A quantum 4-sphere with non central radius and its instanton sheaf

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The Hopf fibration and $SU(2)$-instantons

$$\pi : S^7 \xrightarrow{SU(2)} S^4 \quad \left(\simeq St_{\mathbb{H}}(1,2) \xrightarrow{Sp(1)} Gr_{\mathbb{H}}(1,2) \right)$$

- 2^{nd}-Hopf map [Hopf '31] higher homotopy groups of spheres
- Instanton bundle
  $SU(2)$-Yang-Mills eqs.(1954): Euler-Lagrange eqs. for

$$\mathcal{A}_{YM} = \int_{S^4} |F_A|^2 d\mu$$

Instantons: solutions A of minima of Yang-Mills eqs.:
Topological invariant $k = c_2 \in \mathbb{Z}$ as a lower bound: $8\pi^2|k| \leq \mathcal{A}_{YM}$

Minima reached $\iff$ $*F_A = \pm F_A$ (and then A automatically solution of YM eqs.)

Basic case $k = 1$:

$SU(2)$-instanton: connection A with ASD F_A on the Hopf bundle $S^7 \xrightarrow{SU(2)} S^4$

2002- ... quantum Hopf fibrations on quantum spheres

Various constructions of Hopf-bundles $S^7 \xrightarrow{SU(2)} S^4$ and instantons on quantum spheres:

- Spheres from quantum groups:
  - unitary quantum groups [Bonechi-Ciccoli-Tarlini, '02]
  - symplectic quantum groups [Landi-P.-Reina, '06]
- Spheres from q-deformed $\mathbb{H}^2$ [Brain-Landi, '12]

  } $SU_q(2)$

- Connes-Landi 4-sphere [Landi-van Suijlekom, '05]
  ['08-... Brain, Landi, P., van Suijlekom]

  } $SU(2)$

Principal fibrations in NCG: **Hopf-Galois extensions**

[Kreimer, Takeuchi 1981]

- a $*$-algebra A *'total space'* ;
- a Hopf algebra (H, Δ, ε) as *'structure quantum group'* ;
- *coaction* $\delta : A \to A \otimes H$:

$$(\Delta \otimes id)\delta = (id \otimes \delta)\delta \quad , \quad (\varepsilon \otimes id)\delta = id$$

- a $*$-algebra B *'base space'* of coinvariants $B \simeq A^{co(H)} := \{a \in A | \delta(a) = a \otimes 1\}$

\+ '*condition of principality*': $B \subset A$ is a Hopf-Galois extension:

$$\chi = (m_A \otimes id)(id \otimes_B \delta) : A \otimes_B A \to A \otimes H \qquad \text{(canonical map)}$$

is bijective

$$S^7 \xrightarrow{SU(2)} S^4 \quad \rightsquigarrow \quad \mathcal{A}(S^4) \simeq \mathcal{A}(S^7)^{co\mathcal{A}(SU(2))} \hookrightarrow \mathcal{A}(S^7) \text{ Hopf-Galois}$$

• Quantum spheres: deformations of the algebra of

$$S^n = \{(x_1, \dots x_{n+1}) \in \mathbb{R}^{n+1} | \sum x_i^2 = 1\}$$

The 4-sphere $\mathcal{A}(S^4_q)$ with non central radius

[Cirio-Landi-Szabo, 2011]

The algebra $\mathcal{A}(S^4_q)$ is a one real parameter deformation of the algebra of coordinate functions on the classical four-sphere, where this latter is seen as a real slice of the Klein quadric in $\mathbb{CP}^5$.

Introduced in the context of deformations of toric varieties $(\mathbb{C}^\times)^n$ via Drinfeld 2-cocycle

- The algebra $\mathcal{A}(\mathbb{CP}^5_q)$, $q \in \mathbb{R}$, is generated by 'Plücker coordinates' Λ_{ij}, $i < j$ $(i,j = 1, \dots, 4)$, satisfying

$$\begin{array}{lll} \Lambda_{12}\Lambda_{13} = q^{-2}\Lambda_{13}\Lambda_{12} \quad , & \Lambda_{12}\Lambda_{14} = q^{-2}\Lambda_{14}\Lambda_{12} \quad , & \Lambda_{12}\Lambda_{23} = q^{2}\Lambda_{23}\Lambda_{12} \quad , \\ \Lambda_{12}\Lambda_{24} = q^{2}\Lambda_{24}\Lambda_{12} \quad , & \Lambda_{12}\Lambda_{34} = \Lambda_{34}\Lambda_{12} \quad , & \Lambda_{13}\Lambda_{14} = \Lambda_{14}\Lambda_{13} \quad , \\ \Lambda_{13}\Lambda_{23} = q^{2}\Lambda_{23}\Lambda_{13} \quad , & \Lambda_{13}\Lambda_{24} = q^{2}\Lambda_{24}\Lambda_{13} \quad , & \dots \end{array}$$

