On quadratic symmetric n-ary superalgebras

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Main definitions

$\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $\quad \mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$,

$V = V_{\bar{0}} \oplus V_{\bar{1}}$ is a finite dimensional $\mathbb{Z}_2$-graded vector space over $\mathbb{K}$ with a non-degenerate skew-symmetric even bilinear form $(\,,\,)$.

$(\,,\,)|_{V_{\bar{0}} \times V_{\bar{0}}}$ is non-degenerate skew-symmetric;
$(\,,\,)|_{V_{\bar{1}} \times V_{\bar{1}}}$ is non-degenerate symmetric;

$$(\,,\,) = (\,,\,)|_{V_{\bar{0}} \times V_{\bar{0}}} + (\,,\,)|_{V_{\bar{1}} \times V_{\bar{1}}}$$

$\bar{x} := \bar{i}$ is the parity of a homogeneous element $x \in V_{\bar{i}}$

Main definitions

- An *n-ary superalgebra structure μ on V* is an n-linear map

$$\mu : \underbrace{V \times \cdots \times V}_{n \text{ times}} \to V.$$

Sometimes we will use "bracket notation"
$\mu(a_1, \ldots, a_n) = \{a_1, \ldots, a_n\}$.

- An n-ary superalgebra (V, μ) is called *symmetric* if

$$\{a_1, \ldots, a_i, a_{i+1}, \ldots, a_n\} = (-1)^{\bar{a}_i \bar{a}_{i+1}} \{a_1, \ldots, a_{i+1}, a_i, \ldots, a_n\} \tag{1}$$

for any homogeneous $a_i \in V$.

- A symmetric n-ary superalgebra (V, μ) is called *quadratic* if the following holds:

$$(b, \{a_1, \ldots, a_n\}) = (-1)^{\bar{b}\bar{a}_1} (a_1, \{b, a_2, \ldots, a_n\}). \tag{2}$$

Derived bracket

The "derived bracket" approach was used by B. Kostant and S. Sternberg, Y. Kosmann-Schwarzbach, Th. Voronov and others.

Denote by $S^*V = \bigoplus_n S^n V$ the symmetric power of V. On S^*V there is a natural Poisson superalgebra structure $[\,,]$, defined by:

$$[x, y] := (x, y), \quad x, y \in V;$$

$$[v, w_1 \cdot w_2] := [v, w_1] \cdot w_2 + (-1)^{\bar{v}\bar{w}_1} w_1 \cdot [v, w_2],$$

$$[v, w] = -(-1)^{\bar{v}\bar{w}} [w, v],$$

where $v, w, w_1, w_2 \in S^*V$ are homogeneous elements.

Derived bracket

Let us take any $\mu \in S^{n+1}V$. We can define an n-ary superalgebra structure on V by

$$\{a_1, \ldots, a_n\} := [a_1, [\ldots, [a_n, \mu] \ldots]],$$

where $a_i \in V$.

The n-ary superalgebra structure has two properties:

- This multiplication is *symmetric* and *quadratic*.

  To prove the second statement we use the Jacobi identity:

$$(b, \{a_1, \ldots, a_n\}) = [b, [a_1, [a_2, \ldots, [a_n, \mu] \ldots]]] =$$
$$[[b, a_1], \ldots, [a_n, \mu] \ldots]]] + (-1)^{\bar{b}\bar{a}_1}[a_1, [b, [a_2, \ldots, [a_n, \mu] \ldots]]] =$$
$$(-1)^{\bar{b}\bar{a}_1}(a_1, \{b, a_2, \ldots, a_n\}).$$

Derived bracket

Proposition. Assume that $(\,,\,)$ is non-degenerate. Any symmetric quadratic n-ary superalgebra can be obtained by this construction:

$$\{a_1, \ldots, a_n\} := [a_1, [\ldots, [a_n, \mu] \ldots]],$$

Examples

- (Kostant and Sternberg, Roytenberg) Let $V = V_{\bar{1}}$ and $\mu \in S^3 V$.

$$[\mu, \mu] = 0 \Longleftrightarrow \text{Jacobi identity} + (\ ,\ )\text{-invariance ;}$$

- Let $V = V_{\bar{0}}$ and $\mu \in S^3 V$.

$$[\mu_x, \mu_y] = 0 \Longleftrightarrow \text{associativity} + (\ ,\ )\text{-invariance,}$$

where $\mu_x := [x, \mu]$.

