Higher symmetries of the Laplace and Dirac operators

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Definition of higher symmetries

Let E and F be two vector bundles over the smooth manifold M.

Definition

Let $D \in \mathcal{D}(M; E, F)$ be a differential operator. A diff. op. $A \in \mathcal{D}(M, E)$ is

- *a higher symmetry (HS) of D if: $D \circ A = B \circ D$,*
- *a trivial HS if: $A = A_0 \circ D$.*

- HS preserve the kernel of D,
- the space of HS is a subalgebra of $\mathcal{D}(M, E)$, i.e., an associative and non-commutative filtered algebra.

Determine the algebra structure of $\{\text{HS}\}/\{\text{trivial HS}\}$,
for $D = \Delta$, $\not{D}$, $\Delta \oplus \not{D}$.

HS of Laplacian and conformal geometry

On $\mathbb{R}^n$ (n=p+q), we consider $\Delta = \partial_1^2 + \cdots + \partial_p^2 - \partial_{p+1}^2 - \cdots \partial_{p+q}^2$.

- First order HS of Δ (up to constants) are given by

$$\Delta \circ (X + \tfrac{n-2}{2n}\partial_i X^i) = (X + \tfrac{n+2}{2n}\partial_i X^i) \circ \Delta,$$

  where $L_X \mathrm{g} \in [\mathrm{g}]$.
- The Lie algebra of conformal Killing vector fields is $\mathfrak{g}_{\mathbb{R}} = \mathfrak{o}(p+1, q+1)$.
- For well-chosen density bundles, $\mathcal{E}[\lambda] := \Gamma(\mathrm{Vol}^\lambda)$, Δ is a $\mathfrak{g}_{\mathbb{R}}$-invariant operator

$$\Delta : \mathcal{E}[\tfrac{n-2}{2n}] \to \mathcal{E}[\tfrac{n+2}{2n}].$$

If the metric g is conformally flat

$$\text{HS of } \Delta \text{ on } \mathbb{R}^n \quad \longleftrightarrow \quad \text{HS of } \Delta_Y := \nabla_i \mathrm{g}^{ij} \nabla_j - \tfrac{n-2}{4(n-1)}\mathrm{R} \text{ on } (M, \mathrm{g})$$

HS of Laplacian: results

Let (M, g) be conformally flat, and $\mathfrak{g} = \mathfrak{o}(n+2, \mathbb{C})$.

Theorem

The algebra of HS of Δ_Y is isomorphic to $\mathfrak{U}(\mathfrak{g})/\mathcal{J}$, with $\mathcal{J}$ the Joseph ideal [Eastwood, '02].
The algebra $\mathfrak{U}(\mathfrak{g})/\mathcal{J}$ is the unique invariant star-deformation of $\mathbb{C}[\mathcal{O}_{min}]$ [Arnal et al '94; Astashkevich-Brylinski '01].

- $\mathfrak{U}(\mathfrak{g})/\mathcal{J}$ is an algebra of symmetries in higher spin field theories [Vasiliev et al.];
- On $M = \mathbb{S}^p \times \mathbb{S}^q$, $\ker \Delta_Y$ is the minimal representation of $\mathrm{O}(p+1, q+1)$ [Kobayashi-Orsted, '01];
- Second order symmetries allow to classify separating coordinates for the Laplace equation $\Delta f = 0$ [Boyer-Kalnins-Miller, '76].

Ingredients of proof [M., '11]

Conformally equivariant quantization [Duval-Lecomte-Ovsienko, '99]:

- there exists a unique linear isomorphism

$$\mathcal{Q}_{\lambda,\mu} : \mathrm{Pol}_{\mu-\lambda}(T^*M) \to \mathcal{D}_{\lambda,\mu}(M),$$

which is $\mathfrak{g}$-equivariant and preserves principal symbol (for generic $\lambda, \mu \in \mathbb{R}$).

Classification of conformally invariant operators:

- there exists a unique $\mathfrak{g}$-invariant operator

$$\mathbf{G}_0 : \mathrm{Pol}(T^*M) \to \mathrm{Pol}^{\frac{2}{n}}(T^*M)/(\sigma(\Delta)).$$

- $\mathbf{G}_0 K = \{\sigma(\Delta), K\} \mod (\sigma(\Delta))$, and $\ker \mathbf{G}_0 =$ sym. of null geodesic flow.
- $\{$Traceless conf. Killing k-tensors$\}$ = irrep of $\mathfrak{g}$, included in $\mathrm{S}^k(\mathfrak{g})$.

