Generalized Quantum AdS, a basis for a (new) Standard Model in particle physics?

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[The frameworks and research programs described in this talk are a kind of epitome of my 35 years' collaboration with my friend Moshe Flato and of subsequent developments in that direction, and discussions with many scientists, which would not have been possible without his impact on the community, especially his deep insight on the crucial role of symmetries and deformations in physics.

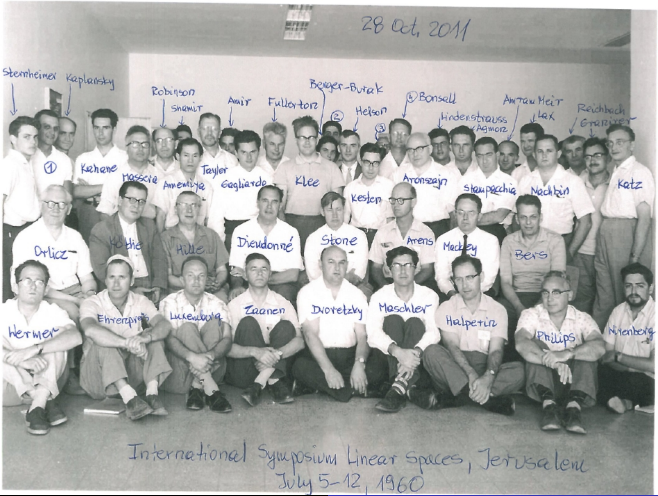
The physical consequences of such an heretic approach might be revolutionary but in any case there are, in the mathematical tools required to jump start the process, potentially important developments to be made.]

NCGaMP, Scalea, 20 June 2014

Abstract

We present a "model generating" multifaceted framework in which the "internal symmetries" on which was based the Standard Model (SM) of strongly interacting elementary particles, could "emerge" by deforming the Anti de Sitter (AdS) deformation of the Poincaré group, i.e. quantizing it, probably at root of unity and possibly in manner not yet mathematically developed (with multiple or noncommutative parameters). We start with a brief reminder of the problem of connection between "internal" and Poincaré symmetries and of how we got to the SM. Then we overview Flato's "deformation philosophy" and review a possible explanation of photons as composites of AdS singletons, and of leptons as similar composites (extending the electroweak model to 3 generations) before presenting our framework and explaining how the SM might be a colossus with clay feet.

Jerusalem 1960



Moshe Flato (17/09/1937 – 27/11/1998) & Noriko Sakurai (20/02/1936 – 16/10/2009)

Modern particle physics: in the beginning; 1961

A cartoon presentation of how it all happened. At first only few particles (mainly nucleons). Isospin (Heisenberg 1932, Wigner 1937). Then "particle explosion" (40's and especially 50's; Fermi botanist quote).
Was noticed that some particles created at high rate decayed back strangely slowly (e.g. Λ^0 produced from π and p with K^0, decays alone with factor 10^{-13}). So in 1953 Gell'Mann and (independently) Nishijima and Nakano suggested new quantum number "strangeness", conserved in strong but violated in weak interactions.
To put some order, in 1956 Sakata suggested that p, n, Λ^0 are "fundamental" and other hadrons are composites.
Early 1961: Rank 2 Lie group for particle spectroscopy (Salam, Sakurai). The UPenn "1961 gang of 4" (Fronsdal, Ben Lee, Behrends, Dreitlein) too thorough RMP paper: "Since it is as yet too early to establish a definite symmetry of the strong interactions, both because of the lack of experimental data and the theoretical uncertainties about the way in which the symmetries will manifest themselves, the formalism developed is left quite flexible in order to accommodate a wide range of conceivable symmetries."
These were SU(n) (in particular SU(3)), and types $C_2 = B_2$ and G_2.
At the same time Ne'eman (subject given by Salam) proposed only SU(3), immediately followed independently by Gell'Mann who coined "eightfold way" for the octet of spin $\frac{1}{2}$ baryons ($p, n, \Sigma^{\pm 1,0}, \Lambda^0, \Xi^{\pm 1}$) and octets of scalar and vector mesons.

The first SU(3), 1964: quarks and color. SM

Initial success of SU(3): There are baryons (spin $\frac{1}{2}$) and scalar and vector mesons octets (spin 0,1) that fit in adjoint representation of SU(3).
Early 50's, big stir. Spin $\frac{3}{2}$ baryons discovered, first $\Delta^{\pm 1,0,++}$ in Fermi group (Fermi: "I will not understand it in my lifetime"; Fermi died in 1954...), then $\Sigma*$, $\Xi*$ families. Fit in dim. 10 rep. of SU(3) with "decuplet" completed with predicted scalar Ω^-, found in 1964 at BNL.
Also in 1964: Gell'Mann and (independently) Zweig suggest that baryons are composites of "quarks", associated with fundamental rep. (dim. 3) of SU(3). "Three quarks for Muster Mark!/ Sure he hasn't got much of a bark/ And sure any he has it's all beside the mark." (from James Joyce's Finnegans Wake). So then had 3 "flavors" (up, down, strange). But quarks must have fractional charge. Being spin $\frac{1}{2}$ they cannot coexist (Fermi exclusion principle for fermions) so Greenberg proposed in 1964 to give them color (now called blue, green and red). Feynman's "parton" interpretation.
Later, in the second generation, strangeness was completed by another flavor (charm) and a third generation was found (2 more flavors, bottom and top), predicted in 1973 by Kobayashi and Maskawa to explain CP violation in kaon decay, "observed" at Fermilab in 1977 and 1995 (resp.), Nobel 2008 with Nambu (for 1960 symmetry breaking),

Hence SM with (for hadrons) 3 generations of quarks in 3 colors, and QCD on the pattern of QED. (Gauge) symmetry $SU(3) \oplus SU(2) \oplus U(1)$ (with electroweak); GUT.