- The algebra $\mathcal{A}(\mathbb{G}r_q(2,4))$ is the quotient of $\mathcal{A}(\mathbb{CP}^5_q)$ by the ideal generated by

$$q\Lambda_{12}\Lambda_{34} - \Lambda_{13}\Lambda_{24} + \Lambda_{14}\Lambda_{23} \ .$$

- The algebra $\mathcal{A}(S^4_q)$ is the real part of $\mathcal{A}(\mathbb{G}r_q(2,4))$ obtained by introducing the $*$-structure

$$\Lambda_{12}^* = \Lambda_{12} \ , \quad \Lambda_{13}^* = q\Lambda_{24} \ , \quad \Lambda_{14}^* = -q\Lambda_{23} \ , \quad \Lambda_{34}^* = \Lambda_{34}$$

By considering the change of generators $X := \frac{1}{2}q(\Lambda_{12} - \Lambda_{34})$ and $R := \frac{1}{2}q(\Lambda_{12} + \Lambda_{34})$, the Klein identity has real form

$$X^2 + \Lambda_{13}\Lambda_{13}^* + \Lambda_{14}\Lambda_{14}^* = R^2 \, .$$

$\longrightarrow$ for $q = 1$, localize by R and retrieve the sphere described in affine coordinates

$\longrightarrow$ for $q \neq 1$, R is not central in $\mathcal{A}(S^4_q)$ and it does not generate a Ore denominator set

$\rightsquigarrow$ not possible to localize by R

$\longrightarrow$ $\mathcal{A}(S^4_q)$ is better described locally

The 'local patches' ${}_N\mathbb{R}^4_q$, ${}_S\mathbb{R}^4_q$ of the sphere $\mathcal{A}(S^4_q)$

- ${}_S\mathbb{R}^4_q$: via left Ore localization of $\mathcal{A}(\mathbb{G}r_\theta(d,n))$ with respect to Λ_{12}
  Generated by elements α_{13}, α_{14}, α_{23} and α_{24} with

$$\alpha_{13}\alpha_{14} = \alpha_{14}\alpha_{13} \,, \qquad \alpha_{13}\alpha_{23} = q^{-2}\alpha_{23}\alpha_{13} \,, \qquad \alpha_{13}\alpha_{24} = q^{-2}\alpha_{24}\alpha_{13} \,,$$
$$\alpha_{14}\alpha_{23} = q^{-2}\alpha_{23}\alpha_{14} \,, \qquad \alpha_{14}\alpha_{24} = q^{-2}\alpha_{24}\alpha_{14} \,, \qquad \alpha_{23}\alpha_{24} = \alpha_{24}\alpha_{23} \,.$$

  and endowed with the $*$-structure $\alpha_{24}^* = q^{-3}\alpha_{13}$, $\alpha_{23}^* = -q^{-3}\alpha_{14}$.
  (Extra redundant generator $\alpha = q^{-4}(\alpha_{13}^*\alpha_{13} + \alpha_{14}^*\alpha_{14})$.)

- ${}_N\mathbb{R}^4_q$: via right Ore localization of $\mathcal{A}(\mathbb{G}r_\theta(d,n))$ with respect to Λ_{34}
  Generated by elements β_{13}, β_{14}, β_{23} and β_{24} with

$$\beta_{13}\beta_{14} = \beta_{14}\beta_{13} \,, \qquad \beta_{13}\beta_{23} = q^{2}\beta_{23}\beta_{13} \,, \qquad \beta_{13}\beta_{24} = q^{2}\beta_{24}\beta_{13} \,,$$
$$\beta_{14}\beta_{23} = q^{2}\beta_{23}\beta_{14} \,, \qquad \beta_{14}\beta_{24} = q^{2}\beta_{24}\beta_{14} \,, \qquad \beta_{23}\beta_{24} = \beta_{24}\beta_{23} \,.$$

  and $*$-structure $\beta_{24}^* = q^{-1}\beta_{13}$, $\beta_{23}^* = -q^{-1}\beta_{14}$. (Extra gener. β redundant.)

