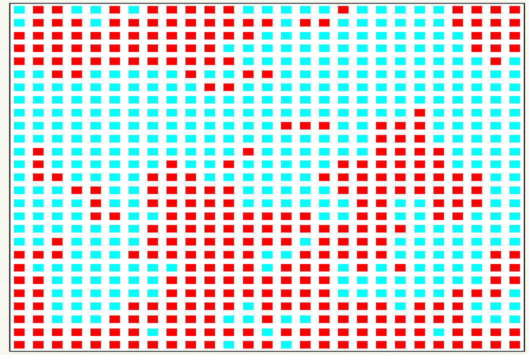

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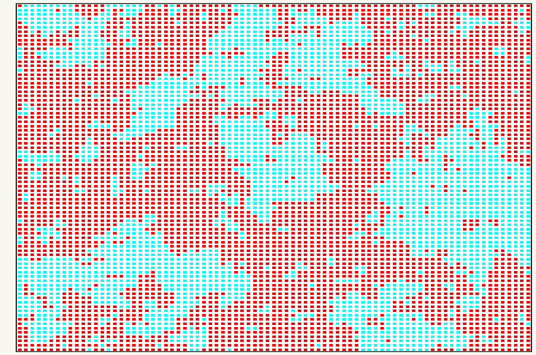

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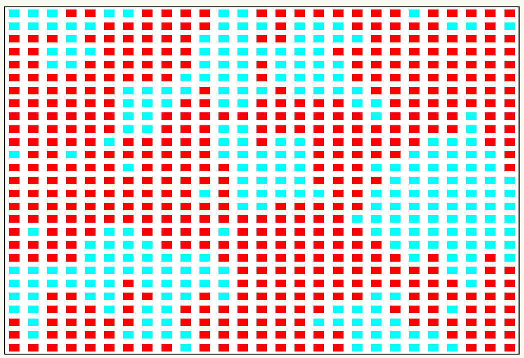

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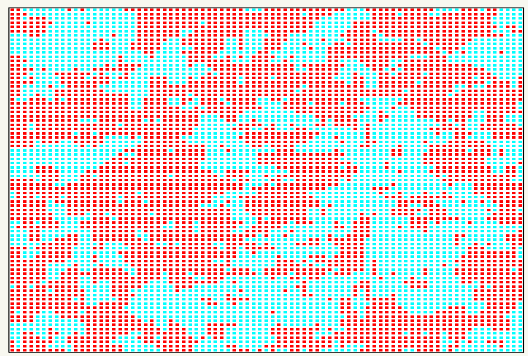

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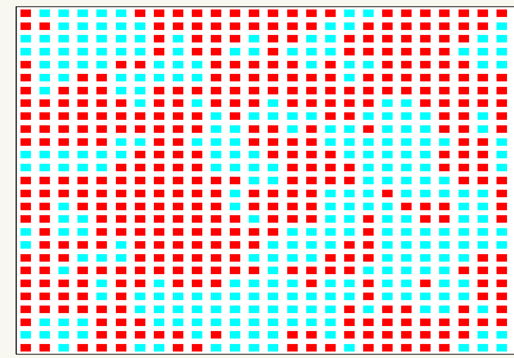

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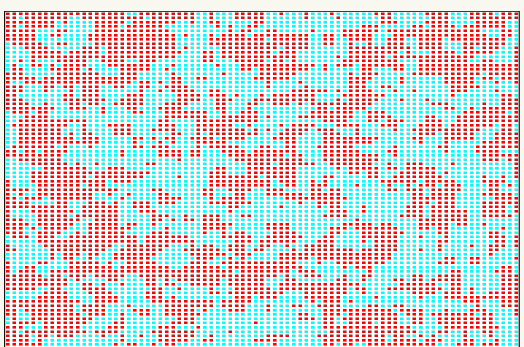

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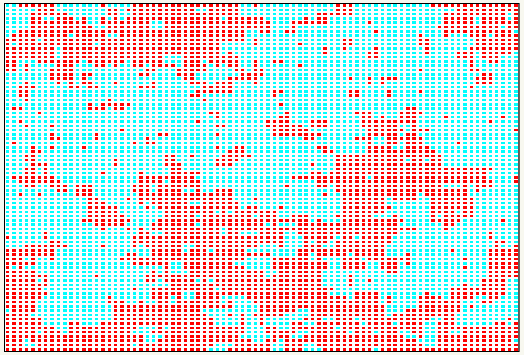
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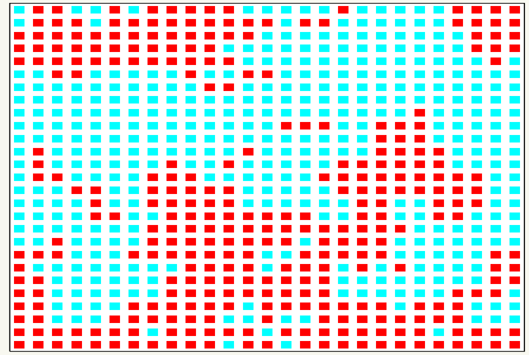
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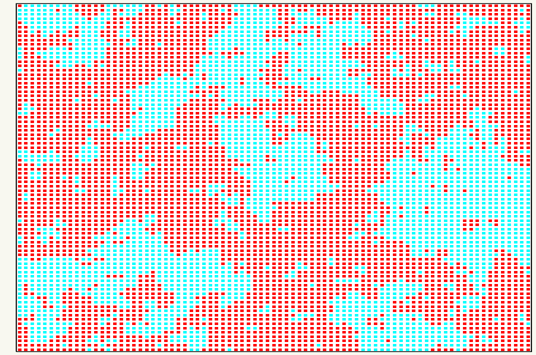
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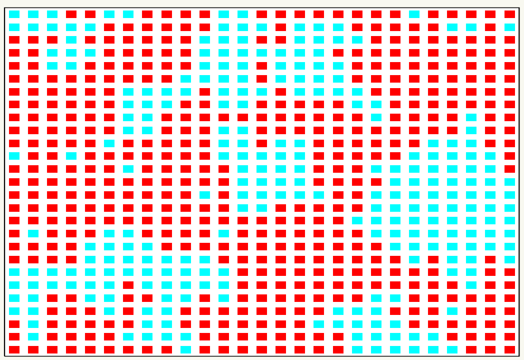
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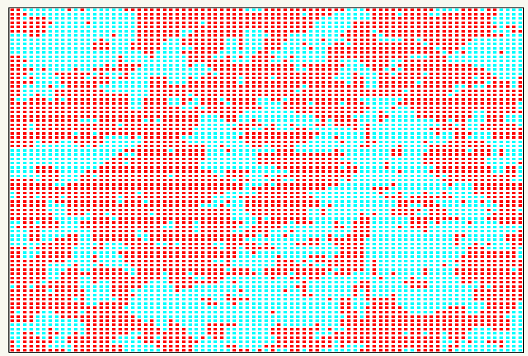
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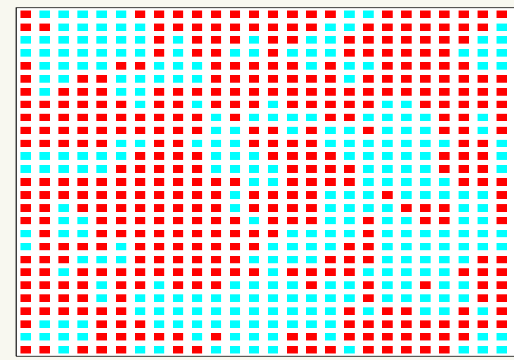
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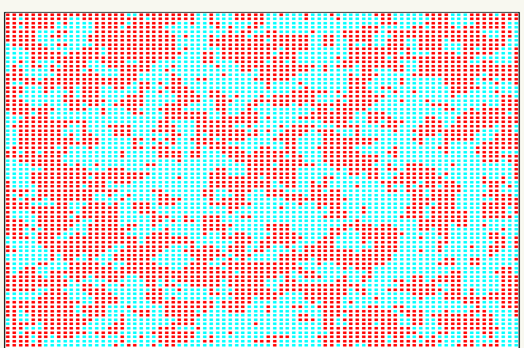
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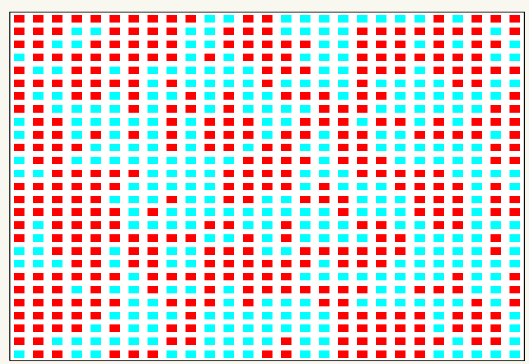
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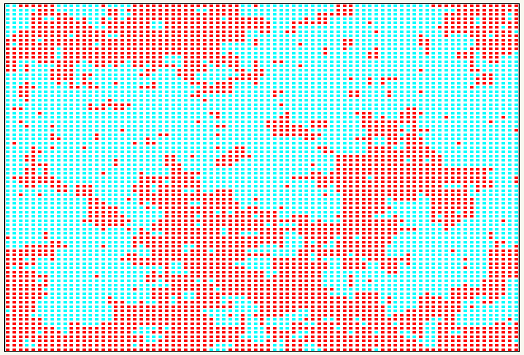
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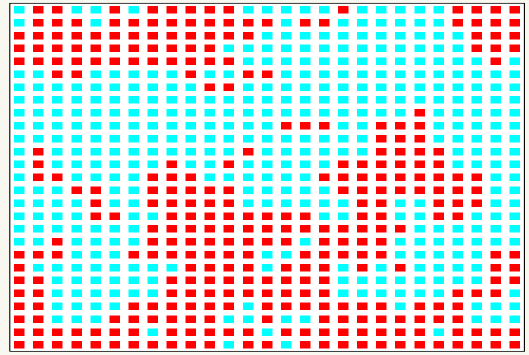
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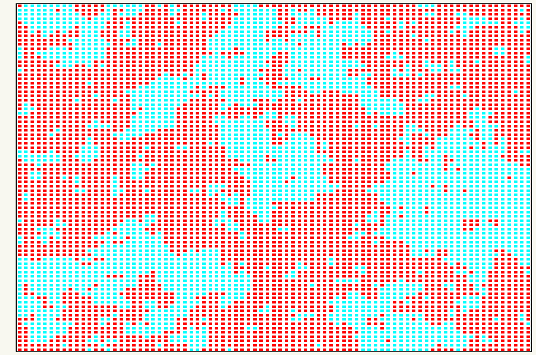
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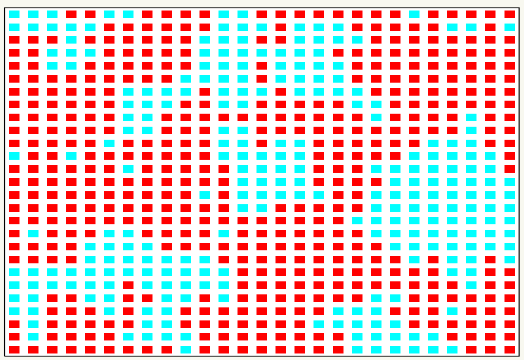
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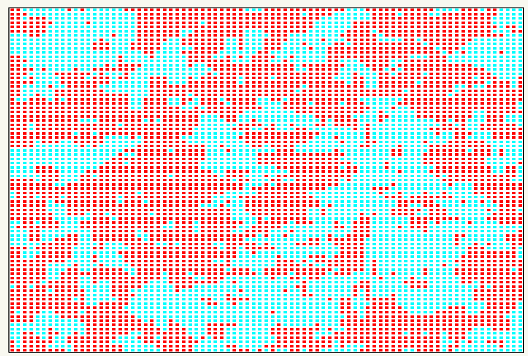
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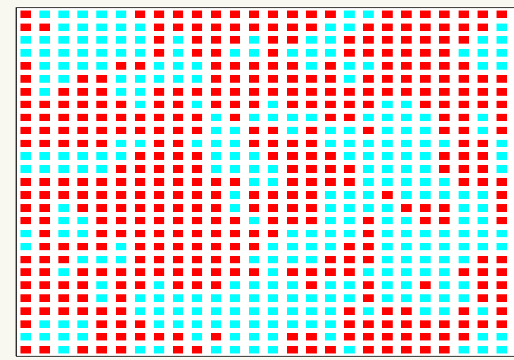
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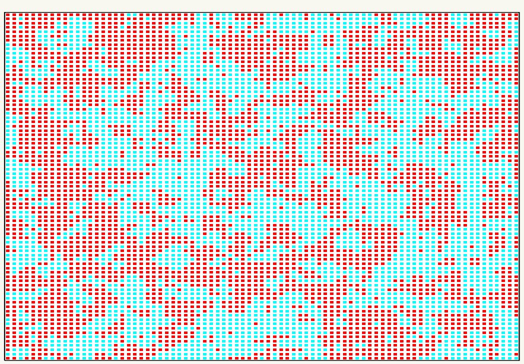
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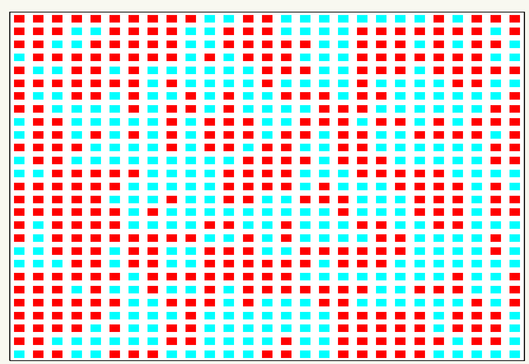
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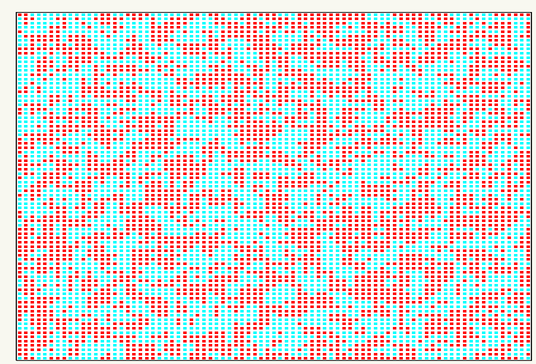
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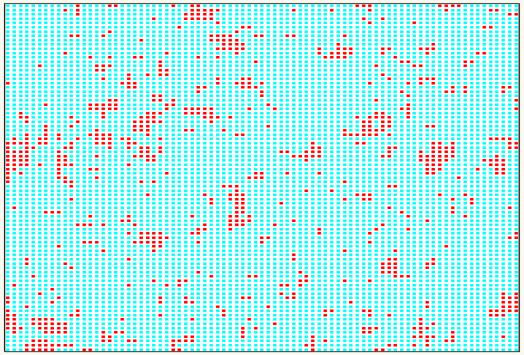
Renormalisation group, $T = T_c * 0.99, L = 6561$



