On Some Constructions of Poisson Structures Related to Lie Bialgebras

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Aim of Talk

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- To explain some general constructions of Poisson structures related to Poisson Lie groups and Lie bialgebras;
- To look at one concrete example.

Outline

1 Review on Lie bialgebras and Poisson Lie groups

2 Three constructions of Poisson structures

3 A Class of Examples

4 A Specific Example

Definitions:

- A **Lie bialgebra** is a pair $(\mathfrak{g}, \delta_{\mathfrak{g}})$, where $\mathfrak{g}$ is a Lie algebra and $\delta_{\mathfrak{g}} : \mathfrak{g} \to \wedge^2 \mathfrak{g}$ is a linear map, such that
  1) $\delta_{\mathfrak{g}}^* : \wedge^2 \mathfrak{g}^* \to \mathfrak{g}^*$ is a Lie bracket on $\mathfrak{g}^*$, and
  2) $\delta_{\mathfrak{g}}$ is a 1-cocycle:
  $$\delta_{\mathfrak{g}}[x, y] = [\delta_{\mathfrak{g}}(x),\ y] + [x,\ \delta_{\mathfrak{g}}(y)], \qquad x, y \in \mathfrak{g}.$$

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- a **Poisson action of a Lie bialgebra** $(\mathfrak{g}, \delta_{\mathfrak{g}})$ on a Poisson manifold (X, π_X) is a Lie algebra homomorphism

$$\sigma : \quad \mathfrak{g} \longrightarrow \mathcal{V}^1(X) \qquad \text{(Vector fields on } X),$$

such that

$$L_{\sigma(x)}\pi_X = (\sigma \wedge \sigma)(\delta_{\mathfrak{g}}(x)), \qquad x \in \mathfrak{g}.$$

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Example. $(\mathfrak{g}, \delta_{\mathfrak{g}} = 0)$ for any Lie algebra $\mathfrak{g}$.

Facts:

- Every Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$ has a **dual Lie bialgebra** $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$, where $\delta_{\mathfrak{g}^*} : \mathfrak{g}^* \to \wedge^2 \mathfrak{g}^*$ is dual to the Lie bracket on $\mathfrak{g}$;

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- Lie bialgebras are in one-to-one correspondence with **Manin triples** via the constructions of their **double Lie algebras**:

$$(\mathfrak{g}\, \delta_{\mathfrak{g}}) \quad \Longrightarrow \quad ((\mathfrak{d}, \langle\ ,\ \rangle_{\mathfrak{d}}), \mathfrak{g}, \mathfrak{g}^*),$$

  where $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ as a vector space, $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lie subalgebras, both isotropic w.r.t. the bilinear form

$$\langle x + \xi,\ y + \eta\rangle_{\mathfrak{d}} = \langle x,\ \eta\rangle + \langle y,\ \xi\rangle, \quad x, y \in \mathfrak{g},\ \xi, \eta \in \mathfrak{g}^*.$$

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$$m: \ (G \times G,\ \pi_G \times \pi_G) \longrightarrow (G, \pi_G), \ \ (g_1, g_2) \longmapsto g_1 g_2$$

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Remark. Poisson groups are semi-classical limits of quantum groups.

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- Have correspondences

$$\begin{aligned} \text{Poisson Lie groups} &\iff \text{Lie bialgebras;} \\ \text{Poisson actions of } (G, \pi_G) &\iff \text{Poisson actions of } (\mathfrak{g}, \delta_{\mathfrak{g}}). \end{aligned}$$

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- A Poisson Lie group (G, π_G) has **dual Poisson Lie groups** and **(Drinfeld) double Poisson Lie groups**

Complex semi-simple Lie bialgebras.

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra, $\mathfrak{t} \subset \mathfrak{g}$ a Cartan subalgebra and $\mathfrak{b} \supset \mathfrak{t}$ a Borel subalgebra. Let $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{t} + \mathfrak{n}$ be the triangular decomposition.

- The **standard** Lie bialgebra structure $(\mathfrak{g}, \delta_{\rm st})$ determined by the standard Manin triple: $\Big((\mathfrak{g} \oplus \mathfrak{g}, \langle\, , \,\rangle_{\mathfrak{g}\oplus\mathfrak{g}}), \ \mathfrak{g}_{\rm diag}, \ \mathfrak{g}^*_{\rm st})\Big)$, where $\langle (x_1, x_2), (y_1, y_2)\rangle_{\mathfrak{g}\oplus\mathfrak{g}} = \langle x_1, y_1\rangle_{\mathfrak{g}} - \langle x_2, y_2\rangle_{\mathfrak{g}}$, with $\langle\, , \,\rangle_{\mathfrak{g}}$ the Killing form of $\mathfrak{g}$, $\mathfrak{g}_{\rm diag} = \{(x, x) : x \in \mathfrak{g}\}$, and

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- Other (quasi-triangular) Lie bialgebra structures on $\mathfrak{g}$ classified by Belavin-Drinfeld.