The deformation philosophy and a new approach to an old question

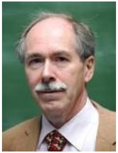
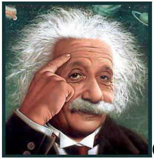
The two major physical theories, relativity and quantization, can now be understood as based on deformations of some algebras. The former became obvious as soon as deformation theory of algebras appeared, but it took a dozen more years before the latter became mathematically understood. Deformations (in the sense of Gerstenhaber) are classified by cohomologies.
On the other hand, in the 60's, the question was raised on whether there is any connection between "external symmetries" (the Poincaré group) and the (empirically found) "internal symmetries" of hadrons. Then (70's) electroweak theory (completed by 't Hooft and Faddeev). For strong interactions dynamics (QCD) built around "color" and SU(3) multiplets (assuming no connection...). That eventually gave the Standard Model (SM) and the dynamics built around it, and GUT (e.g. Yanagida's SU(5)).
My present suggestion is that the question of connection could be a false problem, i.e. maybe "internal symmetries" *emerge* from Poincaré by some kind of deformation, including quantization (probably at root of unity) and possibly generalized deformations (multiparameter or with noncommutative parameters). Which may require going back to the drawing board (present SM all beside the mark??) and raises many questions (phenomenology, re-interpretation of experiments, spin assignments, etc.)

Brief summary of an unorthodox conjectural scheme

In the AdS SO(2,3) deformation of Poincaré, photons can be seen as dynamically composites (of singletons), and (using flavor symmetry) leptons can also be considered there as composite (massified by 5 Higgs). Now quantum groups at root of unity are finite dimensional Hopf algebras. So maybe the symmetries of strong interactions can be obtained from AdS by quantization (and possibly some form of loop AdS algebra to "blow up" field theoretical singularities). Another (possibly complementary) direction is to look at "generalized deformations" where the "parameter", instead of being a scalar (algebra of a one-element group) would be the algebra of a finite group (e.g. "multiparameter" with $\mathbb{Z}/n\mathbb{Z}$ or the Weyl group of some SU(n)) or maybe a quaternion). Whether any of these (conceptually appealing) schemes has any relevance to physics is too early to tell, but in any case that can give nontrivial new maths, and the (new) phenomenology would not require new super-expensive experimental tools. There is work for more than a generation of (daring, at least in the beginning) scientists. Some preliminary results: qAdS as Connes' triples (with Bieliavsky et al.) and possible cosmological implications. Especially qAdS at cubic root of 1 (Jun Murakami: that f.d. Hopf alg. has only 9 irreps, plus contragredients: dim. 1; 4,5; 10, 16, 14; 35, 40; 81).

In this talk we shall present the main lines of the motivating ideas, and give some explanation about the proposal. [Anecdotes: Odessa Rabbi, wake up Lenin.]

’t Hooft on “Salam’s Grand Views”, two Einstein quotes

Gerard ’t Hooft, in “The Grand View of Physics”, Int.J.Mod.Phys.A23: 3755-3759, 2008 (`arXiv:0707.4572 [hep-th]`). To obtain the Grand Picture of the physical world we inhabit, to identify the real problems and distinguish them from technical details, to spot the very deeply hidden areas where there is room for genuine improvement and revolutionary progress, courage is required. Every now and then, one has to take a step backwards, one has to ask silly questions, one must question established wisdom, one must play with ideas like being a child. And one must not be afraid of making dumb mistakes. By his adversaries, Abdus Salam was accused of all these things. He could be a child in his wonder about beauty and esthetics, and he could make mistakes. [...]

Two Einstein quotes: *The important thing is not to stop questioning. Curiosity has its own reason for existing.*

You can never solve a [fundamental] problem on the level on which it was created.

A doubly heretic talk

't Hooft: *My views on the physical interpretation of quantum theory, and its implications for Big Bang theories of the Universe, are rapidly evolving.* [Earlier he had written on his web site: "I have deviating views on the physical interpretation of quantum theory" in relation with a possible underlying classical theory. Now he writes the above sentence and adds:] *I have mathematically sound equations that show how classical models generate quantum mechanics. It's not fake quantum mechanics, with "pilot wave functions" or other such nonsense; it's the real thing. End of argument.*

Deformation quantization, providing a way to develop (when needed, and with care) quantum theories on the phase space of classical theories, without the Procrustean bed of Hilbert space but with deformed composition laws of observables, is sometimes considered as heretic because no Hilbert space is imposed from the start.

Similarly, questioning the conventional simple unitary symmetries used to explain particle spectroscopy (and more) is also heretic. The Grand Views presented in this talk are doubly heretic!

Symmetries in physics: Wigner, Racah, Flato and beyond

The Master

Thesis of Moshé Flato by Maurice Kibler, `arXiv:math-ph/9911016v1`

`http://monge.u-bourgogne.fr/gdito/cmf1999/toc1999.html`

In atomic and molecular physics we know the forces and their symmetries.
Energy levels (spectral lines) classified by UIRs of $SO(3)$ or $SU(2)$, and e.g. with crystals that is refined (broken) by a finite subgroup. [Racah school, Flato's M.Sc.]
And beyond: Symmetries of equations (e.g. Maxwell), of physical states. Classification symmetries ("spectrum generating algebras", nuclear and particle physics), up to "electroweak" ($U(2)$), "standard model" ($\mathfrak{su}(3) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$) with dynamics (QCD) inferred from empirically found symmetries.

and Grand Unified Theories (GUT, e.g. Yanagida $SU(5)$). Phenomenology.

Symmetries and the connection problem

Symmetries play an important role in the use of geometrical methods in physics. Wigner's unreasonable effectiveness of mathematics in natural sciences.

We have symmetries of (covariant) equations (e.g. Poincaré group, conformal group, AdS, etc.)

and, even more directly related, spectroscopical symmetries.

In atomic and molecular physics know the forces and e.g. breaking of SU(2) symmetry by crystalline field is natural (MF's master thesis).

In particle physics, different story. [Many particles, Fermi botanist.] Nice "boxes" to put them in, provided by UIRs of "internal" symmetries, at first SU(2) then SU(3), etc.

Connection? L.O'Raifeartaigh, Jost-Segal, Coleman-Mandula (and super-generalizations), our objections and counterexamples.