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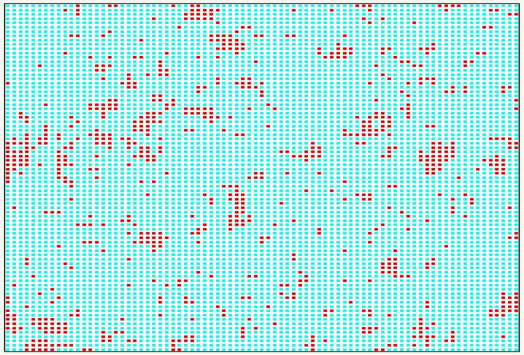
Renormalisation group, $T = T_c * 0.99, L = 6561$



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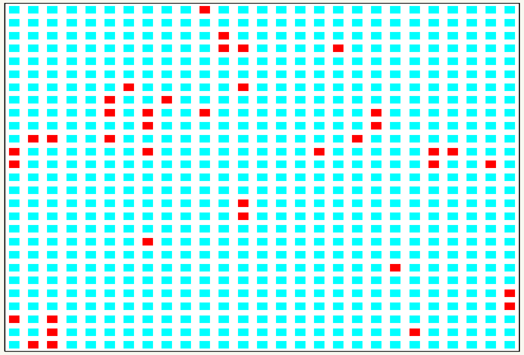

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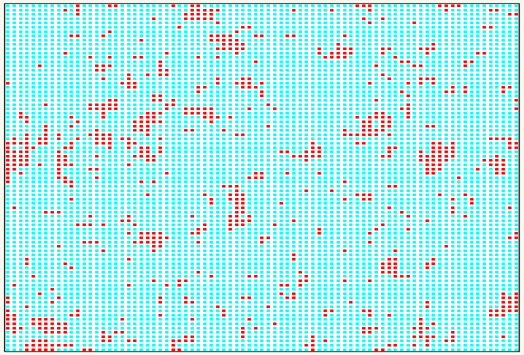

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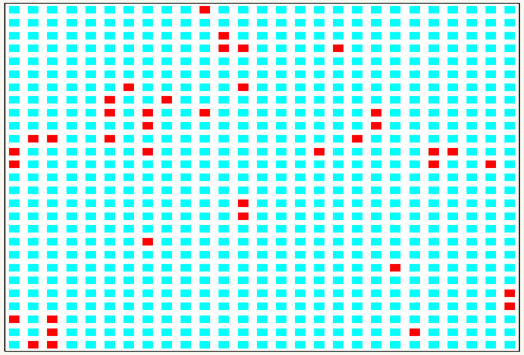

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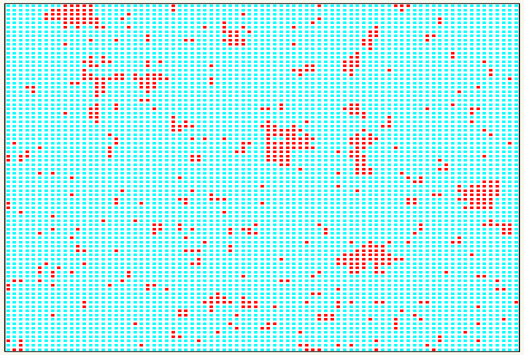

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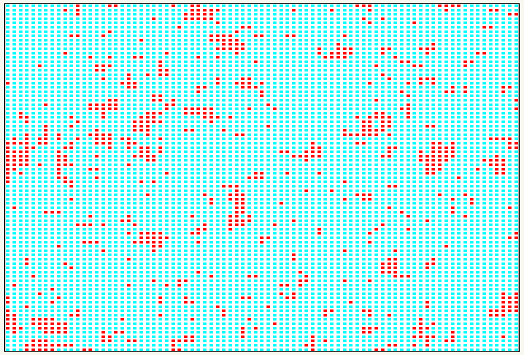
Renormalisation group, $T = T_c * 0.99, L = 6561$



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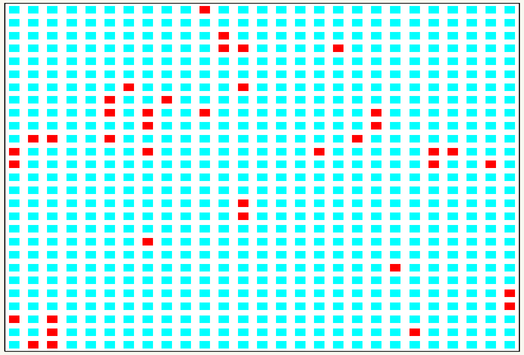

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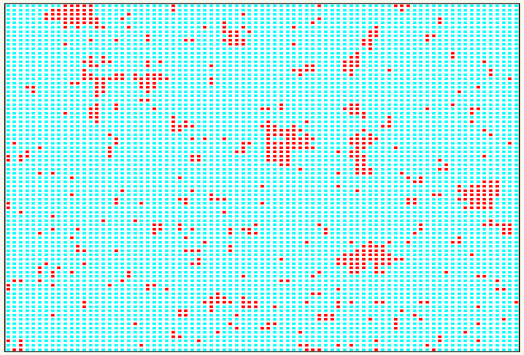

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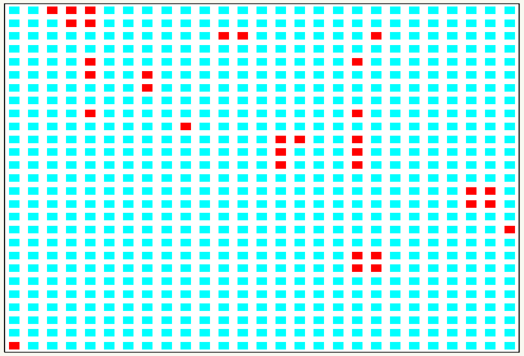

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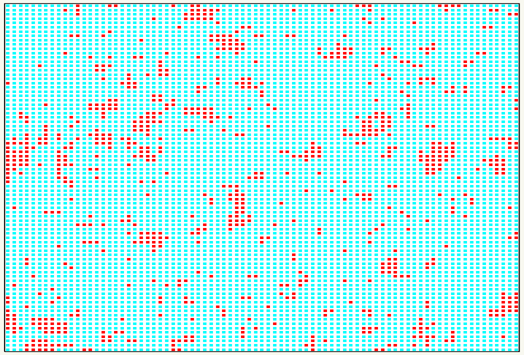

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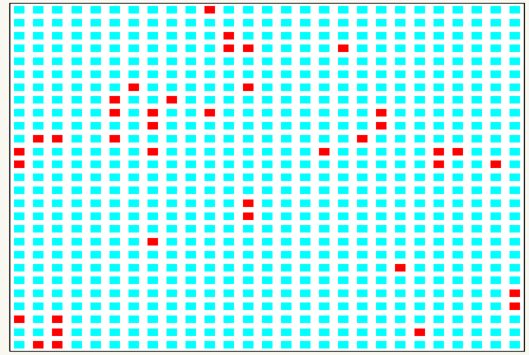