Real semi-simple Lie bialgebras.

Fix a complex semisimple Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}_0$ be a real form of $\mathfrak{g}$, e.g., $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{g}_0 = \mathfrak{sl}(n, \mathbb{R})$ or $\mathfrak{g}_0 = \mathfrak{su}(p, q)$. Let G_0 be the adjoint group of $\mathfrak{g}_0$. Let $\mathcal{B}$ be the variety of all Borel subalgebras of $\mathfrak{g}$.

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- Their dual Lie bialgebras are solvable; Their double Lie algebras are $\mathfrak{g}$.

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Construction I: using actions of the double:

Let $(\mathfrak{d}, \langle \, , \, \rangle_{\mathfrak{d}})$ be the double Lie algebra of a Lie bialgebra $(\mathfrak{g}, \delta_{\mathfrak{g}})$ and its dual $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$. Let $\{x_i\}$ be a basis of $\mathfrak{g}$ and $\{\xi_i\}$ the dual basis of $\mathfrak{g}^*$. Let

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Lemma (Yakimov-L. Li-Bland-Meinrenken)

If $\sigma : \mathfrak{d} \to \mathcal{V}^1(Y)$ is a Lie algebra action of $\mathfrak{d}$ on Y s.t. the stabilizer subalgebra of every $y \in Y$ is coisotropic w. r. t. $\langle \, , \, \rangle_{\mathfrak{d}}$, then

$$\pi_Y \stackrel{\text{def}}{=} \sigma(\Lambda) = \frac{1}{2} \sum_i \sigma(\xi_i) \wedge \sigma(x_i)$$

is a Poisson structure on Y, and σ restricts to Poisson actions of $(\mathfrak{g}, \delta_{\mathfrak{g}})$ and $(\mathfrak{g}^*, \delta_{\mathfrak{g}^*})$ on (Y, π_Y).

Constructing Poisson structures II: taking quotients

Lemma

Let (G, π_G) be a Poisson Lie group and $H \subset G$ a closed **coisotropic subgroup**. *If*

$$\sigma : \ (Y, \pi_Y) \times (G, \pi_G) \longrightarrow (Y, \pi_Y)$$

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Example. If (G, π_G) is a Poisson Lie group and $H \subset G$ a closed **Poisson Lie subgroup**, then π_G projects to a well-defined Poisson structure on G/H.

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Constructing Poisson structures III: mixed products:

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Locally, for $X \times Y$,

$$\pi_{X\times Y} = \sum_{i<j} \alpha_{ij}(x) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} + \sum_{k<l} \beta_{kl}(y) \frac{\partial}{\partial y_k} \wedge \frac{\partial}{\partial y_l} + \sum_{i,k} \gamma_{ik}(x,y) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_k}.$$

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$$\pi_X \times_{(\rho,\lambda)} \pi_Y \stackrel{\text{def}}{=} (\pi_X, 0) + (0, \pi_Y) + \sum_i (\rho(\xi_i), 0) \wedge (0, \lambda(x_i))$$

is a mixed product Poisson structure on $X \times Y$.

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- Symmetric spaces G/K and their wonderful compatifications;
- Real symmetric spaces G/G_0, G_0/K_0, U/K_0.

Examples of Poisson Spaces with Poisson Actions—Cont'd

Common feature of these Poisson structures:

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- Find log-canonical coordinates on each T-leaf;
- Cluster structure on each T-leaf?

One Specific Example: Products of Flag Varieties

Most basic example: the flag variety G/B with the quotient Poisson structure $\pi_{G/B}$ of π_{st} on G. In general, define

$$F_n = \overbrace{G \times_B \cdots \times_B G/B}^{n},$$

where F_n is the quotient manifold of G^n under the action of B^n

$$(g_1, ..., g_n) \cdot (b_1, ..., b_n) = (g_1 b_1, b_1^{-1} g_1 b_2, ..., b_{n-1}^{-1} g_n b_n).$$

Facts

- *The product Poisson structure* $(\pi_{\mathrm{st}})^n$ *on* G^n *projects to a well-defined Poisson structure* π_n *on* F_n*.*
- *Have diffeomorphism* $I_n: F_n \to (G/B)^n$*:*

$$[g_1, g_2, \ldots, g_n] \longmapsto (g_1 B,\ g_1 g_2 B,\ \ldots,\ g_1 g_2 \cdots g_n B).$$

A Class of Examples

In general, let (G, π_G) be a Poisson Lie group and $K_1, K_2, \ldots, K_n$ closed Poisson Lie subgroups. Let

$$Y = G \times_{K_1} G \times_{K_2} \cdots \times_{K_{n-1}} G/K_n,$$

the quotient of G^n by $K_1 \times \cdots \times K_n$ by

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Lemma

The product Poisosn structure $(\pi_G)^n$ on G^n projects to a well-defined Poisson structure π_Y on Y.