About two no-go theorems

Natural question: study the relation (if any) of internal world with space-time (relativity). That was, and still is a hard question. (E.g., combining the present Standard Model of elementary particles with gravitation is until now some quest for a Holy Grail.) Negating any connection, at least at the symmetry level, was a comfortable way out. For many, the proof of a trivial relation was achieved by what is often called the O'Raifeartaigh Theorem, a "no go theorem" stating that any finite-dimensional Lie algebra containing the Poincaré Lie algebra and an "internal" Lie algebra must contain these two as a direct product. The proof was based on nilpotency of Poincaré energy-momentum generators, but implicitly assumed the existence of a common invariant domain of differentiable vectors, which Wigner was careful to state as an assumption in his seminal 1939 paper and was proved later for Banach Lie group representations by "a Swedish gentleman". We showed in a provocative Letter that the result was not proved in the generality stated, then exhibited a number of counterexamples. The more sophisticated Coleman-Mandula attempt to prove a direct product relation contained an implicit hypothesis, hidden in the notation $|p, \alpha >$, that presupposed the result claimed to be proved.
One should be careful with no-go theorems.

In fact the situation is much more complex, especially when dynamics has to be introduced in the theory. One must not rule out a priori any relation between space-time and internal symmetries, nor the bolder idea that the latter emerge from the former.

Flato's deformation philosophy

Physical theories have domain of applicability defined by the relevant distances, velocities, energies, etc. involved. The passage from one domain (of distances, etc.) to another doesn't happen in an uncontrolled way: experimental phenomena appear that cause a paradox and contradict [Fermi quote] accepted theories. Eventually a new fundamental constant enters, the formalism is modified: the attached structures (symmetries, observables, states, etc.) *deform* the initial structure to a new structure which in the limit, when the new parameter goes to zero, "contracts" to the previous formalism. The question is, in which category to seek for deformations?

Physics is conservative: if start with e.g. category of associative or Lie algebras, tend to deform in same category. But there are important generalizations: e.g. quantum groups are deformations of (some commutative) Hopf algebras. And there may be more general deformations, not yet introduced, e.g. with noncommutative "parameters".

Dirac quote

"... One should examine closely even the elementary and the satisfactory features of our Quantum Mechanics and criticize them and try to modify them, because there may still be faults in them. The only way in which one can hope to proceed on those lines is by looking at the basic features of our present Quantum Theory from all possible points of view. Two points of view may be mathematically equivalent and you may think for that reason if you understand one of them you need not bother about the other and can neglect it. But it may be that one point of view may suggest a future development which another point does not suggest, and although in their present state the two points of view are equivalent they may lead to different possibilities for the future. Therefore, I think that we cannot afford to neglect any possible point of view for looking at Quantum Mechanics and in particular its relation to Classical Mechanics. Any point of view which gives us any interesting feature and any novel idea should be closely examined to see whether they suggest any modification or any way of developing the theory along new lines.

A point of view which naturally suggests itself is to examine just how close we can make the connection between Classical and Quantum Mechanics. That is essentially a purely mathematical problem – how close can we make the connection between an algebra of non-commutative variables and the ordinary algebra of commutative variables? In both cases we can do addition, multiplication, division..." **Dirac**, *The relation of Classical to Quantum Mechanics* (2^{nd} Can. Math. Congress, Vancouver 1949). U.Toronto Press (1951) pp 10-31.

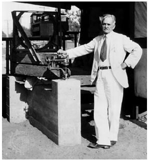
Ultrashort overview of deformations

The deformation philosophy (main paradigms: DQ, quantum groups and NCG; relativity & AdS).

Geometrical examples: Earth non flat (Pythagoras; Aristotle; Eratosthenes).
Relativity: deform Newtonian mechanics (Galilei symmetry $SO(3) \cdot \mathbb{R}^3 \cdot \mathbb{R}^4$) to special relativity (Poincaré, $SO(3,1) \cdot \mathbb{R}^4$), then e.g. dS and AdS.
Analytic examples. Quantization, deformation quantization.
(Quantum) Field theory, cohomological renormalization. Quantizating manifolds.
Quantum groups and Deformation Quantization.

Noncommutative geometry and Deformation Quantization.

Relativity



Paradox coming from Michelson & Morley experiment (1887) resolved in 1905 by Einstein with special theory of relativity. Experimental need triggered theory. In modern language: Galilean geometrical symmetry group of Newtonian mechanics ($SO(3) \cdot \mathbb{R}^3 \cdot \mathbb{R}^4$) is deformed, in Gerstenhaber's sense, to Poincaré group ($SO(3,1) \cdot \mathbb{R}^4$) of special relativity.
A deformation parameter comes in, c^{-1}, c being a *new fundamental constant*, velocity of light in vacuum. General relativity: *deform* Minkowskian space-time with nonzero pseudo-Riemannian curvature. E.g. constant curvature, de Sitter (> 0) or AdS_4 (< 0).

Deformations of algebras

DEFINITION. A deformation of an algebra A over a field $\mathbb{K}$ with deformation parameter ν is a $\mathbb{K}[[\nu]]$-algebra $\tilde{A}$ such that $\tilde{A}/\nu\tilde{A} \approx A$, where A is here considered as an algebra over $\mathbb{K}[[\nu]]$ by base field extension.

Two deformations $\tilde{A}$ and $\tilde{A}'$ are called equivalent if they are isomorphic over $\mathbb{K}[[\nu]]$. A deformation $\tilde{A}$ is trivial if isomorphic to the original algebra A (considered by base field extension as a $\mathbb{K}[[\nu]]$-algebra).

Algebras are generally supposed unital. Bialgebras are associative algebra A where we have in addition a coproduct $\Delta : A \longrightarrow A \otimes A$. Hopf algebras are bialgebras with in addition to the unit $\eta : \mathbb{K} \to A$ one has a counit $\epsilon : A \to \mathbb{K}$ and an antipode $S : A \to A$. All these are supposed with the obvious compatibility relations (commutative diagram). E.g. if $A = C^\infty(G)$, G a Lie group, then $\Delta f(x, y) = f(xy)$, $(Sf)(x) = f(x^{-1})$, $\epsilon(f) = f(1_G)$. Whenever we consider a topology on A, $\tilde{A}$ is supposed to be topologically free. The definition can (cf. e.g. Kontsevich) be extended to operads, so as to apply to the Assoc, Lie, Bialg and maybe Gerst operads, and also to the Hopf category (which cannot be described by an operad), all possibly with topologies.