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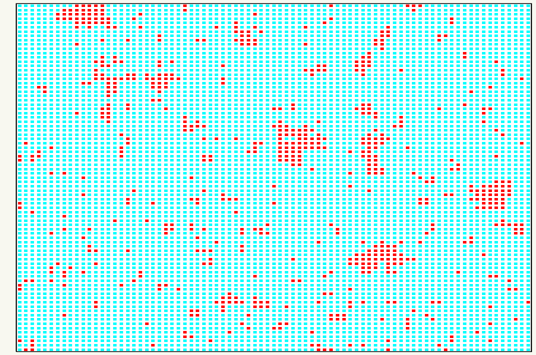

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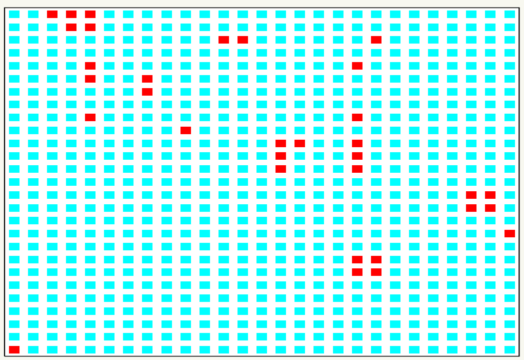

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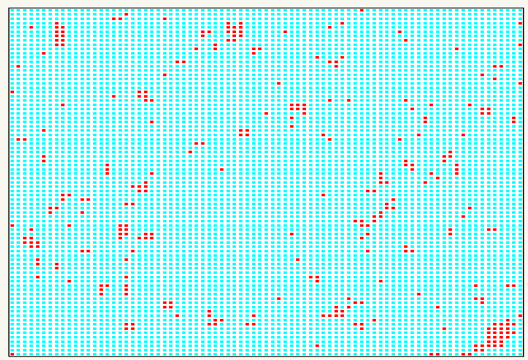

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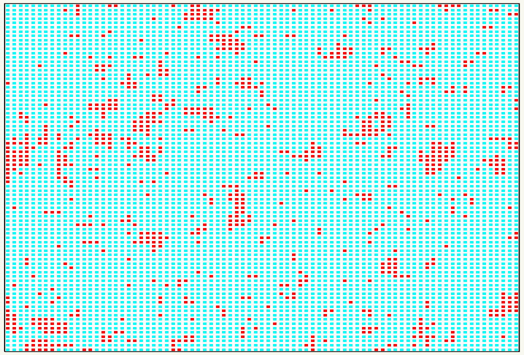
Renormalisation group, $T = T_c * 0.99, L = 6561$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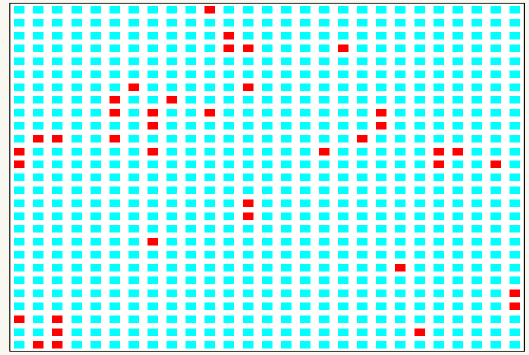
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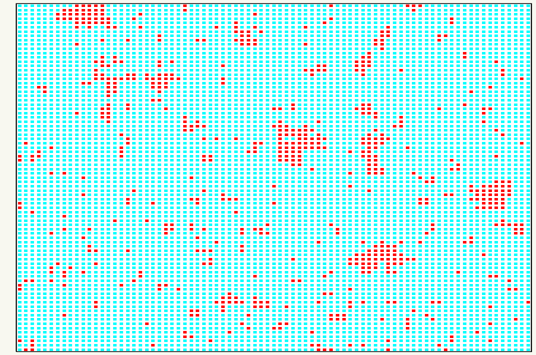
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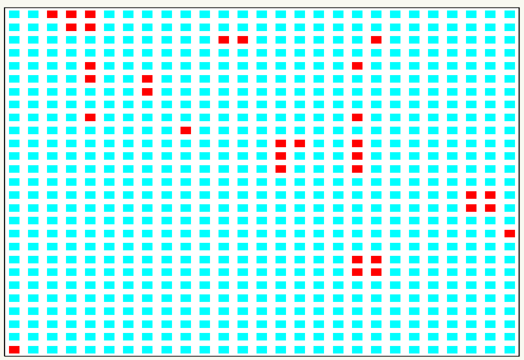
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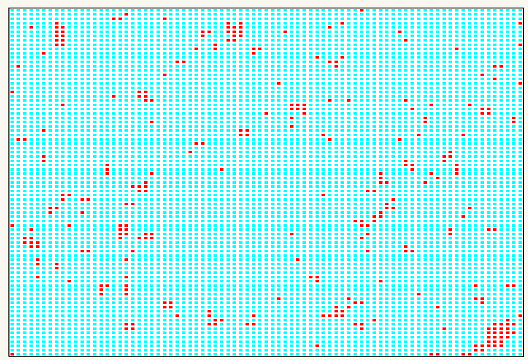
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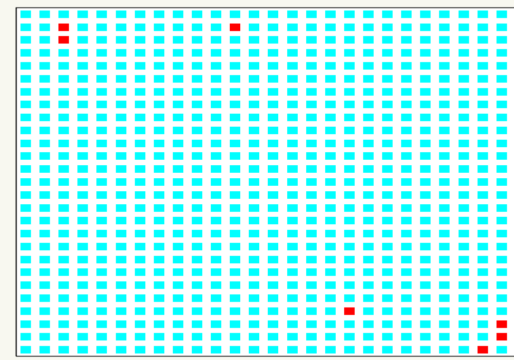
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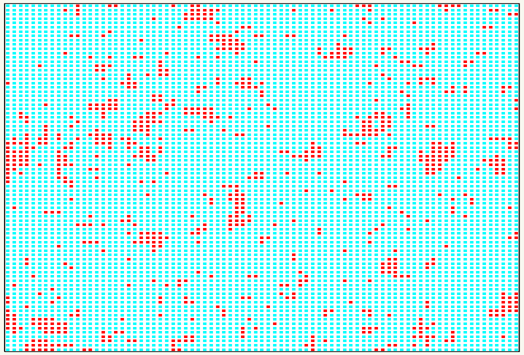
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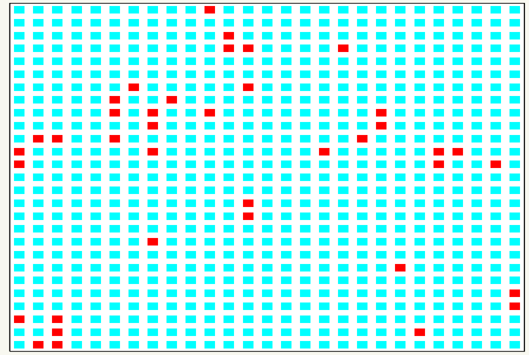
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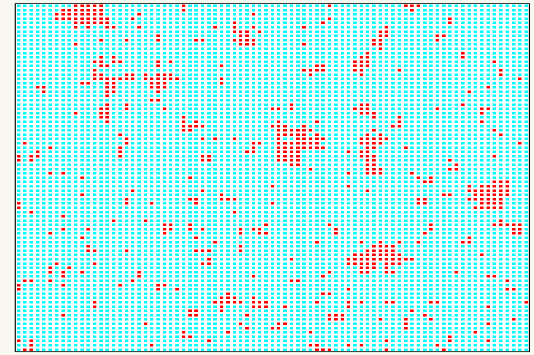
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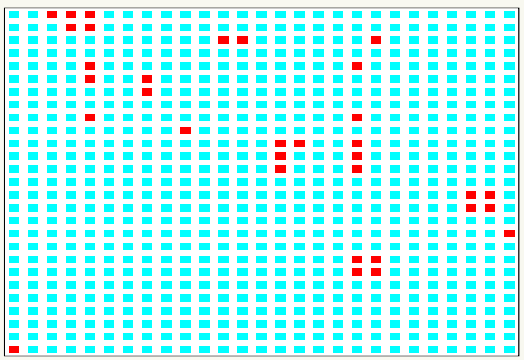
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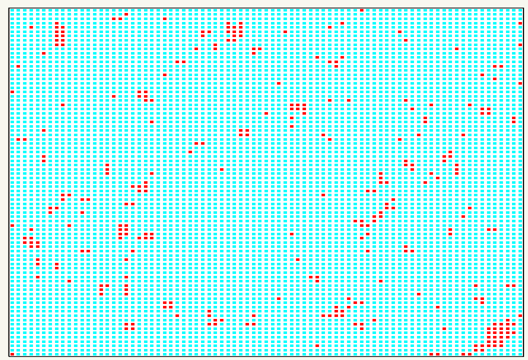
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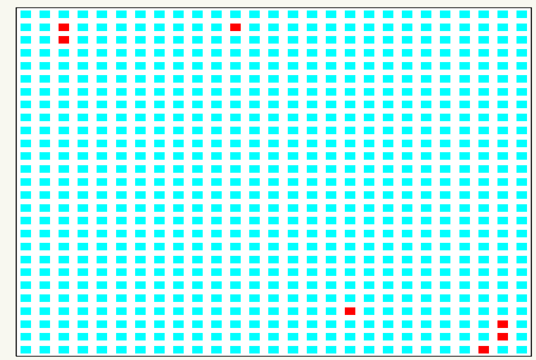
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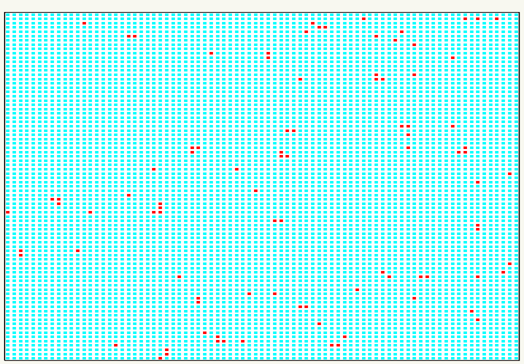
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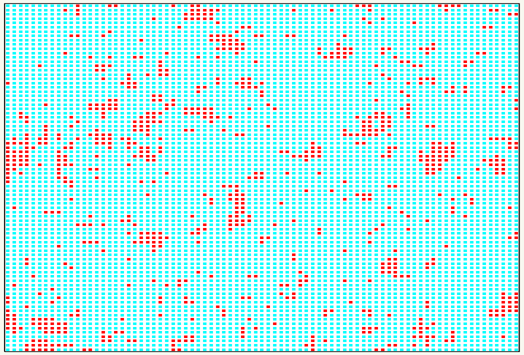
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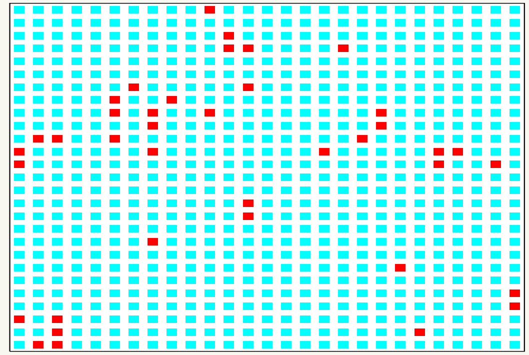
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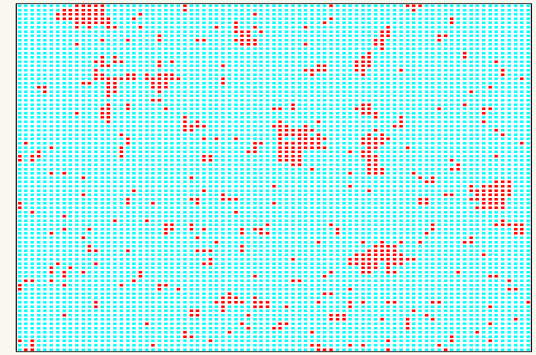
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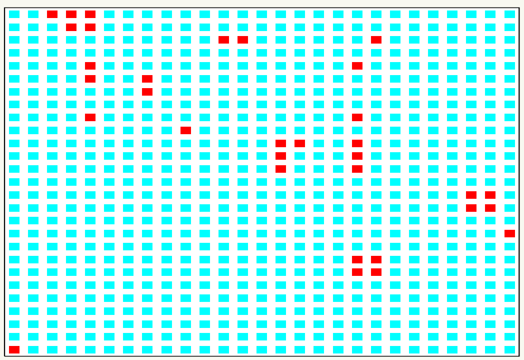
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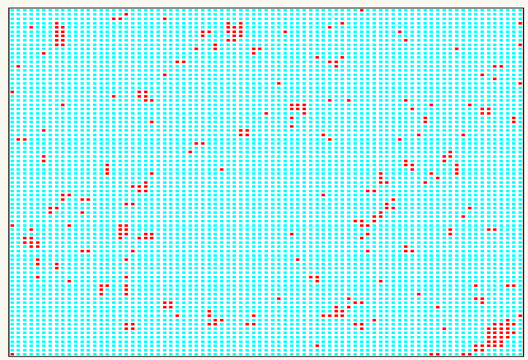
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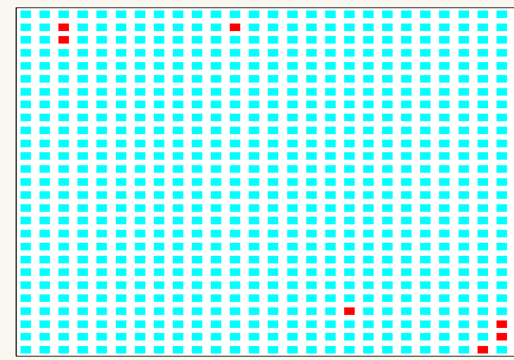
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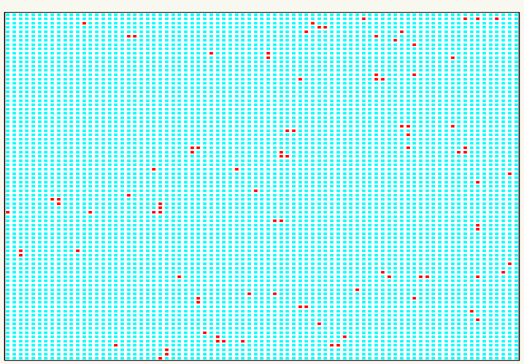
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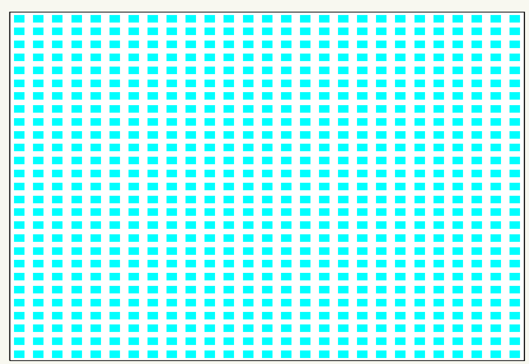
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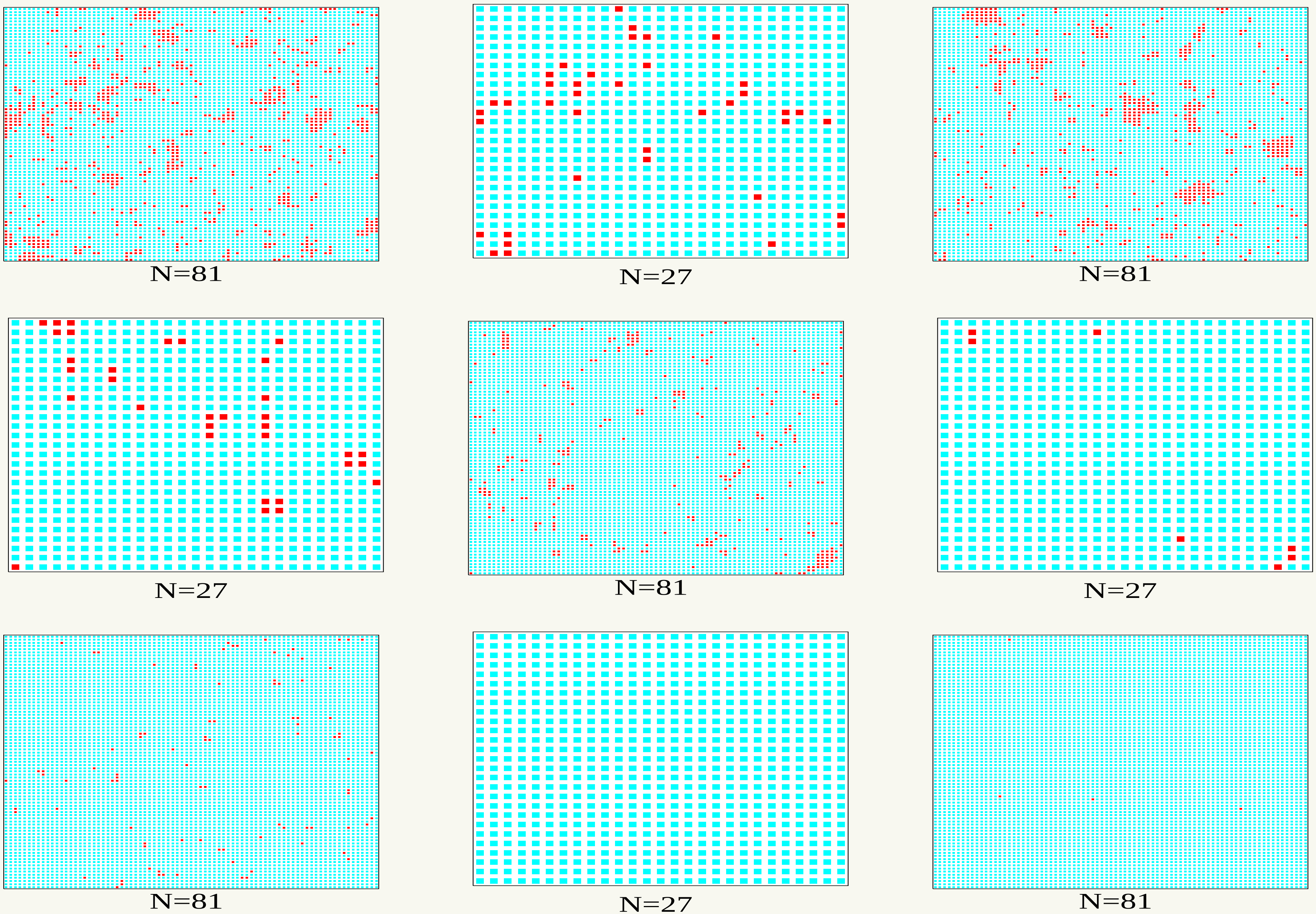
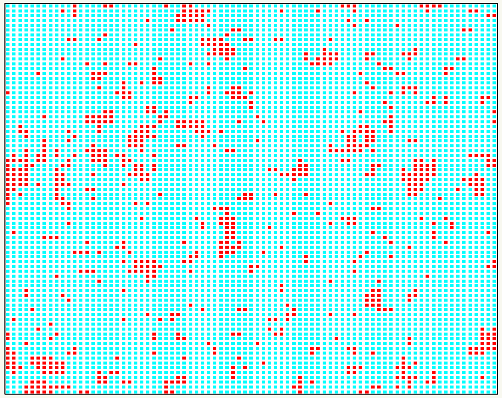
Renormalisation group, $T = T_c * 0.99, L = 6561$