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Lemma

The product Poisosn structure $(\pi_G)^n$ on G^n projects to a well-defined Poisson structure π_Y on Y.

Have diffeomorphism $I_Y : Y \to G/K_1 \times \cdots \times G/K_n$:

$$[g_1, g_2, \ldots, g_n] \longmapsto (g_1 K_1, g_1 g_2 K_2, \ldots, g_1 g_2 \cdots g_n K_n),$$

so $I_Y(\pi_Y)$ is a Poisson structure on the product manifold $G/K_1 \times \cdots \times G/K_n$.

A Class of Examples—Cont'd

Theorem (L.-Mouquin)

$I_Y(\pi_Y)$ is both a mixed product Poisson structure on $G/K_1 \times \cdots \times G/K_n$ and can be defined using a Manin triple.

A Class of Examples—Cont'd

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Application to the case of flag manifolds: Recall the diffeomorphism

$$I_n: \quad G \times_B G \times_B \cdots \times_B G/B \longrightarrow (G/B)^n.$$

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to each factor is Poisson. The mixed terms well-undertood;
- Expressing $I_n(\pi_n)$ using a Manin triple reveals the "symmetries" of the Poisson structure π_n and can apply the general theory.

The Poisson manifold (F_n, π_n)

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Define $\mu_n : F_n \to G/B$, $[g_1, g_1, ..., g_n] \mapsto g_1 g_2 \cdots g_n.B$.

Theorem (L.-Mouquin)

1) The T-leaves of (F_n, π_n) are non-empty intersections

$$\mu_n^{-1}(B_- vB/B) \cap (B\mathbf{w}B/B),$$

where $v \in W$, $\mathbf{w} = (w_1, ..., w_n) \in W^n$, and

$$B\mathbf{w}B/B = (Bw_1B) \times_B \cdots \times_B (Bw_nB/B).$$

2) Over this intersection, π_n has corank

$$\dim \ker(1 + w_1 \cdots w_n v^{-1}).$$

The Poisson manifold (F_n, π_n)—Cont'd

Theorem

Have a decomposition

$$F_n = \bigsqcup_{\alpha \in \mathcal{A}} \mathcal{O}_\alpha \qquad (\textit{disjoint union})$$

where $\mathcal{A}$ is a finite set and each $\mathcal{O}_\alpha$ is a Poisson submanifold of (F_n, π_n) with natural isomorphism to $\mathbb{C}^{m_\alpha}$ for some integer m_α.

Call the $\mathcal{O}_\alpha$'s the pieces of (F_n, π_n).

The Poisson manifold (F_n, π_n)—Cont'd

Theorem

Have a decomposition

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Call the $\mathcal{O}_\alpha$'s the pieces of (F_n, π_n).

Can compute π_n in each piece explicitly.

Example. $G = G_2$. A piece in F_{23} is the following Poisson structure on $A = \mathbb{C}[x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9]$.

$$\{x_1, x_2\} = -3x_1x_2$$
$$\{x_1, x_3\} = 2x_2x_3^2 + x_1x_3$$
$$\{x_1, x_4\} = -4x_2x_3x_4 - x_1x_4 - 2x_2$$
$$\{x_1, x_5\} = -6x_3x_4^3 - 6x_2x_3x_5 - 6x_4^2$$
$$\{x_1, x_6\} = 6x_3x_4^2x_6^2 + 2x_2x_3x_6 + 4x_4x_6^2 - x_1x_6$$
$$\{x_1, x_7\} = -12x_3x_4^2x_6x_7 - 2x_2x_3x_7 - 6x_3x_4^2 - 8x_4x_6x_7 + x_1x_7 - 4x_4$$
$$\{x_1, x_8\} = 12x_3x_4^2x_6x_8 + 2x_2x_3x_8 + 8x_4x_6x_8 - x_1x_8$$