Deformation formulas

For associative (resp. Lie) algebras, the definition tells that there exists a new product $*$ (resp. bracket $[\cdot,\cdot]$) such that the new (deformed) algebra is again associative (resp. Lie). Denoting the original composition laws by ordinary product (resp. $\{\cdot,\cdot\}$) this means that, for $u_1, u_2 \in A$ (we can extend this to $A[[\nu]]$ by $\mathbb{K}[[\nu]]$-linearity) we have:

$$u_1 * u_2 = u_1 u_2 + \sum_{r=1}^{\infty} \nu^r C_r(u_1, u_2) \quad (1)$$

$$[u_1, u_2] = \{u_1, u_2\} + \sum_{r=1}^{\infty} \nu^r B_r(u_1, u_2) \quad (2)$$

where the C_r are Hochschild 2-cochains and the B_r (skew-symmetric) Chevalley-Eilenberg 2-cochains, such that for $u_1, u_2, u_3 \in A$ we have $(u_1 * u_2) * u_3 = u_1 * (u_2 * u_3)$ and $\mathcal{S}[[u_1, u_2], u_3] = 0$, where $\mathcal{S}$ denotes summation over cyclic permutations.

Deformations of bialgebras, Hopf algebras; quantum groups

For a (topological) *bialgebra*, denoting by $\otimes_\nu$ the tensor product of $\mathbb{K}[[\nu]]$-modules we can identify $\tilde{A}\,\hat{\otimes}_\nu\tilde{A}$ with $(A\,\hat{\otimes}A)[[\nu]]$, where $\hat{\otimes}$ denotes the algebraic tensor product completed with respect to some topology (e.g. projective for Fréchet nuclear topology on A). We similarly have a deformed coproduct $\tilde{\Delta} = \Delta + \sum_{r=1}^{\infty} \nu^r D_r$, $D_r \in \mathcal{L}(A, A\hat{\otimes}A)$, satisfying $\tilde{\Delta}(u_1 * u_2) = \tilde{\Delta}(u_1) * \tilde{\Delta}(u_2)$. In this context appropriate cohomologies can be introduced. There are natural additional requirements for Hopf algebras.

"Quantum groups" are e.g. deformations of a Hopf algebra $A = C^\infty(G)$ or "its dual" (in t.v.s. sense) $A' = \mathcal{U}(\mathfrak{g})$ (or some closure of it), where G is a Lie group equipped with a "compatible" Poisson bracket P (making it a Poisson manifold, see below), and $\mathfrak{g}$ its Lie (bi)algebra. The notion arose around 1980 in Faddeev's Leningrad group in relation with inverse scattering and quantum integrable systems, was systematized by Drinfeld and Jimbo, and is now widely used in many contexts.

Quantization in physics

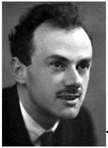


Planck and black body radiation [ca. 1900]. Bohr atom [1913]. Louis de Broglie [1924]: "wave mechanics" (waves and particles are two manifestations of the same physical reality).


Traditional quantization (Schrödinger, Heisenberg) of classical system $(\mathbb{R}^{2n}, \{\cdot,\cdot\}, H)$: Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n) \ni \psi$ where acts "quantized" Hamiltonian $\mathbf{H}$, energy levels $\mathbf{H}\psi = \lambda\psi$, and von Neumann representation of CCR. Define $\hat{q}_\alpha(f)(q) = q_\alpha f(q)$ and $\hat{p}_\beta(f)(q) = -i\hbar\frac{\partial f(q)}{\partial q_\beta}$ for f differentiable in $\mathcal{H}$. Then (CCR) $[\hat{p}_\alpha, \hat{q}_\beta] = i\hbar\delta_{\alpha\beta}I \quad (\alpha, \beta = 1, ..., n)$.

Orderings, Weyl, Wigner; Dirac constraints





The couple $(\hat{q}, \hat{p})$ quantizes the coordinates (q, p). A polynomial classical Hamiltonian H is quantized once chosen an operator ordering, e.g. (Weyl) complete symmetrization of $\hat{p}$ and $\hat{q}$. In general the quantization on $\mathbb{R}^{2n}$ of a function $H(q, p)$ with inverse Fourier transform $\tilde{H}(\xi, \eta)$ can be given by (Hermann Weyl [1927] with weight $\varpi = 1$):

$$H \mapsto \mathbf{H} = \Omega_\varpi(H) = \int_{\mathbb{R}^{2n}} \tilde{H}(\xi, \eta) \exp(i(\hat{p}.\xi + \hat{q}.\eta)/\hbar)\varpi(\xi, \eta) d^n\xi d^n\eta.$$

E. Wigner [1932] inverse $H = (2\pi\hbar)^{-n}\mathrm{Tr}[\Omega_1(H)\exp((\xi.\hat{p} + \eta.\hat{q})/i\hbar)]$. Ω_1 defines an isomorphism of Hilbert spaces between $L^2(\mathbb{R}^{2n})$ and Hilbert–Schmidt operators on $L^2(\mathbb{R}^n)$. Can extend e.g. to distributions. Other orderings: standard (diff. and pseudodiff. ops., "first q then p"), normal (physics): ϖ = exp. of 2^{nd} order polynomial.

Constrained systems (e.g. constraints $f_j(p, q) = 0$): Dirac formalism [1950]. (Second class constraints reduce $\mathbb{R}^{2n}$ to symplectic submanifold, first class to Poisson.)

Classical Mechanics and around

What do we quantize?

Non trivial phase spaces $\rightarrow$ Symplectic and Poisson manifolds.

Symplectic manifold:Differentiable manifold M with nondegenerate closed 2-form ω on M. Necessarily dim $M = 2n$.

Define $\pi^{ij} = \omega_{ij}^{-1}$, then $\{F, G\} = \pi^{ij}\partial_i F \partial_j G$ is a Poisson bracket, i.e. the bracket $\{\cdot,\cdot\}\colon C^\infty(M) \times C^\infty(M) \rightarrow C^\infty(M)$ is a skewsymmetric ($\{F, G\} = -\{G, F\}$) bilinear map satisfying:

- Jacobi identity: $\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0$
- Leibniz rule: $\{FG, H\} = \{F, H\}G + F\{G, H\}$

Examples:$\mathbb{R}^{2n}$ with $\omega = \sum_{i=1}^{i=n} dq^i \wedge dp^i$; Cotangent bundle T^*N, $\omega = d\alpha$, where α is the canonical one-form on T^*N (Locally, $\alpha = -p_i dq^i$).