Renormalisation group, $T = T_c \times 0.99, T_c$ and $T_c \times 1.01$



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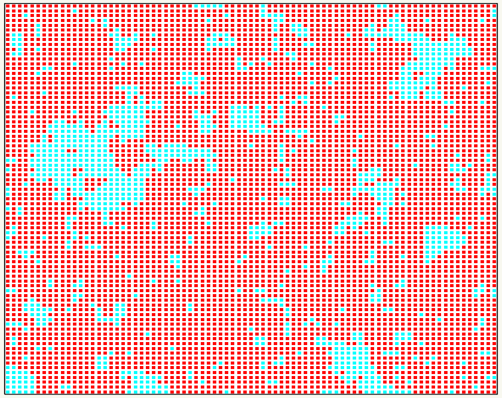

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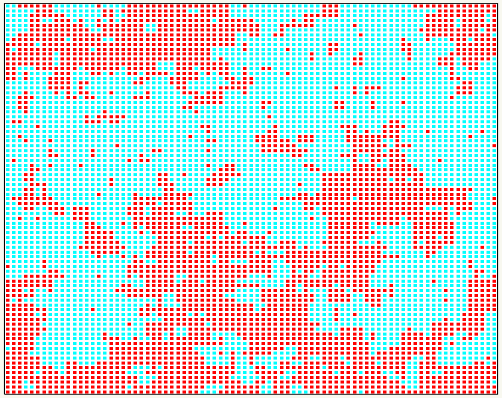

