Quantum Supergroups and Noncommutative Supergeometry

Axel de Goursac

Université Catholique de Louvain

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Motivations

- Supergeometry: anticommuting variables

$\rightarrow$ fermions, BRST, SUSY,...

- Noncommutativity: quantum effects

$\rightarrow$ Heisenberg uncertainty, quantum Hall effect,...

- NC supergeometry: fermionic or SUSY on NC space-time, NC BRST formalism,...

Result (A.G. '10, Bieliavsky A.G. Tuynman '12)

Renormalization is natural for the deformed NCSG of $\mathbb{R}_\theta^{4|1}$.

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2. Deformation quantization of the Heisenberg supergroup

3. Construction of Fréchet quantum supergroups

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flat Supergeometry

(Kostant, Leites, DeWitt, Rogers, Tuynman,...)

- Essence of concrete approach to supergeometry: replace field $\mathbb{R}$ by a real supercommutative algebra $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \simeq \Lambda V = \mathbb{R} \oplus$ nilpotents $\qquad ab = (-1)^{|a||b|} ba$

- Superspace of dim $m|n$: $\mathbb{R}^{m|n} := (\mathcal{A}_0)^m \times (\mathcal{A}_1)^n$ with "body" map $\mathbb{B} : \mathbb{R}^{m|n} \to \mathbb{R}^m$ $\qquad$ (drop nilpotents)

- DeWitt topology: $U \subset \mathbb{R}^{m|n}$ open if $\mathbb{B}U$ is open and $U = \mathbb{B}^{-1}(\mathbb{B}U)$

- Smooth superfunction $f : \mathbb{R}^{m|n} \to \mathcal{A}$ if $\exists f_I \in C^\infty(\mathbb{R}^m)$ $(I \subset \{1,..,n\})$ $\forall (x,\xi) \in \mathbb{R}^{m|n}$, $\qquad (\xi^I = \prod_{i\in I} \xi^i)$

$$f(x,\xi) = \sum_I \tilde{f}_I(x)\xi^I$$

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Supergeometry

- Supermanifold M of dim $m|n$: topological space endowed with a collection of charts (in $\mathbb{R}^{m|n}$) s.t. transition superfunctions are smooth and $\mathbb{B}M$ is a usual manifold.
- Trivial supermanifold: with global odd coordinate system
- Complex smooth superfunctions: $\mathcal{C}^\infty(M) \simeq \mathcal{C}^\infty(\mathbb{B}M) \otimes \bigwedge \mathbb{R}^n$
- Product: $\xi^I \xi^J = \varepsilon(I, J) \xi^{I \cup J} = (-1)^{|I||J|} \xi^J \xi^I$
- Berezin integration: $\int d\xi \; f(x, \xi) = f_{\{1,\dots,n\}}(x)$

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Philosophy of NC Supergeometry

(Gelfand-Naimark point of view)

- Supermanifold M: $\mathcal{C}^\infty(M)$ is $\mathbb{Z}_2$-graded commutative
- NC superspaces: given by NC $\mathbb{Z}_2$-graded associative algebras
- Supergeometrical tools (A.G. Masson Wallet '12)
  - vector fields
  - de Rham forms
  - connections, curvature,...
  - gauge theory
- Analytical objects: Hilbert superspaces, C*-superalgebras
- Symmetries: quantum supergroups

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Hilbert superspaces

Definition (Bieliavsky A.G. Tuynman '12)

A Hilbert superspace of parity n is a $\mathbb{Z}_2$-graded Hilbert space $(\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1, (\cdot,\cdot))$ with $(\mathcal{H}_0, \mathcal{H}_1) = 0$, endowed with a unitary operator $J \in \mathcal{B}(\mathcal{H})$ of degree n, s.t. $J^2(x) = (-1)^{(n+1)|x|} x$.