Poisson manifold: Differentiable manifold M, and skewsymmetric contravariant 2-tensor (not necessarily nondegenerate) $\pi = \sum_{i,j} \pi^{ij}\partial_i \wedge \partial_j$ (locally) such that $\{F, G\} = i(\pi)(dF \wedge dG) = \sum_{i,j} \pi^{ij}\partial_i F \wedge \partial_j G$ is a Poisson bracket.

A Classical System is a Poisson manifold (M, π) with a distinguished smooth function, the Hamiltonian $H\colon M \rightarrow \mathbb{R}$.

The framework of deformation quantization

Poisson manifold (M, π), deformations of product of functions.

Inspired by deformation philosophy, based on Gerstenhaber's deformation theory. [M. Gerstenhaber, Ann.Math. '63 & '64. Flato, Lichnerowicz, Sternheimer; and Vey; mid 70's. Bayen, Flato, Fronsdal, Lichnerowicz, Sternheimer, LMP '77 & Ann. Phys. '78]

- $\mathcal{A}_t = C^\infty(M)[[t]]$, formal series in t with coefficients in $C^\infty(M) = A$. Elements: $f_0 + tf_1 + t^2 f_2 + \cdots$ (t formal parameter, not fixed scalar.)
- Star product $\star_t : \mathcal{A}_t \times \mathcal{A}_t \to \mathcal{A}_t$; $f \star_t g = fg + \sum_{r\geq 1} t^r C_r(f, g)$

- C_r are bidifferential operators null on constants: $(1 \star_t f = f \star_t 1 = f)$.
- $\star_t$ is associative and $C_1(f, g) - C_1(g, f) = 2\{f, g\}$, so that $[f, g]_t \equiv \frac{1}{2t}(f \star_t g - g \star_t f) = \{f, g\} + O(t)$ is Lie algebra deformation.

Basic paradigm. Moyal product on $\mathbb{R}^{2n}$ with the canonical Poisson bracket P:
$F \star_M G = \exp\left(\frac{i\hbar}{2}P\right)(F, G) \equiv FG + \sum_{k\geq 1} \frac{1}{k!}\left(\frac{i\hbar}{2}\right)^k P^k(F, G)$.

Applications and Equivalence

Equation of motion (time τ): $\frac{dF}{d\tau} = [H, F]_M \equiv \frac{1}{i\hbar}(H \star_M F - F \star_M H)$
Link with Weyl's rule of quantization: $\Omega_1(F \star_M G) = \Omega_1(F)\Omega_1(G)$

Equivalence of two star-products $\star_1$ and $\star_2$.

- Formal series of differential operators $T(f) = f + \sum_{r \geq 1} t^r T_r(f)$.
- $T(f \star_1 g) = T(f) \star_2 T(g)$.

For symplectic manifolds, equivalence classes of star-products are parametrized by the 2nd de Rham cohomology space $H^2_{dR}(M)$: $\{\star_t\}/\sim\, = \, H^2_{dR}(M)[[t]]$ (Nest-Tsygan [1995] and others). In particular, $H^2_{dR}(\mathbb{R}^{2n})$ is trivial, all deformations are equivalent.

Kontsevich: $\{$Equivalence classes of star-products$\} \equiv \{$equivalence classes of formal Poisson tensors $\pi_t = \pi + t\pi_1 + \cdots\}$.

Remarks: - The choice of a star-product fixes a quantization rule.
- Operator orderings can be implemented by good choices of T (or ϖ).
- On $\mathbb{R}^{2n}$, all star-products are equivalent to Moyal product (cf. von Neumann uniqueness theorem on projective UIR of CCR).

This is Quantization

Equation of motion (time τ): $\frac{dF}{d\tau} = [H, F]_M \equiv \frac{1}{i\hbar}(H \star_M F - F \star_M H)$
Link with Weyl's rule of quantization: $\Omega_1(F \star_M G) = \Omega_1(F)\Omega_1(G)$.
A star-product provides an *autonomous* quantization of a manifold M.

BFFLS '78: Quantization is a deformation of the composition law of observables of a classical system: $(A, \cdot) \to (A[[t]], \star_t)$, $A = C^\infty(M)$.

Star-product $\star$ ($t = \frac{i}{2}\hbar$) on Poisson manifold M and Hamiltonian H; introduce the star-exponential: $\mathrm{Exp}_\star(\frac{\tau H}{i\hbar}) = \sum_{r\geq 0} \frac{1}{r!}(\frac{\tau}{i\hbar})^r H^{\star r}$.
Corresponds to the unitary evolution operator, is a singular object i.e. belongs not to the quantized algebra $(A[[t]], \star)$ but to $(A[[t, t^{-1}]], \star)$. Singularity at origin of its trace, Harish Chandra character for UIR of semi-simple Lie groups.
Spectrum and states are given by a spectral (Fourier-Stieltjes in the time τ) decomposition of the star-exponential.

Paradigm: Harmonic oscillator $H = \frac{1}{2}(p^2 + q^2)$, Moyal product on $\mathbb{R}^{2\ell}$.
$\mathrm{Exp}_\star(\frac{\tau H}{i\hbar}) = \big(\cos(\frac{\tau}{2})\big)^{-1} \exp\big(\frac{2H}{i\hbar}\tan(\frac{\tau}{2})\big) = \sum_{n=0}^\infty \exp\big(-i(n+\frac{\ell}{2})\tau\big)\pi_n^\ell$.
Here ($\ell = 1$ but similar formulas for $\ell \geq 1$, L_n is Laguerre polynomial of degree n)
$\pi_n^1(q,p) = 2\exp\big(\frac{-2H(q,p)}{\hbar}\big)(-1)^n L_n\big(\frac{4H(q,p)}{\hbar}\big)$.

Conventional vs. deformation quantization

- It is a matter of practical feasibility of calculations, when there are Weyl and Wigner maps to intertwine between both formalisms, to choose to work with operators in Hilbert spaces or with functional analysis methods (distributions etc.) But one should always keep in mind that the Hilbert space formulation is NOT a must for quantization: what characterizes the adjective "quantum" is noncommutativity. Dealing e.g. with spectroscopy (where it all started; cf. also Connes) and finite dimensional Hilbert spaces where operators are matrices, the operatorial formulation may be easier.
- When there are no precise Weyl and Wigner maps (e.g. very general phase spaces, maybe infinite dimensional) one does not have much choice but to work (maybe "at the physical level of rigor") with functional analysis.