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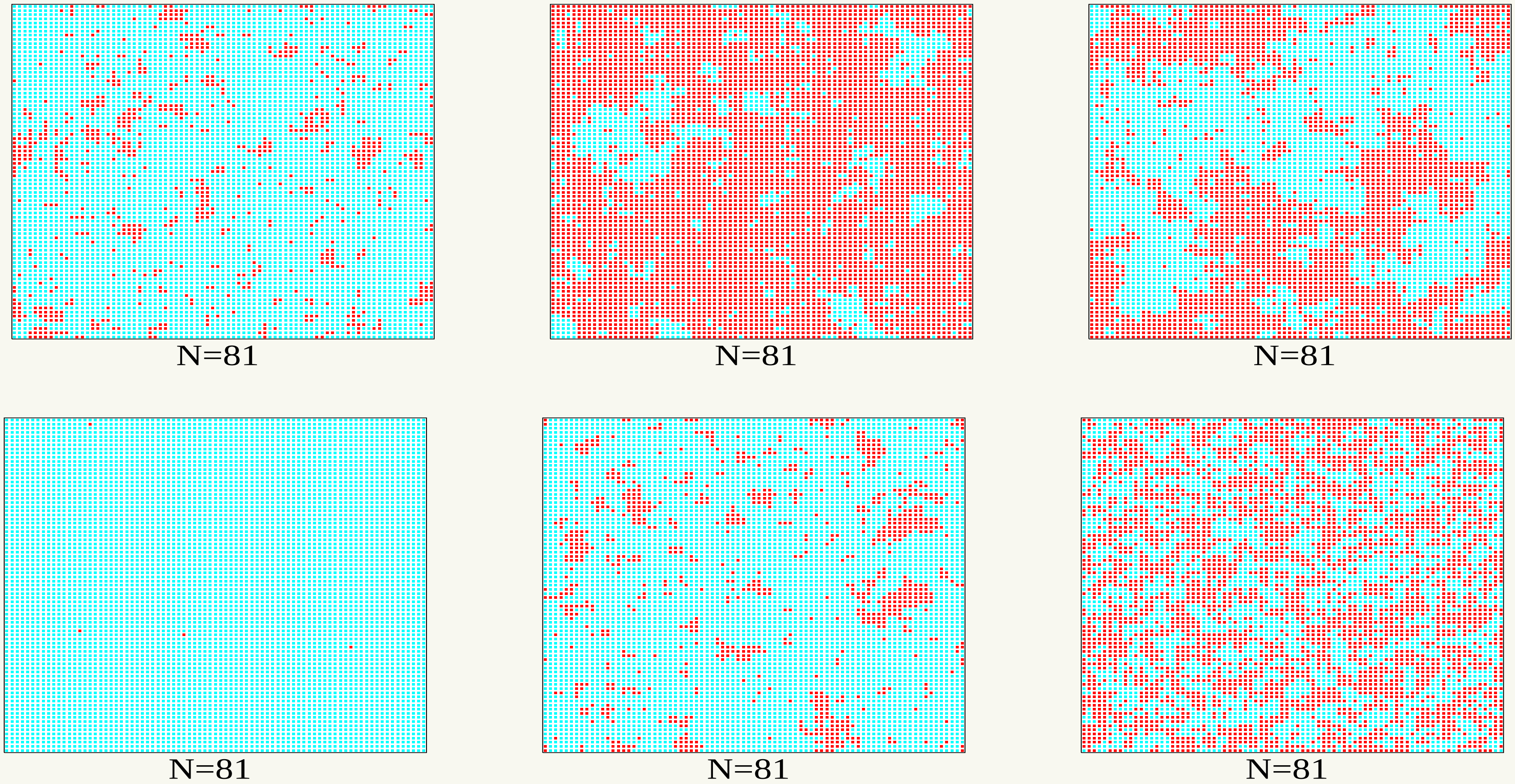
Renormalisation group, $T = T_c \times 0.99, T_c$ and $T_c \times 1.01$

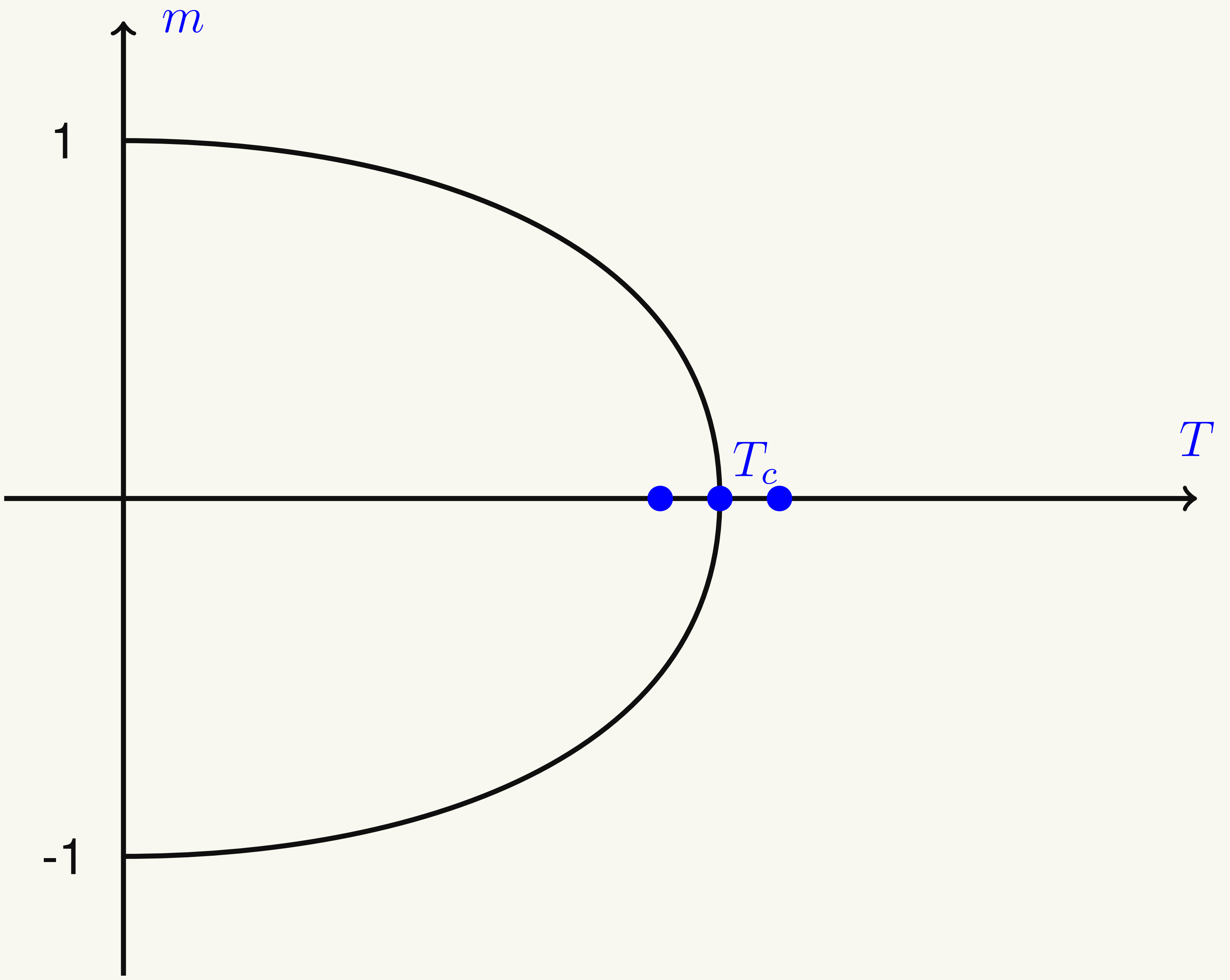


Renormalisation group

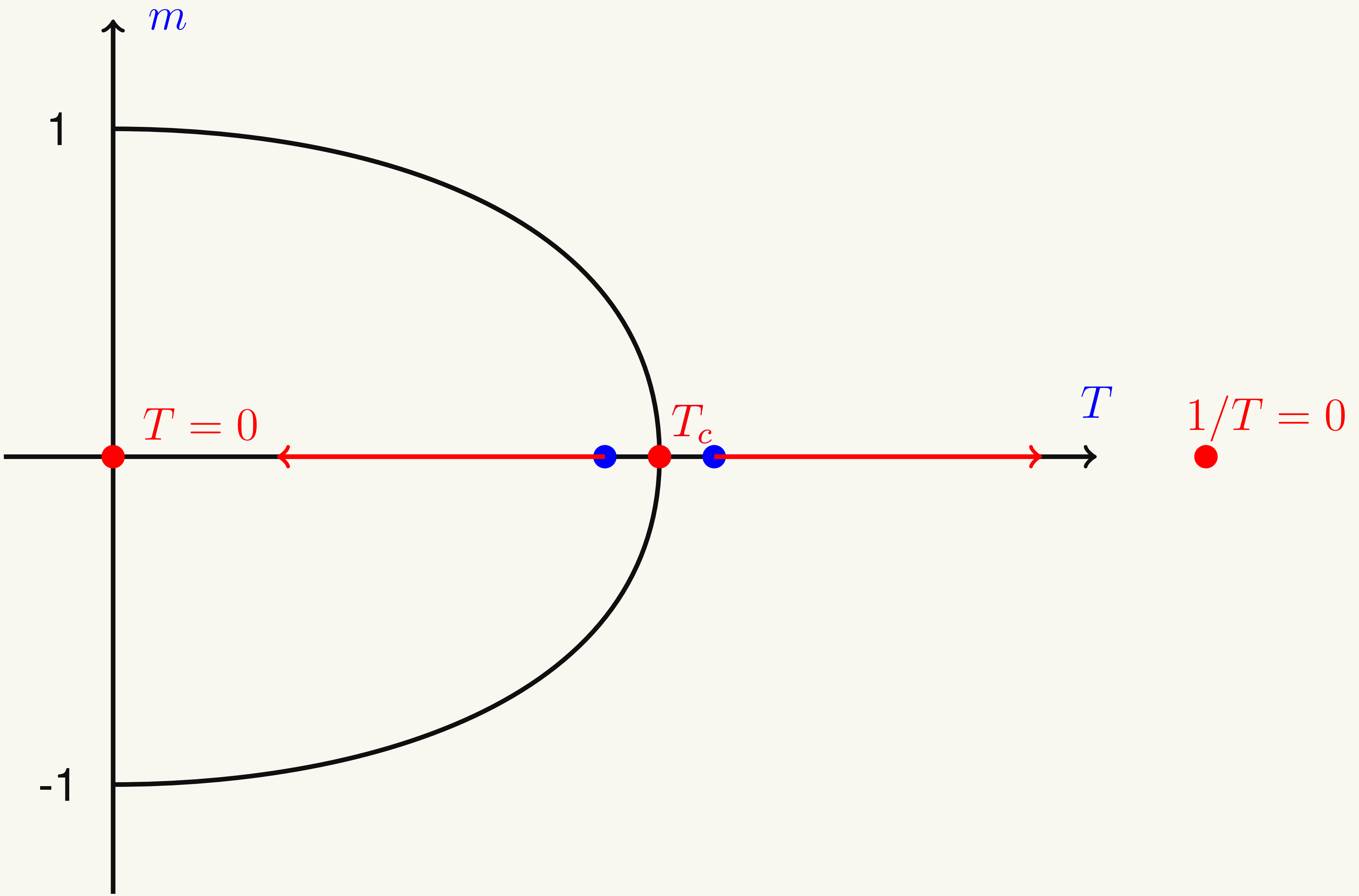
We observed that under rescaling there is three different behaviors under successive rescaling :

- For $T < T_c$ the system becomes more and more magnetized. It flows towards the zero temperature attractive fixed point.
- For $T = T_c$ the system does not change. It is scale invariant. Corresponds to a unstable fixed point.
- For $T > T_c$ the system becomes more and more disorganized, it flows towards the infinite temperature attractive fixed point (paramagnetic).