- Let $V = V_{\bar{0}}$ and $\mu \in S^3 V$.

$$[\mu_x, \mu_{[\mu_x, x]}] = 0 \Longleftrightarrow \text{Jordan identity} + (\ ,\ )\text{-invariance,}$$

where $\mu_x := [x, \mu]$.

Classification of $(m-3)$-ary quadratic algebras

Let $V = V_{\bar{1}}$ and (e_i) be a normalized orthogonal basis of V and

$$T := e_1 \dots e_m,$$

where $m = \dim V$.

For $x_i \in V$, we define a "Hodge operator" $* : S^p \to S^{m-p}$ by:

$$*(x_1 \dots x_p) := [x_1, [\dots [x_p, T]]].$$

We can also use this idea to define "Hodge operator" on Riemannian oriented manifold. We see that this definition depends only on the orientation and the Riemannian metric $(\, , \,)$.

Classification of $(m-3)$-ary quadratic algebras

- $S^2V \simeq \mathfrak{so}(V), \quad x \longmapsto [x, \,] : V \to V$. Therefore, S^2V is an adjoint (or coadjoint) module of $\mathfrak{so}(V)$.
- $* : S^2V \to S^{m-2}V$ is $\mathfrak{so}(V)$-invariant.
- We can integrate everything and replace $\mathfrak{so}(V)$ by $\mathrm{SO}(V)$.

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Theorem. (V.) *Classes of isomorphic real or complex quadratic $(m-3)$-ary algebra structures on V, where $m = \dim V$, are in one-to-one correspondence with coadjoint orbits corresponding to the Lie group $\mathrm{SO}(V)$.*

It is well-known that any real skew-symmetric matrix A can be written in the following form:

$$A = QA'Q^{-1},$$

where

$$A' = \mathrm{diag}(J_{a_1}, \ldots, J_{a_k}, 0, \ldots, 0),$$

$$J_{a_j} = \begin{pmatrix} 0 & a_j \\ -a_j & 0 \end{pmatrix}, \quad a_j \in \mathbb{R},$$

$Q \in \mathrm{O}(V)$ and $0 < a_k \leq \cdots \leq a_1$.

The corresponding to A element in S^2V is

$$A \longmapsto v_A = a_1 e_1 e_2 + \ldots + a_k e_{2k-1} e_{2k}, ,$$

Theorem. (V.) *Real metric $(m-3)$-ary algebra structures on V are parametrized by vectors*

$$v_A = a_1 e_1 e_2 + \ldots + a_k e_{2k-1} e_{2k},$$

where $a_i \in \mathbb{R}$, $0 < a_k \leq \cdots \leq a_1$ and $0 \leq k \leq [\frac{m}{2}]$. Explicitly the metric $(m-3)$-ary algebra structures are given by

$$\mu_v = *(v_A).$$

The algebra (V, μ_{v_A}) is simple if and only if $k > 1$.

- W. X. Ling, On the structure of n-Lie algebras. PhD thesis, Siegen, 1993.
  Classification of simple n-ary algebras of Filippov type.

- Cantarini, Nicoletta, and Victor G. Kac. Classification of Simple Linearly Compact n-Lie Superalgebras. Communications in Mathematical Physics 298.3 (2010): 833-853.
  Classification of n-ary superalgebras of Filippov type, infinite dimensional case.

Other results

- Quasi-Frobenius structures on n-ary algebras;
- Hodge Theory for homogeneous $\mu \in S^*(V)$, where $V = V_{\bar{1}}$;
- Double extension for quadratic superalgebras.
  (For Lie algebras A. Medina, Ph. Revoy)