Contrarily to what Gukov and Witten assert (`arXiv:0809.0305v1` p.10) deformation quantization IS quantization: it permits (in concrete cases) to take for $\hbar$ its value, when there are Weyl and Wigner maps one can translate its results in Hilbert space, and e.g. for the 2-sphere there is a special behavior when the radius of the sphere has quantized values related to the Casimir values of $SO(3)$.

Star-representations, wavelets and potential applications

Star representations. G Lie group acts on symplectic (M, P),
Lie algebra $\mathfrak{g} \ni x$ realized by $u_x \in C^\infty(M)$, with
$P(u_x, u_y) = [u_x, u_y] \equiv \frac{1}{2\nu}(u_x \star u_y - u_y \star u_x)$ (preferred observables)
Define (group element) $E(e^x) = \text{Exp}(x) \equiv \sum_{n=0}^{\infty}(n!)^{-1}(u_x/2\nu)^{\star n}$.
Star Representation: $\mathrm{Im}E$-valued distribution on M
(test functions on M) $D \ni f \mapsto \mathcal{E}(f) = \int_G f(g)E(g^{-1})dg$.

Wavelets (a kind of NC Fourier transform) can be viewed as analysis on *-reps. of $ax + b$ group; that was generalized to the 3-dim. solvable groups $[a, b] = b, [a, c] = \theta c$

Fedosov algorithm and Kontsevich formula for star products. Fedosov builds from a symplectic connection ∇ on M (symplectic) a flat connection D on the Weyl bundle W on M such that the algebra of horizontal sections for D induces a star-product on M. Kontsevich shows that the map $\star\colon C^\infty(\mathbb{R}^d) \times C^\infty(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d)[[\lambda]]$ defined by $(f, g) \mapsto f \star g = \sum_{n\geq 0} \lambda^n \sum_{\Gamma \in G_{n,2}} w(\Gamma)B_\Gamma(f, g)$ defines a star-product on $(\mathbb{R}^d, \alpha)$, and globalizes that. The suggestion is to use Kontsevich's formula in applications.

Some related mathematical topics

* Sém. Cartan–Schwartz 1963/64, 1963 Atiyah-Singer index thm. In parallel with Palais' sem. in Pctn. (Gelfand conj.) My share: mult. ppty of anal. index, allows dim. reduction. ∃ many extensions.

* Early 70's: Geometric quantization Good for reps. of solvable gps. but ...

* Berezin quantization.(some kind of quantization for mfds. but deformation aspect absent),

* Anal. vect. for Lie alg. reps. in t.v.s. (FSSS Ann.ENS 1972).

* Deformation quantization since 1976. Comp. of symbols of ΨDO is star product. KMS states and DQ: 2-param. def., symplectic form with conf. factor $\exp^{-\beta\frac{H}{2}}$.

* Quantum groups (since 1980) are Hopf alg. defs. Topological q. gps.

* NC Geom., since 1980. Idea: characterize diff. mfds by properties of algebras, then deform algebras. Based on works by Connes on C^* algebras in 70's. (Closed) star products (CFS 1992, OMY 1993): another example. Algebraic Index thms.

Poincaré and Anti de Sitter "external" symmetries

1930's: Dirac asks Wigner to study UIRs of Poincaré group. 1939: Wigner paper in Ann.Math. UIR: particle with positive and zero mass (and "tachyons"). Seminal for UIRs (Bargmann, Mackey, Harish Chandra etc.)

Deform Minkowski to AdS, and Poincaré to AdS group SO(2,3). UIRs of AdS studied incompletely around 1950's. 2 (most degenerate) missing found (1963) by Dirac, the singletons that we call Rac$= D(\frac{1}{2}, 0)$ and Di$= D(1, \frac{1}{2})$ (massless of Poincaré in 2+1 dimensions). In normal units a singleton with angular momentum j has energy $E = (j + \frac{1}{2})\rho$, where ρ is the curvature of the AdS_4 universe (they are naturally confined, fields are determined by their value on cone at infinity in AdS_4 space).

The massless representations of $\mathrm{SO}(2,3)$ are defined (for $s \geq \frac{1}{2}$) as $D(s+1, s)$ and (for helicity zero) $D(1,0) \oplus D(2,0)$. There are many justifications to this definition. They are kinematically composite:
$(\mathrm{Di} \oplus \mathrm{Rac}) \otimes (\mathrm{Di} \oplus \mathrm{Rac}) = (D(1,0) \oplus D(2,0)) \oplus 2\bigoplus_{s=\frac{1}{2}}^{\infty} D(s+1, s)$.
Also dynamically (QED with photons composed of 2 Racs, FF88).

Composite electrodynamics

Photon (composite QED) and new infinite dimensional algebras. Flato, M.; Fronsdal, C. *Composite electrodynamics*. J. Geom. Phys. 5 (1988), no. 1, 37–61.
Singleton theory of light, based on a pure gauge coupling of scalar singleton field to electromagnetic current. Like quarks, singletons are essentially unobservable. The field operators are not local observables and therefore need not commute for spacelike separation, hence (like for quarks) generalized statistics. Then a pure gauge coupling generates real interactions – ordinary electrodynamics in AdS space. Singleton field operator $\phi(x) = \sum_j \phi^j(x) a_j$ +h.c. Concept of normal ordering in theory with unconventional statistics is worked out; there is a natural way of including both photon helicities.
Quantization is a study in representation theory of certain infinite-dimensional, nilpotent Lie algebras (generated by the a_j), of which the Heisenberg algebra is the prototype (and included it it for the photon). Compatible with QED.

Singleton field theory and neutrino oscillations in AdS

Singletons, Physics in AdS Universe and Oscillations of Composite Neutrinos,
Lett. Math. Phys. 48 (1999), no. 1, 109–119. (MF, CF, DS)
The study starts with the kinematical aspects of singletons and massless particles. It extends to the beginning of a field theory of composite elementary particles and its relations with conformal field theory, including very recent developments and speculations about a possible interpretation of neutrino oscillations and CP violation in this context. The "singleton" framework was developed mainly during the last two decades of the last century. Based on our deformation philosophy of physical theories, it deals with elementary particles composed of singletons in anti-De Sitter spacetime.