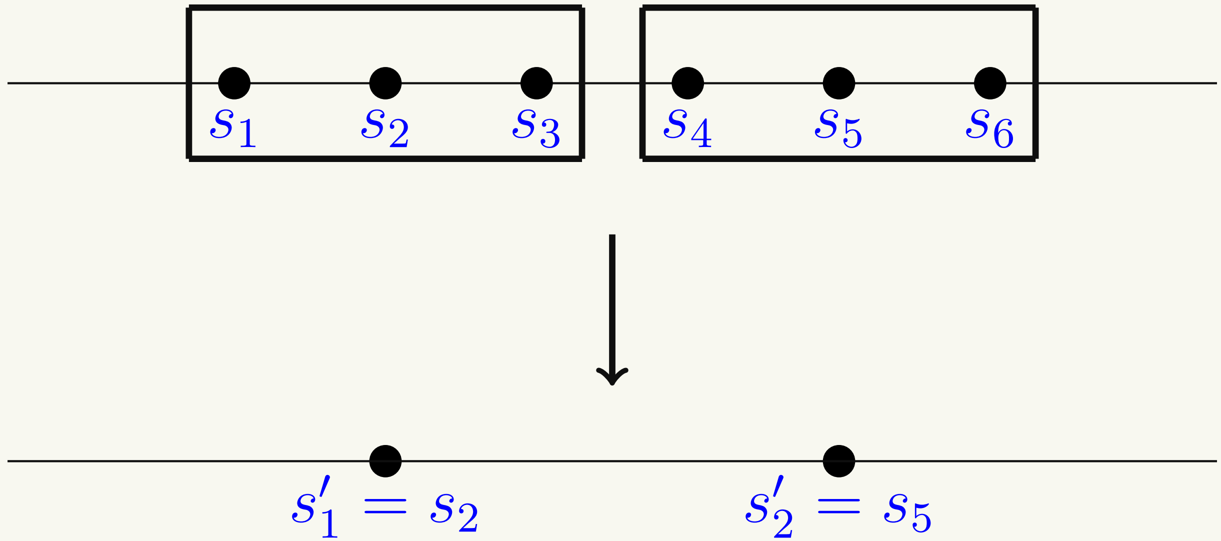
Renormalisation group



Renormalisation group



- We will show how this works for the one dimensional Ising model using decimation (simple version than block spins).



- We search a fixed point of the decimation transformation such that

$$Z = \sum_s e^{K s_i s_{i+1}} = \sum_{s'} e^{K s'_i s'_{i+1}} \tag{31}$$

Renormalisation group, $1d$ Ising model

- We use the relation $e^{Ks_is_j} = \cosh K(1+\tanh K s_i s_j)$ to obtain
$$e^{Ks_2s_3}e^{Ks_3s_4}e^{Ks_4s_5} = (\cosh K)^3(1+\tanh K s_2s_3)\times \times(1+\tanh K s_3s_4)(1+\tanh K s_4s_5) \tag{32}$$

- For the summation over s_3 and s_4, only terms with even powers have a non zero contribution, so
$$\sum_{s_3,s_4} e^{Ks_2s_3}e^{Ks_3s_4}e^{Ks_4s_5} = 2^2(\cosh K)^3(1+(\tanh K)^3 s_2s_5) \tag{33}$$

- Up to some multiplicative factor, which depends only on K, we got
$$\sum_{s_3,s_4} e^{Ks_2s_3}e^{Ks_3s_4}e^{Ks_4s_5} \simeq e^{K' s_2s_5} \tag{34}$$

- We obtain the condition

$$\tanh K' = (\tanh K)^3 \,, \tag{35}$$

  which is a renormalisation group equation.

- There is two fixed points for this equation :
  i) $\tanh K = 1$ which corresponds to $K = \infty$. Since $K = \beta J \simeq 1/T$, this is the zero temperature fixed point. Unstable fixed point.
  ii) $\tanh K = 0$ which corresponds to $K = 0$ or $T = \infty$. Stable fixed point.

Scaling theories

- We consider again the Landau-Ginzburg-Wilson Hamiltonian. We had obtained the following result :

$$\mathcal{H} = \int d^d r [\frac{1}{2}(\nabla S(r))^2 + t a^{-2} S^2(r) + u a^{d-4} S^4 + h a^{-d/2-1} S] \quad (36)$$

with a a dimension parameter (the lattice spacing).

- We want to rescale this parameter and impose invariance of the Hamiltonian : $a \to ba$. This will then impose the following redefinitions of the parameters t, u and h.

$$\begin{aligned} t' &= b^2 t \\ h' &= b^{d/2+1} h \\ u' &= b^{4-d} u \end{aligned} \quad (37)$$

- In any dimension, the t parameter will increase. So the criticality has to be associated to the condition $t = 0$.

Scaling theories

- The condition $t = 0$ is the same condition we already obtained from the mean field approach (the quadratic term in M was $(1 - \beta J) \to T_c = J$).
- The same is also true for the linear term. The system will be critical only at zero magnetic field.
- The relevance of the quartic term will depend on the dimension :
  For $d > 4$, $u \to 0$ under rescaling. The term is irrelevant and we can forget it.
  For $d < 4$, $u \to \infty$ under rescaling. The term is relevant :
  $\epsilon = 4 - d$ expansion, Wilson-Fisher fixed point.

- We will consider the case with no external field, *i.e.* $h = 0$ and with $u = 0$ but allowing a thermal deviation.
- This corresponds to the Gaussian theory with a mass term :

$$\mathcal{H} = \int d^d r[(\nabla S(r))^2 + m^2 S^2(r) + a_1 S(r) + a_2 S^2(r) + a_3 S^3(r) + a_4 S^4(r) + \cdots] \,. \tag{38}$$

The last terms with $a_1, a_2, a_3, a_4, \cdots$ are perturbations of the Gaussian theory.

- Propagator of this theory is simply $\frac{1}{k^2+m^2}$.

- We can compute the correlation function of the field $S(r)$ as

$$< S(0)S(r) > \simeq \int d^d k \frac{e^{i\vec{k}.\vec{r}}}{k^2 + m^2} \simeq \frac{e^{-rm}}{r^{(d-1)/2}} \tag{39}$$

(for $r >> 1/m$...).

- The mass term is the inverse of a length. We replace this term by $m \simeq 1/\xi(T)$ which defines the correlation length $\xi(T)$. The critical point corresponds to the cancelation of the mass m and is the point at which $\xi(T)$ diverges. In that case, we can check that the correlation function is given by :

$$< S(0)S(r) > \simeq \frac{1}{r^{d-2}} \tag{40}$$

Scaling theories

- This correlation length gives a scale to the problem. For $r >> \xi(T)$, the correlation decreases very quickly.
- For a given temperature, if we rescale the system by $r \to br$, then the correlation is changed by

$$< S(0)S(r) > \to \frac{e^{-(b-1)\frac{r}{\xi(T)}}}{b^{(d-1)/2}} < S(0)S(r) > \tag{41}$$

- Then the correlation function is modified and in the limit of large b (or of many small step) goes to zero.
- This correspond to the situation for $T > T_c$ or $T < T_c$ as we have seen before for the $2d$ Ising model.
- The case $T = T_c$ corresponds to $\xi(T_c) = \infty$ for which the correlation function is a pure power law

Scaling theories

- Scale invariance is one of the particular symmetry that we can impose.
- We can consider more general operators, like $S(r)$ or $S^2(r)$, $S^3(r)$ etc. Each operator A will have some scaling dimension Δ_A.
- Invariance under scale invariance (+ translation + rotation + normalisation + inversion) will impose the following general result

$$
\begin{aligned}
< A(\vec{r_1})B(\vec{r_2}) > &= f(\vec{r_1} - \vec{r_2}) = f(|\vec{r_1} - \vec{r_2}|) \\
&= \frac{C_{A,B}\delta_{A,B}}{r^{\Delta_A + \Delta_B}} = \frac{\delta_{A,B}}{r^{2\Delta_A}}
\end{aligned} \tag{42}
$$

- We can even do better by imposing conformal invariance or local scale invariance $\rightarrow$ Conformal Field Theories

Renormalisation and scaling theory

Renormalisation and scaling theory

- In the two previous sections, we show that at the critical point there is scale invariance. Away from this point, under a rescaling, the parameters controlling the deviation to the critical point are rescaled.
- In general, one can have more than one parameter (*i.e.* temperature and magnetic field, density of vacancies, etc.)
- The transformation under a rescaling is written as

$$\{K'\} = \mathcal{R}_b\{K\} \ , \tag{43}$$

with $\{K\}$ the set of parameters, b the scaling parameter and $\mathcal{R}$ the transformation under a rescaling. For the 1d Ising model, we had the transformation $\tanh K' = (\tanh K)^3$

Renormalisation and scaling theory

- We suppose that there exists a fixed point of the RG transformation $\{K^*\}$. We will assume that $\mathcal{R}$ is differentiable at the fixed point. Then we can linearize the RG equations close to the fixed point.

$$K'_a - K^*_a \simeq \sum_b T_{ab}(K_b - K^*_b) \ , \tag{44}$$

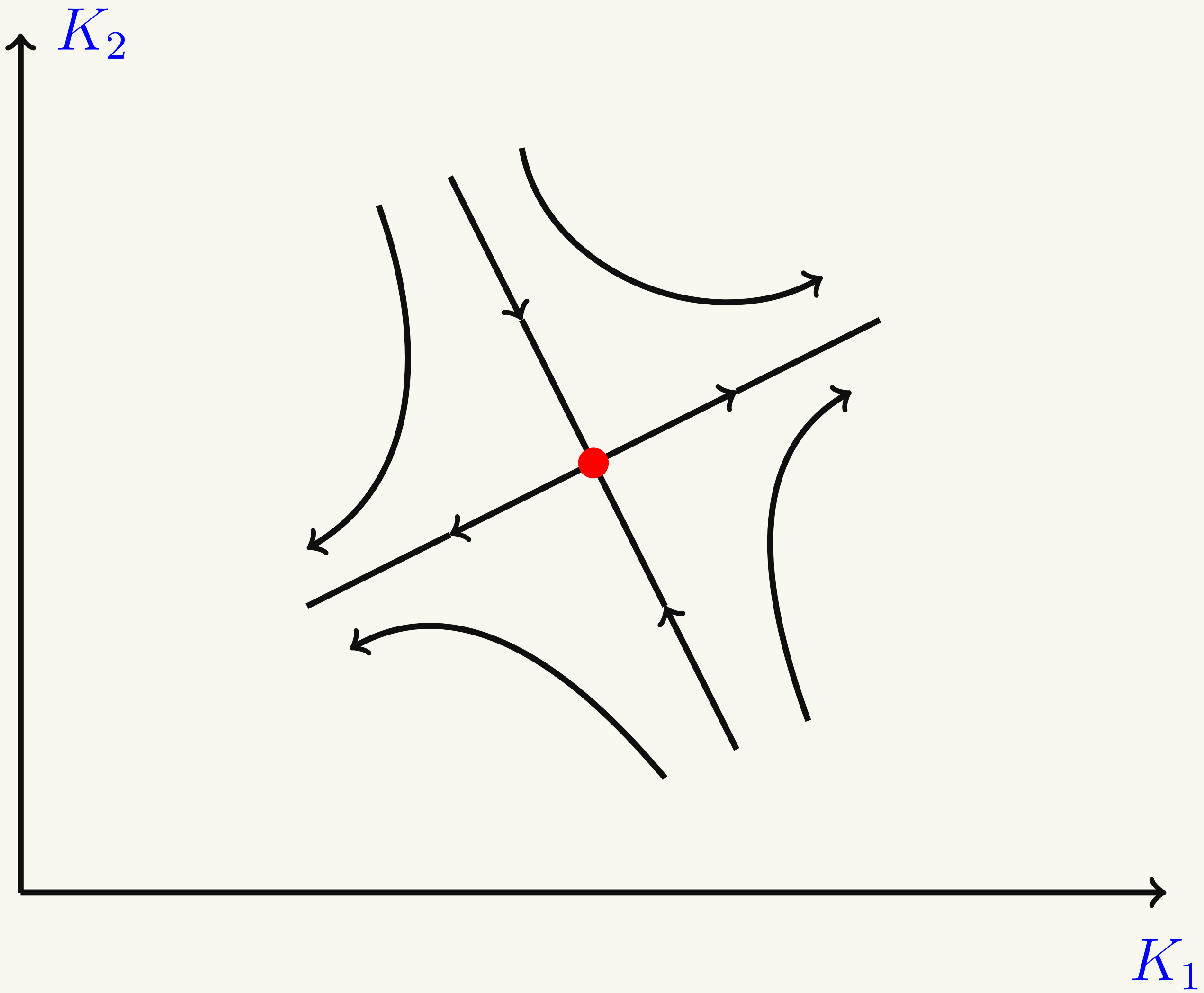
  with $T_{ab} = \frac{\partial K'_a}{\partial K_b}_{|K=K^*}$