The intersection of the two 'charts' ${}_N\mathbb{R}^4_q$ and ${}_S\mathbb{R}^4_q$

- ${}_S\mathbb{R}^4_q$ with generators $\alpha_{13}, \ \alpha_{14}, \ \alpha_{23}$ and α_{24} (and α)
- ${}_N\mathbb{R}^4_q$ with generators $\beta_{13}, \ \beta_{14}, \ \beta_{23}$ and β_{24} (and β)

The intersection of the two 'charts' ${}_N\mathbb{R}^4_q$ and ${}_S\mathbb{R}^4_q$ is geometrically obtained by removing the 'origin' in each patch. Algebraically:

- ${}_{NS}\mathbb{R}^4_q$ by extending ${}_S\mathbb{R}^4_q$ with an element α^{-1} as inverse of α
- ${}_{SN}\mathbb{R}^4_q$ by extending ${}_N\mathbb{R}^4_q$ with an element β^{-1}, inverse of β

In the overlap of the two patches there are two sets of generators to describe 'points',

$$\exists \ *\text{-algebra isomorphism } \mathcal{Q} : \ {}_{SN}\mathbb{R}^4_q \to {}_{NS}\mathbb{R}^4_q$$

$\mathcal{Q}$ describes how to pass from the coordinates β_{ij} to the coordinates α_{ij}.

Sheaf of Hopf-Galois extensions

Goal: construct a (quantum) Hopf bundle over $\mathcal{A}(S^4_q)$ and its instanton connection

Fact: previous approaches used to construct bundles on quantum spheres (e.g. globally defined instanton projector) cannot be used for $\mathcal{A}(S^4_q)$.

Definition

Let X be a topological space. Let $\mathcal{F}$ be a sheaf of (not necessarily commutative) algebras over X and H a Hopf algebra. We say that $\mathcal{F}$ is a sheaf of H-Hopf-Galois extensions if for each $U \subseteq X$ open set,

(i) *$\mathcal{F}$ is a sheaf of H-comodule algebras*

$$\delta_U : \mathcal{F}(U) \to \mathcal{F}(U) \otimes H \qquad \text{(coaction)}$$

and for each $W \subset U$ the restriction map $\rho_{UW} : \mathcal{F}(U) \to \mathcal{F}(W)$ is a morphism of H-comodule algebras;

(ii) *$\mathcal{F}(U)^{co(H)} \subseteq \mathcal{F}(U)$ is a Hopf-Galois extension.*

Let M be a topological space with an open covering $(U_i)_i$. Let **A** a subcategory of the category of all associative $\mathbb{C}$-algebras.

- [Pflaum (1994)] A quantum principal bundle $(\mathcal{P}, \mathcal{M}, H)$ over M is the data of:
  - a sheaf $\mathcal{M}$ over M with objects in **A** (the base quantum space),
  - a sheaf $\mathcal{P}$ over M with objects in **A** (the total quantum space),
  - a Hopf algebra H called the structure quantum group,
  - a family of sheaf morphisms $\Omega_i : \mathcal{M}(U_i)\#_i H \to \mathcal{P}(U_i)$ (local trivializations) + conditions for $\Omega_{ji} := \Omega_j^{-1}\Omega_i$ on $U \subset U_i \cap U_j$.

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Proposition

A sufficient condition for a quantum principal bundle $\mathcal{P}$ to be a sheaf of Hopf-Galois extensions (in fact, locally cleft) is that $\mathcal{P}$ is a flabby sheaf. In the opposite direction, every locally cleft sheaf of Hopf-Galois extensions is a quantum principal bundle.

The base quantum space $\mathcal{A}(S^4_q)$

Base of the topology: $U_N := S^4 \setminus \{NP\}$, $U_S := S^4 \setminus \{SP\}$ and their intersection U_{SN}

The quantum space of the noncommutative 4-sphere S^4_q is the sheaf of noncommutative $*$-algebras $\mathcal{O}_{S^4_q}$ over the classical 4-sphere S^4 defined by the assignment

$$\mathcal{O}_{S^4_q}(U_N) := {}_N\mathbb{R}^4_q\,, \qquad \mathcal{O}_{S^4_q}(U_S) := {}_S\mathbb{R}^4_q\,, \qquad \mathcal{O}_{S^4_q}(U_{SN}) := \widetilde{{}_{SN}\mathbb{R}^4_q}$$

together with restriction maps

$$\begin{aligned} \rho_{N,SN} &: {}_N\mathbb{R}^4_q \to \widetilde{{}_{SN}\mathbb{R}^4_q}\,, \qquad \beta_{ij} \mapsto \beta_{ij} \\ \rho_{S,SN} &: {}_S\mathbb{R}^4_q \to \widetilde{{}_{SN}\mathbb{R}^4_q}\,, \qquad \alpha_{ij} \mapsto \widetilde{\mathcal{Q}}(\alpha_{ij}) \end{aligned}$$