Composite neutrinos oscillations

Developing a field theory of composite neutrinos (neutrinos composed of singleton pairs with, e.g., three flavors of singletons) it might be possible to correlate oscillations between the three kinds of neutrinos with the AdS_4 description of these 'massless' particles. Of course any reasonable estimate of the value of the cosmological constant rules out a direct connection to the value of experimental parameters like PC violation coupling constants or neutrino masses. PC violation is a feature of composite electrodynamics and any direct observation of singletons, even at infinity, will imply PC violation. If more than one singleton flavor is used, as is appropriate in the context of neutrinos, then PC invariance can be restored in the electromagnetic sector, but in that case, neutrino oscillations will imply PC violation. The structure of Anti de Sitter field theory, especially that of singleton field theory, may provide a natural framework for a description of neutrino oscillations.

Composite leptons and flavor symmetry

The electroweak model is based on "the weak group", $S_W = SU(2) \times U(1)$, on the Glashow representation of this group, carried by the triplet $(\nu_e, e_L; e_R)$ and by each of the other generations of leptons.

Now make the following phenomenological Ansatz:
(a) There are three bosonic singletons $(R^N R^L; R^R) = (R^A)_{A=N,L,R}$ (three "Rac"s) that carry the Glashow representation of S_W;

(b) There are three spinorial singletons $(D_\varepsilon, D_\mu; D_\tau) = (D_\alpha)_{\alpha=\varepsilon,\mu,\tau}$ (three "Di"s). They are insensitive to S_W but transform as a Glashow triplet with respect to another group S_F (the "flavor group"), isomorphic to S_W;

(c) The vector mesons of the standard model are Rac-Rac composites, the leptons are Di-Rac composites, and there is a set of vector mesons that are Di-Di composites and that play exactly the same role for S_F as the weak vector bosons do for S_W: $W^B_A = \bar{R}^B R_A$, $L^A_\beta = R^A D_\beta$, $F^\alpha_\beta = \bar{D}_\beta D^\alpha$.

These are initially massless, massified by interaction with Higgs.

Composite leptons massified

Let us concentrate on the leptons ($A = N, L, R$; $\beta = \varepsilon, \mu, \tau$)

$$(L^A_\beta) = \begin{pmatrix} \nu_e & e_L & e_R \\ \nu_\mu & \mu_L & \mu_R \\ \nu_\tau & \tau_L & \tau_R \end{pmatrix} . \qquad (3)$$

A factorization $L^A_\beta = R^A D_\beta$ is strongly urged upon us by the nature of the previous phenomenological Ansatz. Fields in the first two columns couple horizontally to make the standard electroweak current, those in the last two pair off to make Dirac mass-terms. Particles in the first two rows combine to make the (neutral) flavor current and couple to the flavor vector mesons. The Higgs fields have a Yukawa coupling to lepton currents, $\mathcal{L}_{\rm Yu} = -g_{\rm Yu}\bar{L}^\beta_A L^B_\alpha H^{\alpha A}_{\beta B}$. The electroweak model was constructed with a single generation in mind, hence it assumes a single Higgs doublet. We postulate additional Higgs fields, coupled to leptons in the following way, $\mathcal{L}'_{\rm Yu} = h_{\rm Yu} L^A_\alpha L^B_\beta K^{\alpha\beta}_{AB} + {\rm h.c.}$. This model predicts 2 new mesons, parallel to the W and Z of the electroweak model (Frønsdal, LMP 2000). But too many free parameters.

Do the same for quarks (and gluons), adding color?

Questions and facts

Even if know "intimate structure" of particles (as composites of quarks etc. or singletons): How, when and where happened "baryogenesis"? [Creation of 'our matter', now 4% of universe mass, vs. 74% 'dark energy' and 22 % 'dark matter'; and matter–antimatter asymetry, Sakharov 1967.] Everything at "big bang"?! [Shrapnel of 'stem cells' of initial singularity?]

Facts: $SO_q(3,2)$ at even root of 1 is f.dim. Hopf alg. has f.dim.UIRs ("compact"?).
Black holes à la 't Hooft: can communicate with them, by interaction at surface.
Noncommutative (quantized) manifolds. E.g. quantum 3- and 4-spheres (Connes with Landi and Dubois-Violette); spectral triples $(\mathcal{A}, \mathcal{H}, D)$).
Connes' Standard Model with neutrino mixing, minimally coupled to gravity.
Space-time is Riemannian compact spin 4-manifold $\times$ finite (32) NCG. It predicted at first a higher Higgs mass, but they had forgotten a quadratic term which corrects that. (Barrett has Lorentzian version.) A main issue is that mathematicians interested in physics ask physicists what they are doing, not why.

[Dark matter models with sterile neutrinos, Kusenko.]

Conjectures and a speculative answer

[Odessa Rabbi anecdote] Space-time could be, at very small distances, not only deformed (to AdS_4 with tiny negative curvature ρ, which does not exclude at cosmological distances to have a positive curvature or cosmological constant, e.g. due to matter) but also "quantized" to some $qAdS_4$. Such $qAdS_4$ could be considered, in a sense to make more precise (e.g. with some measure or trace) as having "finite" (possibly "small") volume (for q even root of unity). At the "border" of these one would have, for most practical purposes at "our" scale, the Minkowski space-time, obtained by $q\rho \to 0$. They could be considered as some "black holes" from which "q-singletons" would emerge, create massless particles that would be massified by interaction with dark matter or dark energy. That could (and should, otherwise there would be manifestations closer to us, that were not observed) occur mostly at or near the "edge" of our universe in accelerated expansion.

These "qAdS black holes" ("inside" which one might find compactified extra dimensions) could be "stem cells" resulting from Big Bang from which matter would be continuously created.