- $\lambda_i, \{\varphi^i\}$ are the eigenvalues and eigenvectors of T_{ab}.
- $u_i = \sum_a \varphi^i_a (K_a - K^*_a)$ are defined as the scaling variables.
- Under a RG transformation, their transform as

$$u'_i = \lambda_i u_i \ . \tag{45}$$

  The relation $\lambda_i = b^{y_i}$ defines the RG eigenvalues y_i.

Renormalisation and scaling theory



Renormalisation and scaling theory

- $y_i > 0 \to u_i$ is relevant
- $y_i < 0 \to u_i$ is irrelevant
- $y_i = 0 \to u_i$ is marginal
- We consider again the Ising model : it has two scaling variables : the thermal u_t, y_t and the magnetic u_h, y_h.
- For the Ising model, no mixing of these parameters : Symmetry plays a role !!!
- Under a RG transformation we have

$$\begin{aligned} \mathcal{Z} &= \sum_S e^{-\mathcal{H}(S)} = \sum_{S'} e^{-\mathcal{H}'(S')} \\ &= e^{-Nf(\{K\})} \end{aligned} \tag{46}$$

- This then implies the relation

$$f(\{K\}) = g(\{K\}) + b^{-d} f(\{K'\}) \tag{47}$$

- To explain this relation, remember for the 1d Ising model :

$$\sum_{s_3,s_4} e^{K s_2 s_3} e^{K s_3 s_4} e^{K s_4 s_5} = 2^2 (\cosh K)^3 (1 + (\tanh K)^3 s_2 s_5)\ . \tag{48}$$

  This $(\cosh K)^3$ term will give something proportional to N so the g part, while the $(\tanh K)^3$ will give something proportional to $b^{-1}N$ so the f part.

- Only the homogeneous f part will be important, the other one will give an analytical function of the parameters and thus does not contribute to the computation of the critical exponents. This will be denoted by f_s in the following.

Renormalisation and scaling theory

- We can then iterate n times the RG transformations

$$f_s(u_t, u_h) = b^{-d} f_s(b^{y_t} u_t, b^{y_h} u_h) = b^{-nd} f_s(b^{n y_t} u_t, b^{n y_h} u_h) \qquad (49)$$

- We choose n such that $b^{n y_t} u_t = u_{t_0}$ with u_{t_0} a fixed value.

$$f_s(u_t, u_h) = |u_t/u_{t_0}|^{d/y_t} f_s(u_{t_0}, u_h (u_t/u_{t_0})^{-y_h/y_t}) \ . \qquad (50)$$

Or

$$f_s(t, h) = |t/t_0|^{d/y_t} \varphi(\frac{h/h_0}{|t/t_0|^{y_h/y_t}}) \ , \qquad (51)$$

with φ some scaling function.

- Form there, it is very easy to deduce a relation of all the critical exponents with y_t and y_h.

Renormalisation and scaling theory

- Specific heat :

$$\frac{\partial^2 f}{\partial t^2}_{|h=0} = |t|^{d/y_t - 2} \to \alpha = 2 - \frac{d}{y_t} \tag{52}$$

- Spontaneous magnetization :

$$\frac{\partial f}{\partial h}_{|h=0} = |t|^{(d-y_h)/y_t} \to \beta = \frac{d - y_h}{y_t} \tag{53}$$

- Susceptibility :

$$\frac{\partial^2 f}{\partial h^2}_{|h=0} = |t|^{(d-2y_h)/y_t} \to \gamma = \frac{2y_h - d}{y_t} \tag{54}$$

- All these exponents depend only of y_t and y_h : Scaling relations

$$\alpha + 2\beta + \gamma = 2 \;\; ; \;\; \alpha + \beta(1 + \delta) = 2 \tag{55}$$

Perturbative renormalisation group

Perturbative renormalisation group

- We have seen that a rather general theory can be described as a Gaussian fixed point + perturbations

$$\mathcal{H} = \int d^d r [\frac{1}{2}(\nabla S(r))^2 + ta^{-2} S^2(r) + ua^{d-4} S^4 + ha^{-d/2-1} S] \quad (56)$$

  The Gaussian fixed point is the point in parameters space with $t = u = h = 0$. At this point the theory is very simple and we can compute any correlation function (free field theory !!!).

- We have also seen that the perturbation associated to t is always going to be relevant close to the fixed point. If we start from $t \neq 0$, under rescaling, we will end up in a massive field theory. So we need to fine tune this quantity to zero. The same is also true for the magnetic perturbation.

- As for the quartic perturbation, it is irrelevant for $d > 4$ and relevant for $d < 4$. We will now consider the case when d is slightly lower than 4 and define a parameter $\epsilon = 4 - d$.
- We will then compute close to the fixed point corresponding to $t = u = h = 0$.
- More generally, we can consider a theory for which we have an "exact" solution with an Hamiltonian $\mathcal{H}_0$ and a set of operators ϕ_i :

$$\mathcal{Z} = \int d\phi e^{-\mathcal{H}_0 - \sum_i g_i \int \phi_i(r) \frac{d^d r}{a^{d-x_i}}} . \quad (57)$$

with a scaling dimensions x_i (*i.e.* the scaling dimension defined earlier from the two point correlation functions).
$\phi_1 = S \rightarrow x_1 = (d-2)/2$; $\phi_2 = S^2 \rightarrow x_2 = (d-2)$; $\phi_3 = S^4 \rightarrow x_3 = 2(d-2)$

Perturbative renormalisation group

- We then start the perturbative development :

$$\begin{aligned}\mathcal{Z} = \mathcal{Z}_0 \quad \times \quad & [1 - \sum_i g_i \int < \phi_i(r) > \frac{d^d r}{a^{d-x_i}} \\ + \quad & \frac{1}{2} \sum_{ij} g_i g_j \int < \phi_i(r_1)\phi_j(r_2) > \frac{d^d r_1 d^d r_2}{a^{2d-x_i-x_j}} \\ - \quad & \frac{1}{6} \sum_{ijk} g_i g_j g_k \int < \phi_i(r_1)\phi_j(r_2)\phi_j(r_2) > \frac{d^d r_1 d^d r_2 d^d r_3}{a^{3d-x_i-x_j-x_k}} \\ + \quad & \cdots]\end{aligned} \tag{58}$$

  Here, the correlation functions are computed with the original fixed point corresponding to $\mathcal{H}_0$.

- The next step is to evaluate the correlation functions. We will evaluate them by using Operator Product Expansion (OPE).

Perturbative renormalisation group

$$< \phi_i(r_1)\phi_j(r_2)\Phi >= \sum_k C_{ijk}(r_1 - r_2) < \phi_k((r_1 + r_2)/2)\Phi > \quad (59)$$

- Here Φ is any combination of operators located far from r_1 and r_2.
- OPE can be proved in some simple examples (2d Ising model). For the case we consider here, it is rather easy to derive the values of C_{ijk}.
- $C_{ijk}(r_1 - r_2)$ does not depend on the choice of Φ. We can then write

$$\phi_i(r_1)\phi_j(r_2) = \sum_k C_{ijk}(r_1 - r_2)\phi_k((r_1 + r_2)/2) \quad (60)$$

but we must remember that this is only true when inserted in a correlation function.

Perturbative renormalisation group

- We need also to specify how we perform the integrations with multiple variables.
- We start with

$$\sum_i \phi(x_i) \to \int_{a<r<L} \frac{d^d r}{a^{d-x}} \phi(r) \ , \tag{61}$$

so we explicitly add a small distance cut-off a and a large distance cut-off L.

- This can be interpreted as the lattice spacing and the size of the system.
- If we have multiple integrations, we need to choose a way with dealing when two operators become close one to the other one.
- Hard-core : operators have to remain at a distance larger than a.

Perturbative renormalisation group

Renormalisation Group : We change the microscopic cut-off

$$a \to a(1+\delta l) \qquad with \;\; \delta l << 1 \tag{62}$$

- The first term will change as

$$g_i \to (1+\delta l)^{d-x_i} g_i \simeq g_i + (d-x_i) g_i \delta l \tag{63}$$

- The second term in the expansion will contain

$$\int_{|r_1-r_2|>a(1+\delta l)} = \int_{|r_1-r_2|>a} - \int_{a(1+\delta l)>|r_1-r_2|>a} \tag{64}$$