The Poisson manifold (F_n, π_n)—Cont'd

$$\begin{aligned}\{x_1, x_9\} = &-6x_3x_5x_6^3x_7^3x_8^3x_9^2 + 18x_3x_4x_6^2x_7^3x_8^3x_9^2 + 18x_3x_5x_6^3x_7^2x_8^2x_9^2\\ &- 18x_3x_5x_6^2x_7^2x_8^3x_9^2 - 54x_3x_4x_6^2x_7^2x_8^2x_9^2 + 36x_3x_4x_6x_7^2x_8^3x_9^2\\ &+ 6x_6^2x_7^3x_8^3x_9^2 - 18x_3x_5x_6^3x_7x_8x_9^2 + 36x_3x_5x_6^2x_7x_8^2x_9^2\\ &- 18x_3x_5x_6x_7x_8^3x_9^2 + 54x_3x_4x_6^2x_7x_8x_9^2 - 72x_3x_4x_6x_7x_8^2x_9^2\\ &+ 18x_3x_4x_7x_8^3x_9^2 - 18x_6^2x_7^2x_8^2x_9^2 + 12x_6x_7^2x_8^3x_9^2\\ &+ 6x_3x_5x_6^3x_9^2 - 18x_3x_5x_6^2x_8x_9^2 + 18x_3x_5x_6x_8^2x_9^2\\ &- 6x_3x_5x_8^3x_9^2 - 18x_3x_4x_6^2x_9^2 + 36x_3x_4x_6x_8x_9^2 - 18x_3x_4x_8^2x_9^2\\ &+ 18x_6^2x_7x_8x_9^2 - 24x_6x_7x_8^2x_9^2 + 6x_7x_8^3x_9^2 - 18x_3x_4^2x_6x_9\\ &- 6x_6^2x_9^2 + 12x_6x_8x_9^2 - 6x_8^2x_9^2 - 6x_2x_3x_9 - 12x_4x_6x_9\end{aligned}$$

The Poisson manifold (F_n, π_n)—Cont'd

$$\{x_2, x_3\} = 3x_2x_3$$
$$\{x_2, x_4\} = -3x_2x_4$$
$$\{x_2, x_5\} = -6x_4^3 - 3x_2x_5$$
$$\{x_2, x_6\} = 6x_4^2x_6^2$$
$$\{x_2, x_7\} = -12x_4^2x_6x_7 - 6x_4^2$$
$$\{x_2, x_8\} = 12x_4^2x_6x_8$$
$$\begin{aligned}\{x_2, x_9\} = {} & -6x_5x_6^3x_7^3x_8^3x_9^2 + 18x_4x_6^2x_7^3x_8^3x_9^2 + 18x_5x_6^3x_7^2x_8^2x_9^2 \\ & - 18x_5x_6^2x_7^2x_8^3x_9^2 - 54x_4x_6^2x_7^2x_8^2x_9^2 + 36x_4x_6x_7^2x_8^3x_9^2 \\ & - 18x_5x_6^3x_7x_8x_9^2 + 36x_5x_6^2x_7x_8^2x_9^2 - 18x_5x_6x_7x_8^3x_9^2 \\ & + 54x_4x_6^2x_7x_8x_9^2 - 72x_4x_6x_7x_8^2x_9^2 + 18x_4x_7x_8^3x_9^2 \\ & + 6x_5x_6^3x_9^2 - 18x_5x_6^2x_8x_9^2 + 18x_5x_6x_8^2x_9^2 - 6x_5x_8^3x_9^2 \\ & - 18x_4x_6^2x_9^2 + 36x_4x_6x_8x_9^2 - 18x_4x_8^2x_9^2 - 18x_4^2x_6x_9 \\ & - 3x_2x_9\end{aligned}$$

The Poisson manifold (F_n, π_n)—Cont'd

$$
\begin{aligned}
\{x_3, x_4\} &= -2x_3x_4 \\
\{x_3, x_5\} &= -3x_3x_5 \\
\{x_3, x_6\} &= x_3x_6 \\
\{x_3, x_7\} &= -x_3x_7 \\
\{x_3, x_8\} &= x_3x_8 \\
\{x_3, x_9\} &= -3x_3x_9 \\
\{x_4, x_5\} &= -3x_4x_5 \\
\{x_4, x_6\} &= 2x_5x_6^2 + x_4x_6 \\
\{x_4, x_7\} &= -4x_5x_6x_7 - x_4x_7 - 2x_5 \\
\{x_4, x_8\} &= 4x_5x_6x_8 + x_4x_8 \\
\{x_4, x_9\} &= 6x_6x_7^3x_8^3x_9^2 - 18x_6x_7^2x_8^2x_9^2 + 6x_7^2x_8^3x_9^2 + 18x_6x_7x_8x_9^2 \\
&\quad - 12x_7x_8^2x_9^2 - 6x_5x_6x_9 - 6x_6x_9^2 + 6x_8x_9^2 - 3x_4x_9
\end{aligned}
$$