- Superhermitian scalar product: $\langle x, y \rangle := (J(x), y)$
- Superadjoint: $\forall T \in \mathcal{B}(\mathcal{H})$, $\exists T^\dagger \in \mathcal{B}(\mathcal{H})$, $\forall x, y \in \mathcal{H}$,

$$\langle T^\dagger(x), y \rangle = (-1)^{|T||x|} \langle x, T(y) \rangle$$

- $\mathcal{H} = L^2(X)$, for X a trivial supermanifold of dim $m|n$, with Hodge operation: $J(\xi^I) = \varepsilon(I, \complement I)\xi^{\complement I}$, and scalar products $\langle f, g \rangle = \int_M \mathrm{d}z\, \overline{f(z)} g(z)$, $(f, g) = \int_M \mathrm{d}z\, \overline{f(z)} J(g)(z)$

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C*-superalgebras

Superinvolution on a $\mathbb{Z}_2$-graded algebra $\mathbf{A}$: antilinear $^\dagger : \mathbf{A} \to \mathbf{A}$ of degree 0 s.t. $(a^\dagger)^\dagger = a$ and $(a \cdot b)^\dagger = (-1)^{|a||b|} b^\dagger \cdot a^\dagger$

Définition (Bieliavsky A.G. Tuynman '12)

A C*-superalgebra is a superinvolutive $\mathbb{Z}_2$-graded algebra $\mathbf{A}$ faithfully represented on a Hilbert superspace $\mathcal{H}$, and closed for the operator norm topology:

$$\varrho : \mathbf{A} \hookrightarrow \mathcal{B}(\mathcal{H})$$

- Example: $\mathbf{A} = L^\infty(X)$ is a C*-superalgebra for the complex conjugation: $\overline{\xi^I \xi^J} = \xi^I \xi^J = (-1)^{|I||J|} \xi^J \xi^I$, represented by multiplication on $\mathcal{H} = L^2(X)$.

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Results

- Categories: direct sum, tensor product, morphisms,...
- Harmonic analysis of Heisenberg and Poincaré supergroups (Kirillov, Stone-von Neumann)
- Non-formal deformation quantization of Heisenberg and Poincaré supergroups (star-product $\star_\theta$, quantization map, pseudodiff. calculus)
- C*-Universal deformation formula (UDF) for Heisenberg supergroup G:

Theorem (Bieliavsky A.G. Tuynman '12)

Let $\mathbf{A}$ be a C*-superalgebra acted by G. Then $\star_\theta$ permits to deform the structure of $\mathbf{A}$ into a new C*-superalgebra $\mathbf{A}_\theta$.

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Quantization

(Bieliavsky Gayral '12, Bieliavsky A.G. Tuynman '12)

- Symplectic superspace $\mathbb{R}^{2m|n}$ with $\omega = \begin{pmatrix} 0 & I_m & 0 \\ -I_m & 0 & 0 \\ 0 & 0 & I_n \end{pmatrix}$
- Heisenberg supergroup $G = \mathbb{R}^{2m|n} \times \mathbb{R}^{1|0}$ with

$$(z_1, t_1)\cdot(z_2, t_2) = (z_1 + z_2, t_1 + t_2 + \frac{1}{2}\omega(z_1, z_2))$$

- Coadjoint orbit $M \simeq G/\mathbb{R}^{1|0} \simeq \mathbb{R}^{2m|n}$
- Kirillov's orbit method: Schrödinger representation $U_\theta : G \to \mathcal{B}(L^2(\mathbb{R}^{m|n})) \otimes \mathcal{A}$ superunitary
- Weyl-type quantization map $\Omega_\theta : L^1(M) \to \mathcal{B}(L^2(\mathbb{R}^{m|n}))$

$$\mathcal{B}^1(M) = \{f \in C^\infty(M), \forall D^\alpha, \|f\|_\alpha = \sup_{x \in \mathbb{B}M} \sum_I |D^\alpha f_I(x)| < \infty\}$$

- Extension $\Omega_\theta : \mathcal{B}^1(M) \to \mathcal{B}(L^2(\mathbb{R}^{m|n}))$ (oscillatory integral)