(Here $\widetilde{{}_{SN}\mathbb{R}^4_q}$ the $*$-algebra generated by elements $\{\beta_{13}, \beta_{14}, \beta_{23}, \beta_{24}, r^{-1}\}$, for $\beta^{-1} = r^{-2}$).

$$\begin{aligned} \mathcal{O}_{S^4_q}(S^4) &= \mathcal{O}_{S^4_q}(U_N \cup U_S) \\ &= \{(a_N, a_S, a_{SN}) \in \mathcal{O}_{S^4_q}(U_N) \oplus \mathcal{O}_{S^4_q}(U_S) \oplus \mathcal{O}_{S^4_q}(U_{SN}) \mid \rho_{N,SN}(a_N) = \rho_{S,SN}(a_S) = a_{SN}\} \\ &\simeq \{(a_N, a_S) \in \mathcal{O}_{S^4_q}(U_N) \oplus \mathcal{O}_{S^4_q}(U_S) \mid \rho_{N,SN}(a_N) = \rho_{S,SN}(a_S)\}. \end{aligned}$$

Aim: (re)construct the sheaf $\mathcal{P}$ of the total space from transition functions

$$\tau_{ji} : SU(2) \to \mathcal{M}(U_i \cap U_j)$$

Strategy: recognize an $SU(2)$ inside the intersection of the two patches

$\widetilde{{}_{SN}\mathbb{R}^4_q}$ can be factorized into a product of a 3-sphere $S^3 \simeq SU(2)$ and a 1-dim interval:

- Let $\mathcal{A}$ be the $*$-subalgebra of $\widetilde{{}_{SN}\mathbb{R}^4_q}$ generated by the elements

$$x_{23} := \beta_{23} r^{-1}, \quad x_{24} := \beta_{24} r^{-1}, \quad x_{23}^* = r^{-1}\beta_{23}^*, \quad x_{24}^* = r^{-1}\beta_{24}^*.$$

  Then the sphere relation $x_{23}^* x_{23} + x_{24}^* x_{24} = 1$ holds and $\mathcal{A}$ is commutative. It can be endowed with a Hopf algebra structure s.t.

$$\mathcal{A} \simeq \mathcal{A}(SU(2))$$

- Let $\mathcal{I}$ be the $*$-algebra generated by $\{r, r^{-1}\}$ satisfying the relation $rr^{-1} = r^{-1}r = 1$

$\longrightarrow$ Both $\mathcal{A}$ and $\mathcal{I}$ are commutative, the noncommutativity emerges from their tensor product.

We denote by $\mathcal{A} \otimes_\Psi \mathcal{I}$ the twisted tensor product algebra consisting of the vector space $\mathcal{A} \otimes \mathcal{I}$ endowed with the multiplication

$$m_\theta := (m_\mathcal{A} \otimes m_\mathcal{I})(\mathrm{id}_\mathcal{A} \otimes \Psi \otimes \mathrm{id}_\mathcal{I})$$

where $\Psi : \mathcal{I} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{I}$ is the linear map defined on the vector space base elements by

$$\Psi\left(r^{\pm n} \otimes x_{23}^a\ (x_{23}^*)^b\ x_{24}^c\ (x_{24}^*)^d\right) := q^{\pm n(a+c) \mp n(b+d)}\ x_{23}^a\ (x_{23}^*)^b\ x_{24}^c\ (x_{24}^*)^d \otimes r^{\pm n}$$

for all integers $n, a, b, c, d \in \mathbb{N} \cup \{0\}$.

Proposition

∃ a ∗-algebra isomorphism:

$$f_{SN} : \widetilde{{}_{SN}\mathbb{R}_q^4} \xrightarrow{\simeq} \mathcal{A} \otimes_\Psi \mathcal{I}$$

The (sheaf of the) total space $\mathcal{P}$ of the quantum Hopf bundle

We introduce 'transition functions' $\mathcal{A} \to \mathcal{O}_{S^4_q}(U_N \cap U_S) \simeq \mathcal{A} \otimes_\Psi \mathcal{I}$, for $\mathcal{A} \simeq SU(2)$

$$\tau_{NN} : \mathcal{A} \to \mathcal{A} \otimes_\Psi \mathcal{I}\,, \quad h \mapsto \varepsilon(h) 1 \otimes 1\,, \qquad \tau_{SS} : \mathcal{A} \to \mathcal{A} \otimes_\Psi \mathcal{I}\,, \quad h \mapsto \varepsilon(h) 1 \otimes 1\,,$$

$$\tau_{NS} : \mathcal{A} \to \mathcal{A} \otimes_\Psi \mathcal{I}\,, \quad h \mapsto h \otimes 1\,, \qquad \tau_{SN} : \mathcal{A} \to \mathcal{A} \otimes_\Psi \mathcal{I} \quad h \mapsto S(h) \otimes 1$$