Symplectic reduction:

- $T^*(\mathbb{S}^p \times \mathbb{S}^q)//\langle\sigma(\Delta)\rangle \cong \mathcal{O}_{\min}$,
- $\ker \mathbf{G}_0/(\sigma(\Delta)) \cong \mathbb{C}[\mathcal{O}_{\min}]$

Higher symmetries of Dirac operator

Let S be the spinor bundle over the pseudo-Euclidean space $(\mathbb{R}^n, \mathrm{g})$. The Dirac operator is $\mathfrak{g}$-invariant,

$$\not{D} : \mathcal{S}[\tfrac{n-1}{2n}] \longrightarrow \mathcal{S}[\tfrac{n+1}{2n}].$$

We have $\mathcal{D}(\mathbb{R}^n, S) \cong \mathcal{D}(\mathbb{R}^n) \otimes \mathbb{C}\mathrm{l}(n)$, with symbol space $\mathrm{Pol}(T^*\mathbb{R}^n \times \Pi\mathbb{R}^n)$.

Conformally invariant operator:

- $\mathbf{G}_0 K = \{\sigma(\not{D}), K\} \mod (\sigma(\not{D}))$,
- $\ker \mathbf{G}_0 \supset \{\text{conf. Killing } \kappa\text{-forms}\} \cong \wedge^\kappa \mathbb{C}^{n+2}$ give first order HS.

Theorem ([M., '12])

For all $w \in \mathbb{R}$, there exists a unique conformally equivariant quantization

$$\mathcal{Q}_w : \mathrm{Pol}(T^*\mathbb{R}^n \times \Pi\mathbb{R}^n) \to \mathcal{D}(\mathbb{R}^n, S[w]).$$

The map $\mathcal{Q}_{\frac{n-1}{2n}}$ induces a bijection between $\ker \mathbf{G}_0$ and HS of the Dirac operator.

HS of the system Laplace + Dirac operator

Determine the algebra of HS of the system of differential operators

$$\begin{array}{ccc} \mathcal{E}[\frac{n-2}{2n}] \oplus \mathcal{S}[\frac{n-1}{2n}] & \to & \mathcal{S}[\frac{n+1}{2n}] \oplus \mathcal{E}[\frac{n+2}{2n}] \\ \begin{pmatrix} f \\ \phi \end{pmatrix} & \mapsto & \begin{pmatrix} \Delta f \\ \not{D}\phi \end{pmatrix} \end{array}$$

The HS read as

$$\begin{pmatrix} \Delta & 0 \\ 0 & \not{D} \end{pmatrix} \begin{pmatrix} a & \alpha^- \\ \alpha^+ & A \end{pmatrix} = \begin{pmatrix} b & \beta^+ \\ \beta^- & B \end{pmatrix} \begin{pmatrix} \Delta & 0 \\ 0 & \not{D} \end{pmatrix}$$

with new symmetries:

$$\Delta \alpha^- = \beta^+ \not{D} \quad \text{on } \mathcal{S}[\tfrac{n-1}{2n}] \qquad \text{and} \qquad \not{D} \alpha^+ = \beta^- \Delta \quad \text{on } \mathcal{E}[\tfrac{n-2}{2n}]$$

Let $\varepsilon \in S^* \otimes S^*$ be the invariant pairing on S.

Examples: if $\Lambda \in \mathcal{S}[\frac{1}{2}]$ is a twistor spinor, $\nabla_i \Lambda = -\frac{1}{n}\gamma_i(\not{D}\Lambda)$, the following are symmetries [Wess-Zumino, '74]:

$$\Delta \alpha_\Lambda^- = \beta_\Lambda^+ \not{D}, \quad \begin{cases} \alpha_\Lambda^-(\phi) = \varepsilon(\Lambda, \phi), \\ \beta_\Lambda^+(\not{D}\phi) = \varepsilon(\Lambda, \not{D}^2\phi) + \frac{2}{n}\varepsilon(\not{D}\Lambda, \not{D}\phi), \end{cases} \quad \phi \in \mathcal{S}[-\tfrac{n-1}{2}];$$

$$\not{D}\alpha_\Lambda^+ = \beta_\Lambda^- \Delta, \quad \begin{cases} \alpha_\Lambda^+(f) = \gamma^i(\Lambda)\partial_i f + \frac{n-2}{n}(\not{D}\Lambda)\cdot f, \\ \beta_\Lambda^-(\Delta f) = \Lambda \cdot \Delta f, \end{cases} \quad f \in \mathcal{E}[-\tfrac{n-2}{2}].$$

Proposition ([M.-Šilhan, '14])

The matrix of operators $\begin{pmatrix} a & \alpha^- \\ \alpha^+ & A \end{pmatrix}$ *is a HS iff*

- *a is a HS of Δ and A is a HS of $\not{D}$,*
- *$\alpha^- = \sum_i a_i \circ \alpha_{\Lambda_i}^-$, with a_i HS of Δ and $\alpha_{\Lambda_i}^-$ as above,*
- *$\alpha^+ = \sum_i \alpha_{\Lambda_i}^+ \circ a_i$, with a_i HS of Δ and $\alpha_{\Lambda_i}^+$ as above.*

Composition of twistor spinors actions

Lie (super-)algebra ?