A NCG model for qAdS$_4$

To AdS$_n$, $n \geq 3$, we associate *naturally* a symplectic symmetric space (M, ω, s). The data of any invariant (formal or not) deformation quantization on (M, ω, s) yields canonically universal deformation formulae (procedures associating to a topological algebra $\mathbb{A}$ having a symmetry $\mathcal{G}$ a deformation $\mathbb{A}_\theta$ in same category) for the actions of a non-Abelian solvable Lie group $\mathcal{R}_0$ (one-dimensional extension of the Heisenberg group $\mathcal{H}_n$), given by an oscillatory integral kernel.
Using it we (P.Bieliavsky, LC, DS & YV) define a noncommutative Lorentzian spectral triple $(\mathcal{A}^\infty, \mathcal{H}, D)$ where $\mathcal{A}^\infty := (L^2_{\mathrm{right}}(\mathcal{R}_0))^\infty$ is a NC Fréchet algebra modelled on the space $\mathcal{H}^\infty$ of smooth vectors of the regular representation on the space $\mathcal{H}$ of square integrable functions on $\mathcal{R}_0$, and D a Dirac operator acting as a derivation of the noncommutative bi-module structure, and for all $a \in \mathcal{A}^\infty$, the commutator $[D, a]$ extends to $\mathcal{H}$ as a bounded operator. The underlying commutative limit is endowed with a causal black hole structure (for $n \geq 3$) encoded in the $\mathcal{R}_0$-group action.
Cf. (also for the following) the two papers in ref. 12, available as:
`http://wwwen.uni.lu/content/download/56018/661547/file/sternheimer.pdf`
`http://monge.u-bourgogne.fr/dsternh/papers/sternheimer2WGMPd.pdf`

The latter is being published by Birkhäuser in the Proceedings of WGMP32,

Białowieża, July 2013.

Some cosmological perspectives and speculations

1. Define within the present Lorentzian context the notion of causality at the operator algebraic level.
2. Representation theory for $\mathrm{SO}_q(2, n)$ (e.g. new reps. at root of unity, analogs of singletons, 'square root' of massless reps. of AdS or Poincaré, etc.)
3. Define a kind of trace giving finite "q-volume" for qAdS at even root of unity (possibly in TVS context).
4. Find analogs of all the 'good' properties (e.g. compactness of the resolvent of D) of Connes' spectral triples in compact Riemannian case, possibly with quadruples $(\mathcal{A}, \mathcal{E}, D, \mathcal{G})$ where $\mathcal{A}$ is some topological algebra, $\mathcal{E}$ an appropriate TVS, D some (bounded on $\mathcal{E}$) "Dirac" operator and $\mathcal{G}$ some symmetry.
5. Limit $\rho q \to 0$ ($\rho < 0$ being AdS curvature)?
6. Unify (groupoid?) Poincaré in Minkowski space (possibly modified locally by the presence of matter) with these $\mathrm{SO}_q(2, n)$ in the qAdS "black holes".
7. Field theory on such q-deformed spaces, etc.

Quantum (loop?) groups at root of unity

Fact: quantum groups at root of unity have finite dimensional UIRs. (The Hopf algebra is *finite dimensional*. But can be tricky; bialgebras should generically behave well w.r.t tensor products; pbs. at root of 1 for $\mathfrak{sl}_2$). Natural to start with Poincaré symmetry, or its (simple) AdS deformation, and "deform it" by quantization (to quantum AdS, taken at root of 1). One can also quantize some form of "loop AdS" algebra to "blow up" field theoretical singularities. By "loop" I mean maps to AdS ($\mathfrak{so}(3,2)$) from a closed string S^1, i.e. "affine" simple Lie algebra, or possibly (something not yet studied mathematically) maps from a higher dimensional object, e.g. a K_3 surface or a Calabi-Yau manifold. Maybe the successes of the SM can be derived (or the SM built) by starting with such procedures, e.g. qAdS at 6^{th} root of 1. There could be a part of self-fulfilling prophecy when experimental data are phenomenologically interpreted in the framework of a model.

At present the pieces of the "puzzle" fit remarkably well, but it could be that different interpretations of the present experimental data fit even better. E.g. interpretations based on the above framework. Further experiments, using the presently available apparati, could tell.

Generalized deformations and the deformation conjecture

Pinczon, Nadaud: the deformation "parameter" acts on the algebra. Still a cohomological theory. E.g. G-rigid Weyl algebra deformable to $\mathfrak{osp}(2,1)$.

More generalized deformations where the "parameter", instead of being a scalar (the algebra of a one-element group) would belong to the algebra of a finite group (e.g. the center $\mathbb{Z}/n\mathbb{Z}$ or the Weyl group (S_n) of some SU(n)) or be quaternionic. Most of these theories have yet to be properly defined and studied. (Might also be useful in quantum computing.)

It is likely that the core of the success of unitary groups as classification symmetries, appearing in the SM, is number-theoretic. It should thus be possible to develop similar (or better) explanations from suitably deformed (and quantized) space-time symmetries. Or supersymmetries for that matter. That would give a conceptual basis to the SM, or some variation of it, including the dynamics built from it. Or alternatively a totally new interpretation as deformations could prove more effective.
In any case the mathematical problems raised by both approaches are worthy of attack (and are likely to prove their worth by hitting back). And maybe that will permit to base the interpretation of the present data on firmer "space-time ground".

THE DEFORMATION CONJECTURE. *Internal symmetries of elementary particles arise from their relativistic counterparts by some form of deformation (including quantization).*

A tentative "road map"

1. "Mathematical homework".
a. Study representations and (some of) their tensor products for qAdS at (some) root of unity. Maybe start with $q\mathfrak{sl}(3)$ instead of qB_2 (or qC_2, which could be different, especially for AdS real forms.
b. Multiparameter quantum groups at roots of 1. E.g. qAdS with 3 Abelian parameters at some roots of 1 (e.g. sixth for all 3, but maybe different), their representations and (some of) the tensor products of these.

c. Reshetikihin-Turaev (& Quantum Chern-Simons) theories with such gauges (Andersen).

d. Define & study "quantum deformations" with quaternionic "parameters", or in the group algebra of e.g. S_n. (All are problems of independent mathematical interest.)

2. Possible physical applications.

a. Try to use 1 (with some qAdS) to (step by step) re-examine the phenomenological classification of elementary particles. We might not need quarks, except perhaps if we get them with quaternionic deformations.
b. Build a new dynamics based on such deformed relativistic symmetries.
c. Re-examine half a century of particle physics, from the points of view of theory, experiments and phenomenology. Connection with the "String Framework"?

Problems worthy of attack prove their worth by hitting back.

Very few references

[See also references in all these, and more]
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"The important thing is not to stop questioning", including the symmetries on which is based the Standard Model.