Then we can construct the quantum total space out of the transition functions:

$$\mathcal{P}(U_N) := \Big\{ (b^N, b^{SN}) \in \Big(\mathcal{O}_{S^4_q}(U_N) \otimes \mathcal{A} \Big) \oplus \Big(\mathcal{O}_{S^4_q}(U_S \cap U_N) \otimes \mathcal{A} \Big)$$
$$s.t.\ (\rho_{N,SN} \otimes id)(b^N) = (m \otimes id)(id \otimes f_{SN}^{-1} \circ \tau_{NS} \otimes id)(id \otimes \Delta)(b^{SN}) \Big\}$$

and similarly $\mathcal{P}(U_S)$ while on the intersection $\mathcal{P}$ is simply given by

$$\mathcal{P}(U_S \cap U_N) := \mathcal{O}_{S^4_q}(U_S \cap U_N) \otimes \mathcal{A}\,.$$

The (sheaf of the) total space $\mathcal{P}$ of the quantum Hopf bundle

Finally

$$\mathcal{P}(S^4) := \Big\{(b^N, b^S) \in \Big(\mathcal{O}_{S^4_q}(U_N) \otimes \mathcal{A}\Big) \oplus \Big(\mathcal{O}_{S^4_q}(U_S) \otimes \mathcal{A}\Big)$$
$$s.t.\ (\rho_{N,SN} \otimes id)(b^N) = (m \otimes id)(id \otimes f_{SN}^{-1} \circ \tau_{NS} \otimes id)(id \otimes \Delta)(\rho_{S,SN} \otimes id)(b^S)\Big\}\ .$$

- $\mathcal{P}$ is a sheaf of $\mathcal{A}$-comodule algebras:

the regular corepresentation Δ of $\mathcal{A} = \mathcal{A}(SU(2))$ on itself induces right coactions

$$\delta = (id \otimes \Delta)$$

of $\mathcal{A}(SU(2))$ on the two summands ${}_N\mathbb{R}^4_q \otimes \mathcal{A}$ and ${}_S\mathbb{R}^4_q \otimes \mathcal{A}$ and then on $\mathcal{P}(S^4)$

Theorem

The subalgebra of coinvariants $\mathcal{B} := (\mathcal{P}(S^4))^{co(SU(2))}$ is

$$\mathcal{B} = \{(x \otimes 1, y \otimes 1) \in \mathcal{P}(S^4)\ /\ \rho_{N,SN}(x) = \rho_{S,SN}(y)\} = \mathcal{O}_{S^4_q}(S^4)\,.$$

The extension $\mathcal{B} \subset \mathcal{P}(S^4)$ is Hopf-Galois.

The instanton connection

- NC differential calculus of S^4_q: a sheaf of noncommutative algebras $\Omega^\bullet_{S^4_q}$ over the classical 4-sphere S^4.
- Hodge operator: the sheaf of anti-selfdual 2-forms $\Omega^{2,-}_q$
- On the local patch ${}_N\mathbb{R}^4_q$: extend ${}_N\Omega^\bullet_q$ by a generator t and its differential dt and quotient by the relation

$$t(1+r^2) = 1 = (1+r^2)t \; .$$

- gauge potential

$$\mathsf{A} := \frac{1}{2} t \begin{pmatrix} \eta_1 & \eta_2 \\ -\eta_2^* & \eta_1^* \end{pmatrix} \in su(2) \otimes {}_N\Omega^1_q$$

Theorem

The curvature $F_\mathsf{A} = \mathrm{d}\mathsf{A} + \mathsf{A} \wedge_q \mathsf{A}$ of the $SU(2)$-potential A has the expression

$$F_\mathsf{A} = t^2 \begin{pmatrix} \mathrm{d}\beta_{23}^*\mathrm{d}\beta_{23} + q^2\mathrm{d}\beta_{24}\mathrm{d}\beta_{24}^* & 2\mathrm{d}\beta_{23}^*\mathrm{d}\beta_{24} \\ -2\mathrm{d}\beta_{24}^*\mathrm{d}\beta_{23} & -\mathrm{d}\beta_{23}^*\mathrm{d}\beta_{23} - q^2\mathrm{d}\beta_{24}\mathrm{d}\beta_{24}^* \end{pmatrix}$$

and it is anti-selfdual, $\star_q F_\mathsf{A} = -F_\mathsf{A}$.