Candidate: Conf. Killing vector fields $\oplus$ Twistor-spinors.

Hint from Rep. Theory:

- in odd dimension, $\mathrm{TwSp} \otimes \mathrm{TwSp} \cong \wedge^+ \mathbb{C}^{n+2} \cong$ space of conf. Killing odd forms;
- in even dimension, $\mathrm{TwSp} \otimes \mathrm{TwSp} \cong \wedge \mathbb{C}^{n+2} \cong$ space of all conf. Killing forms, whereas $\mathrm{TwSp}^+ \otimes \mathrm{TwSp}^- \cong \wedge^+ \mathbb{C}^{n+2} \cong$ space of conf. Killing odd forms;

Fact: the composition $\alpha^+_{\Lambda'} \circ \alpha^-_{\Lambda}$ gives indeed rise to all HS of 1st order of $\not{D}$.

$\Rightarrow$ Algebra of HS is not generated by a Lie (super-)algebra!

Idea: If dimension=3 (or 4 and $\not{D} : S^+ \to S^-$), then the 1st order symmetries of $\not{D}$ are all given by the Lie algebra $\mathfrak{o}(n+2, \mathbb{C}) \oplus \mathbb{C}$ and we have

$$\mathfrak{o}(5) \cong \mathfrak{sp}(4) \qquad \text{and} \qquad \mathfrak{o}(6) \cong \mathfrak{sl}(4).$$

Supergeometric reformulation

$\Pi S^* \cong \mathbb{R}^{3|2}$ is a supermanifold with sheaf of functions

$$\mathcal{O}(\Pi S^*) = \mathcal{E} \oplus \mathcal{S} \oplus \wedge^2 \mathcal{S}.$$

The pairing ε is a distinguished element of $\wedge^2 \mathcal{S} \cong \mathcal{E}$. We define

$$\boxslash : \mathcal{O}\Big(\Pi S^*[-\tfrac{1}{2}]\Big)[-\tfrac{n-2}{2}] \to \mathcal{O}\Big(\Pi S^*[-\tfrac{1}{2}]\Big)[-\tfrac{n}{2}]$$

by the formula $\boxslash := \varepsilon\Delta + \not{D} + \varepsilon^* = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \not{D} & 0 \\ \Delta & 0 & 0 \end{pmatrix}$.

Proposition ([M.-Šilhan, '14])

The algebra of HS of $\boxslash$ is isomorphic to the one of $\begin{pmatrix} \Delta & 0 \\ 0 & \not{D} \end{pmatrix}$.

Twistor-spinors as odd vector fields

Let (x^i, θ^a) be coordinates on $\mathbb{R}^{3|2}$ and $(\partial_i, \partial_{\theta^a})$ the corresponding derivatives.

We define on $\mathcal{O}\Big(\Pi S^*[-\frac{1}{2}]\Big)[w]$

- $L_X = X^i\partial_i - \frac{1}{2}\gamma(\boldsymbol{d}X^\flat)_a^b\theta^a\partial_{\theta^b} - (\frac{w}{n} - \frac{1}{2n}\theta^a\partial_{\theta^a})(\partial_i X^i)$,
- $L^+_\Lambda = \gamma^{i\,a}_{\;b}\Lambda_a\theta^b\partial_i - 2(\frac{w}{n} + \frac{1}{n}\theta^a\partial_{\theta^a})(\not{D}\Lambda)$,
- $L^-_\Lambda = \varepsilon^{ab}\Lambda_a\partial_{\theta^b}$.

We have

$$\boxslash L_X = L_X\boxslash, \qquad \boxslash L^+_\Lambda = L^-_\Lambda\boxslash, \qquad \boxslash L^-_\Lambda = L^+_\Lambda\boxslash\,.$$

Proposition

The space $\langle c, L_X\rangle \oplus \langle L^+_\Lambda, L^-_\Lambda\rangle$ is stable under the commutator in $\mathcal{D}_w(\mathbb{R}^{3|2})$ and isomorphic to the Lie superalgebra $\mathrm{spo}(4|2)$.

Main result

Theorem ([M.-Šilhan, '14])

The algebra of HS of $\boxslash$ *and* $\begin{pmatrix} \Delta & 0 \\ 0 & \not{D} \end{pmatrix}$ *is isomorphic to* $\mathfrak{U}(\mathfrak{spo}(4|2))/\mathcal{J}$, *with* $\mathcal{J}$ *the Joseph-like ideal* [Coulembier-Somberg-Souček, '13].