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- Symplectic superspace $\mathbb{R}^{2m|n}$ with $\omega = \begin{pmatrix} 0 & I_m & 0 \\ -I_m & 0 & 0 \\ 0 & 0 & I_n \end{pmatrix}$
- Heisenberg supergroup $G = \mathbb{R}^{2m|n} \times \mathbb{R}^{1|0}$ with

$$(z_1, t_1)\cdot(z_2, t_2) = (z_1 + z_2, t_1 + t_2 + \frac{1}{2}\omega(z_1, z_2))$$

- Coadjoint orbit $M \simeq G/\mathbb{R}^{1|0} \simeq \mathbb{R}^{2m|n}$
- Kirillov's orbit method: Schrödinger representation $U_\theta : G \to \mathcal{B}(L^2(\mathbb{R}^{m|n})) \otimes \mathcal{A}$ superunitary
- Weyl-type quantization map $\Omega_\theta : L^1(M) \to \mathcal{B}(L^2(\mathbb{R}^{m|n}))$

$$\mathcal{B}^1(M) = \{f \in C^\infty(M), \forall D^\alpha, \|f\|_\alpha = \sup_{x \in \mathbb{B}M} \sum_I |D^\alpha f_I(x)| < \infty\}$$

- Extension $\Omega_\theta : \mathcal{B}^1(M) \to \mathcal{B}(L^2(\mathbb{R}^{m|n}))$ (oscillatory integral)

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Star-product

Product on M given by $(f_1 \star_\theta f_2)(z) = \mathrm{Tr}(\Omega_\theta(f_1)\Omega_\theta(f_2)\Omega_\theta(z))$

$$= \int_{M^2} \mathrm{d}z_1 \mathrm{d}z_2\, K_\theta(z_1, z_2) R^*_{z_1} f_1(z) R^*_{z_2} f_2(z)$$

with kernel $K_\theta(z_1, z_2) \propto e^{-\frac{2i}{\theta}\omega(z_1,z_2)}$ and $R^*_z f(z') = f(z' + z)$

Proposition (Bieliavsky A.G. Tuynman '12)

- $\Omega_\theta(f_1 \star_\theta f_2) = \Omega_\theta(f_1)\Omega_\theta(f_2)$
- $(\mathcal{B}^1(M), \star_\theta)$ is a Fréchet associative algebra and $\star_\theta$ is G-invariant.
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UDF for Fréchet algebras

Non-graded (Rieffel '92, Bieliavsky Gayral '12)

- Let $(\mathbf{A}, \|\cdot\|_j)$ be a Fréchet algebra.

$\rightarrow$ Product $\star_\theta$ can be extended to $\mathcal{B}^1_{\mathbf{A}}(M)$.

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$$\rho_{z_1 z_2} = \rho_{z_1}\rho_{z_2}, \quad \rho_0 = \mathrm{id}, \quad \rho_z(ab) = \rho_z(a)\rho_z(b), \ldots$$

- Space of smooth vectors $\mathbf{A}^\infty$ is dense in $\mathbf{A}$

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Fréchet quantum supergroups

Definition (A.G. '14)

A Fréchet quantum supergroup is a $\mathbb{Z}_2$-graded Fréchet space with graded topo. tensor product and continuous homogeneous of degree 0 Hopf algebra structure.

ex: $H := \mathcal{B}^1(M)$ without deformation

- Product $\mu : H\hat{\otimes}H \to H$, $\mu(f_1 \otimes f_2)(z) = f_1(z)f_2(z)$
- Unit $\mathbb{1} : \mathbb{C} \to H$, $\mathbb{1}(\lambda)(z) = \lambda$
- Coproduct $\Delta : H \to H\hat{\otimes}H$, $\Delta(f)(z_1, z_2) = f(z_1 z_2)$
- Counit $\varepsilon : H \to \mathbb{C}$, $\varepsilon(f) = f(0)$
- Antipode $S : H \to H$, $S(f)(z) = f(z^{-1})$

Representation: Fréchet comodule-algebras with continuous homogeneous coaction

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Solvable examples of FQSG

Non-graded (Rieffel '92)

- Action $\pi : \mathbb{R}^{1|0} \to Sp(M,\omega)$
  Supergroup $G' := \mathbb{R}^{1|0} \ltimes_\pi M$ with
  $$(a,z)\cdot(a',z') = (a+a', \pi(a')z + z')$$