- The last integral has to be taken in account :

$$\frac{1}{2}\sum_{ij}\sum_{k} C_{ijk} a^{x_k-x_i-x_j} \int_{a(1+\delta l)>|r_1-r_2|>a} \frac{d^d r_1 d^d r_2}{a^{2d-x_i-x_j}} \tag{65}$$

$$< \phi_k((r_1+r_2)/2) >$$

Perturbative renormalisation group

- The integral can be evaluated and gives $S_d a^d \delta l$ with $S_d = (2\pi)^{\frac{d}{2}}\Gamma(d/2)$ the volume of the sphere of radius one in d dimensions. Thus the second term can be absorbed in the redefinition

$$g_k \to g_k - \frac{1}{2} S_d \sum_{ij} C_{ijk} g_i g_j \delta l \ , \tag{66}$$

  and we obtain

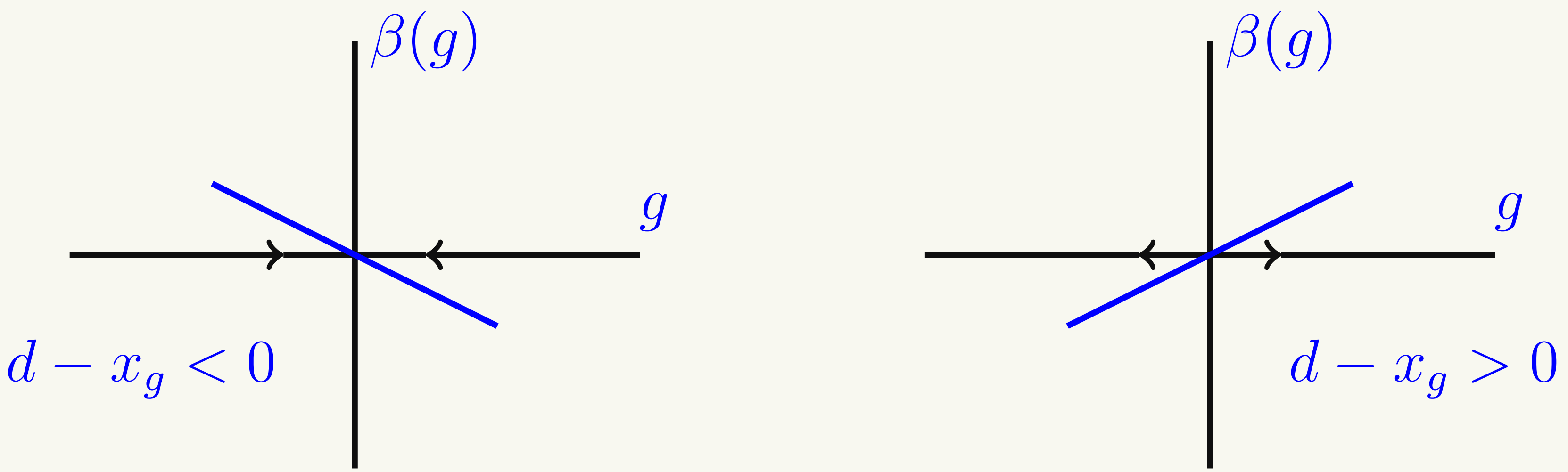
$$\beta_k(g_i) = \frac{dg_k}{dl} = (d - x_k) g_k - \sum_{ij} C_{ijk} g_i g_j + \cdots \tag{67}$$

  after a rescaling to absorb $S_d/2$.

- From there, the general strategy is to check the zero's of the β-functions.

Perturbative renormalisation group

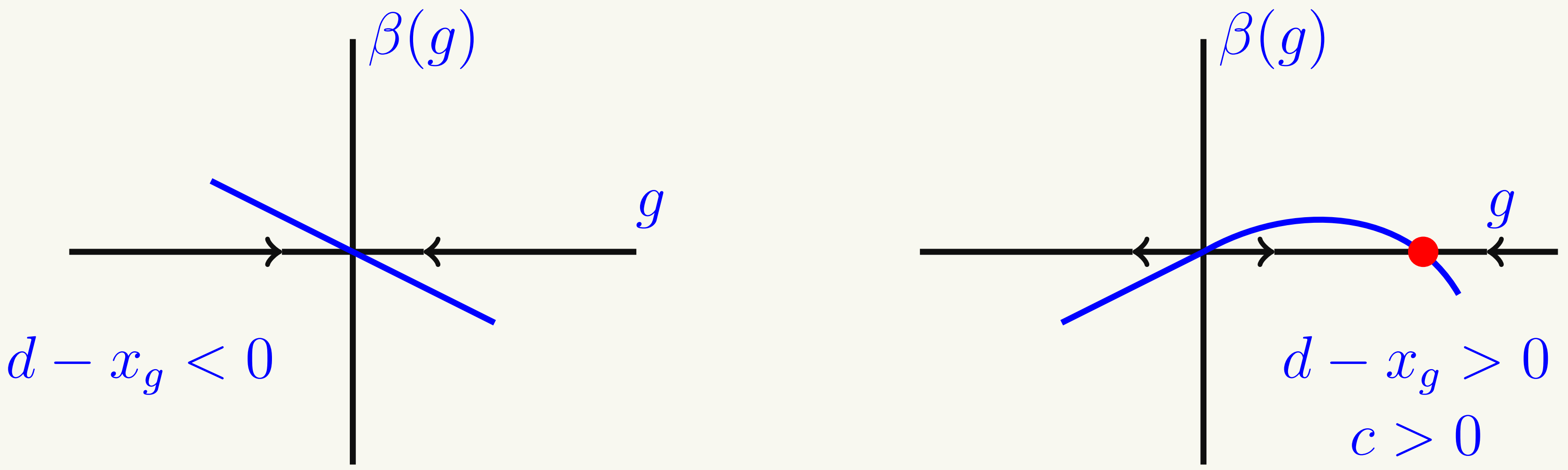
- We start from the solution corresponding to $g_i = 0$ (Free Gaussian theory in our case).
- For each operator, we can check the relevance. We illustrate this for the case with one operator and with one coupling g.
- We first check the lowest order $\beta(g) = \frac{dg}{dl} = (d - x_g)g + \cdots$



- Note that $d - x_g = y_g$, the RG eigenvalue defined earlier.

Perturbative renormalisation group

- Next we move, again for a single operator, to the next term : $\beta(g) = \frac{dg}{dl} = (d - x_g)g - cg^2 + \cdots$



- If $d - x_g > 0$ (relevant perturbation) and $c > 0$, we have, at this order in perturbation a fixed point at the value $g_* = (d - x_g)/c$

Perturbative renormalisation group

- In general, there is more than one perturbation and then a corresponding number of coupling constants and beta functions (and we need to diagonalize as seen above).
- The complicated part is to evaluate the C_{ijk}.
- We go back to our original problem in which we had three perturbations, corresponding either to S, S^2 or S^4. In that case, the starting problem is the Gaussian model for which we can compute the C_{ijk} rather easily, by contraction of operators (Wick contraction).

$$S \times S \simeq 1 + S^2 \;\; ; \;\; S \times S^2 = 2S + S^3 \;\; ; \;\; S \times S^4 \simeq 4S^3 + \cdots$$
$$S^2 \times S^2 = 2 + 4S^2 + S^4 \;\; ; \;\; S^2 \times S^4 \simeq 12S^2 + 8S^4 + \cdots$$
$$S^4 \times S^4 = 24 + 96S^2 + 72S^4 + \cdots$$

Perturbative renormalisation group

- We ignore term of order S^5 or larger order or with more derivatives since these would be irrelevant terms (see later).
- Note that a term S^3 appears in the OPE. This can be removed by noticing that, under a redefinition $S \to S + \alpha$, then

$$tS + uS^2 + hS^4 \to cst + t'S + u'S^2 + r'S^3 + u'S^4 \tag{68}$$

  We can absorb one of the powers by a choice of α. So we can get rid of the cubic term.
- The OPE coefficients can be read from the previous expressions. For example, $C_{uuu} = 72$.
- Collecting all the OPE coefficients, we get :

Perturbative renormalisation group

$$\begin{aligned}
\frac{dh}{dl} &= (d-(d-2)/2)h - 4ht + \cdots \\
&= (d/2+1)h - 4ht + \cdots \\
\frac{dt}{dl} &= (d-(d-2))t - h^2 - 4t^2 - 24tu - 96u^2 + \cdots \\
&= 2t - h^2 - 4t^2 - 24tu - 96u^2 + \cdots \\
\frac{du}{dl} &= (d-(2d-4))u - t^2 - 16tu - 72u^2 + \cdots \\
&= \epsilon u - t^2 - 16tu - 72u^2 + \cdots
\end{aligned} \tag{69}$$

- We will assume now that $\epsilon = 4 - d$ is small. We will expend h, t and u in powers of ϵ
  $h = h_1\epsilon + h_2\epsilon^2 + \cdots \;;\; t = t_1\epsilon + t_2\epsilon^2 + \cdots \;;\; u = u_1\epsilon + u_2\epsilon^2 + \cdots$

- We impose the condition that all the β's functions are zero. A simple inspection shows that we need to have
  $h = 0(\epsilon^2) \;\; ; \;\; t = 0(\epsilon^2) \;\; ; \;\; u = 0(\epsilon)$
- Then at this order in the ϵ expansion, we get :

$$\begin{aligned} \frac{dh}{dl} &= (d/2+1)h_2\epsilon^2 + O(\epsilon^3) \\ \frac{dt}{dl} &= 2t_2\epsilon^2 - 96u_1^2\epsilon^2 + 0(\epsilon^3) \\ \frac{du}{dl} &= \epsilon u_1\epsilon - 72u_1^2\epsilon^2 + 0(\epsilon^3) \end{aligned}$$

with the simple solution

$$u = \frac{\epsilon}{72} + O(\epsilon^2) \;\; ; \;\; t = O(\epsilon^2) \;\; ; \;\; h = O(\epsilon^2) \tag{70}$$

Wilson-Fisher fixed point

Perturbative renormalisation group

- At the fixed point, we can reexpress

$$\begin{aligned}\beta_t = \frac{dt}{dl} &= 2t - h^2 - 4t^2 - 24tu - 96u^2 + \cdots \\ &= 2t - 24ut + \cdots = (2 - \frac{24}{72}\epsilon)t + \cdots \\ &= (d - x_t)t + \cdots \end{aligned} \quad (71)$$

with the new dimension associate to the thermal perturbation $x_t = d - 2 + \frac{24}{72}\epsilon$ compared to the original dimension $x_t = d - 2$.

- This dimension is associated to the exponent corresponding to the correlation length by the relation

$$\nu = 1/y_t = 1/(d - x_t) = 1/(2 - \frac{24}{72}\epsilon) = 1/2 + \frac{1}{12}\epsilon + O(\epsilon^2) \quad (72)$$

Perturbative renormalisation group

- Let's consider the term S^6 with a scaling dimension $x_6 = 3d - 6$. It's RG eigenvalue $y_6 = d - x_6 = 6 - 2d$ is negative for any dimension larger than 3. But in fact, even at $d = 3$ it will be irrelevant. Indeed, it is easy to see that

$$\frac{dg_6}{dl} = (6 - 2d)g_6 - 360ug_6 + \cdots \tag{73}$$

But we have to consider this term at the Wilson-Fisher fixed point, at which value $u = \epsilon/72$. Then

$$\begin{aligned} \frac{dg_6}{dl} &= (6 - 2d)g_6 - (360/72)\epsilon g_6 + O(\epsilon^2) \\ &= (-2 + 2\epsilon)g_6 - 5\epsilon g_6 + O(\epsilon^2) \\ &= (-3\epsilon - 2)g_6 + O(\epsilon^2)\ . \end{aligned} \tag{74}$$

So we see that this term is *always* irrelevant for any dimension !!!

Universality and comparison with experimental systems :

Transition type	Material	α	β	γ	ν
Ferro. (n=3)	Fe, Ni	-0.1	0.34	1.4	0.7
Superfluid (n=2)	He^4	0	0.3	1.2	0.7
Liquid-gas (n=1)	CO_2, Xe	0.11	0.32	1.24	0.63
Superconductors		0	1/2	1	1/2
Mean Field		0	1/2	1	1/2
ϵ (3d)	$O(\epsilon^5)$		0.3268 (3)		0.631 (3)
IM Monte Carlo			0.32644 (2)		0.6300 (1)
ϵ (2d)	$O(\epsilon^5)$		0.130 (25)		0.99 (4)
IM Exact (2d)			0.125		1.0