- M acts on the Fréchet algebra $H := \mathcal{C}^\infty(\mathbb{R}^{1|0})\hat{\otimes}\mathcal{B}^1(M)$

$\to$ UDF with parameter a:
$(f_1 \star f_2)(a,z) = \int \mathrm{d}z_1 \mathrm{d}z_2\, K_a(z_1,z_2) R^*_{(0,z_1)} f_1(a,z) R^*_{(0,z_2)} f_2(a,z)$

Proposition (A.G. '14)

Endowed with standard other structures, $(H, \star, \mathbb{1}, \Delta, \varepsilon, S)$ is a Fréchet quantum supergroup.

In particular, $\Delta(f_1 \star f_2) = \Delta(f_1) \star \Delta(f_2)$

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Multiplicative superunitary

Non-graded (Baaj-Skandalis, Woronowicz, Rieffel,...)

- Operator $W : H\hat{\otimes}H \to H\hat{\otimes}H$ associated to the FQSG, defined by $W(f_1 \otimes f_2) = (\Delta f_1) \star (\mathbb{1} \otimes f_2)$
- Expression $W(F)(a, z, a', z') = \int \mathrm{d}z_1 \mathrm{d}z_2 \, K_{a'}(z_1, z_2) R^*_{(a', z_1, 0, z_2)} F(a, z, a', z')$
- W is continuous homogeneous of degree 0 and satisfies $W_{12} W_{13} W_{23} = W_{23} W_{12}$.

Proposition (A.G. '14)

W is superunitary on the Hilbert superspace $L^2(G' \times G', \mathrm{d}^R g \otimes \mathrm{d}^R g)$:

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Examples of FQSG with supertoral subgroup

Non-graded (Rieffel '93)

- G' compact connected Lie supergroup with $\Gamma = \mathbb{T}^{2m|n} = e^M$ supertoral subgroup
- $M \times M$ acts on $H := \mathcal{C}^\infty(G')$ by $\rho_{(z,z')} f(g) = f(e^{-z} g e^{z'})$

$\rightarrow$ UDF with parameter $(\theta, -\theta)$: $(f_1 \star_\theta f_2)(g) = \int dz_1 dz_2 dz_3 dz_4 \, K_\theta(z_1, z_2) K_{-\theta}(z_3, z_4) \rho_{(z_1,z_3)} f_1(g) \rho_{(z_2,z_4)} f_2(g)$

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Proposition (A.G. '14)

Endowed with standard other structures, $(H, \star_\theta, \mathbb{1}, \Delta, \varepsilon, S)$ is a Fréchet quantum supergroup, with undeformed subgroup Γ.

In particular, $\Delta(f_1 \star_\theta f_2) = \Delta(f_1) \star_\theta \Delta(f_2)$

Examples of FQSG with supertoral subgroup

Non-graded (Rieffel '93)

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Multiplicative superunitary 2

- Operator $W : H\hat{\otimes}H \to H\hat{\otimes}H$ associated to the FQSG, defined by $W(f_1 \otimes f_2) = (\Delta f_1) \star_\theta (\mathbb{1} \otimes f_2)$
- Expression $W(F)(g, g') = \int dz_1 dz_2 dz_3 dz_4 \, K_\theta(z_1, z_2) K_{-\theta}(z_3, z_4) \rho_{(z_1,z_3,z_2,z_4)} F(gg', g')$
- W is continuous homogeneous of degree 0 and satisfies $W_{12} W_{13} W_{23} = W_{23} W_{12}$.

Proposition (A.G. '14)

If G' is unimodular, W is superunitary on the Hilbert superspace $L^2(G' \times G')$:

$$\langle W(F_1), W(F_2) \rangle = \langle F_1, F_2 \rangle$$

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Results:

- Natural definition of Fréchet Quantum Supergroups.
- Def. quant. of the Heisenberg supergroup and its UDF.
- With UDF, construction of solvable FQSG and FQSG with supertoral subgroups.
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Perspectives:

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- Symmetries of UDF of other supergroups.
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