The Poisson manifold (F_n, π_n)—Cont'd

$$\{x_5, x_6\} = 3x_5x_6$$
$$\{x_5, x_7\} = -3x_5x_7$$
$$\{x_5, x_8\} = 3x_5x_8$$
$$\{x_5, x_9\} = 6x_7^3x_8^3x_9^2 - 18x_7^2x_8^2x_9^2 + 18x_7x_8x_9^2 - 6x_5x_9 - 6x_9^2$$
$$\{x_6, x_7\} = -2x_6x_7$$
$$\{x_6, x_8\} = 2x_6x_8$$
$$\{x_6, x_9\} = -3x_6x_9$$
$$\{x_7, x_8\} = 2x_7x_8 - 2$$
$$\{x_7, x_9\} = -3x_7x_9$$
$$\{x_8, x_9\} = 3x_8x_9$$

The Poisson manifold (F_n, π_n)—Cont'd

$$
\begin{aligned}
&\{x_1, \{x_5, x_9\}\} \\
&= -72x_3x_5x_6^3x_7^6x_8^6x_9^3 + 216x_3x_4x_6^2x_7^6x_8^6x_9^3 + 432x_3x_5x_6^3x_7^5x_8^5x_9^3 \\
&- 216x_3x_5x_6^2x_7^5x_8^6x_9^3 - 1296x_3x_4x_6^2x_7^5x_8^5x_9^3 + 432x_3x_4x_6x_7^5x_8^6x_9^3 \\
&+ 72x_6^2x_7^6x_8^6x_9^3 - 1080x_3x_5x_6^3x_7^4x_8^4x_9^3 + 1080x_3x_5x_6^2x_7^4x_8^5x_9^3 \\
&- 216x_3x_5x_6x_7^4x_8^6x_9^3 + 3240x_3x_4x_6^2x_7^4x_8^4x_9^3 - 2160x_3x_4x_6x_7^4x_8^5x_9^3 \\
&+ 216x_3x_4x_7^4x_8^6x_9^3 - 432x_6^2x_7^5x_8^5x_9^3 + 144x_6x_7^5x_8^6x_9^3 \\
&+ 36x_3x_5^2x_6^3x_7^3x_8^3x_9^2 + 1440x_3x_5x_6^3x_7^3x_8^3x_9^3 - 2160x_3x_5x_6^2x_7^3x_8^4x_9^3 \\
&+ 864x_3x_5x_6x_7^3x_8^5x_9^3 - 72x_3x_5x_7^3x_8^6x_9^3 - 108x_3x_4x_5x_6^2x_7^3x_8^3x_9^2 \\
&- 4320x_3x_4x_6^2x_7^3x_8^3x_9^3 + 4320x_3x_4x_6x_7^3x_8^4x_9^3 - 864x_3x_4x_7^3x_8^5x_9^3 \\
&+ 1080x_6^2x_7^4x_8^4x_9^3 - 720x_6x_7^4x_8^5x_9^3 + 72x_7^4x_8^6x_9^3 - 216x_3x_4^2x_6x_7^3x_8^3x_9^2 \\
&- 108x_3x_5^2x_6^3x_7^2x_8^2x_9^2 + 108x_3x_5^2x_6^2x_7^2x_8^3x_9^2 - 1080x_3x_5x_6^3x_7^2x_8^2x_9^3
\end{aligned}
$$

The Poisson manifold (F_n, π_n)—Cont'd

$$
\begin{aligned}
&+2160x_3x_5x_6^2x_7^2x_8^3x_9^3 - 1296x_3x_5x_6x_7^2x_8^4x_9^3 + 216x_3x_5x_7^2x_8^5x_9^3\\
&+324x_3x_4x_5x_6^2x_7^2x_8^2x_9^2 - 216x_3x_4x_5x_6x_7^2x_8^3x_9^2 + 3240x_3x_4x_6^2x_7^2x_8^2x_9^3\\
&-4320x_3x_4x_6x_7^2x_8^3x_9^3 + 1296x_3x_4x_7^2x_8^4x_9^3 - 36x_5x_6^2x_7^3x_8^3x_9^2\\
&-1440x_6^2x_7^3x_8^3x_9^3 + 1440x_6x_7^3x_8^4x_9^3 - 288x_7^3x_8^5x_9^3 - 72x_2x_3x_7^3x_8^3x_9^2\\
&+648x_3x_4^2x_6x_7^2x_8^2x_9^2 - 108x_3x_4^2x_7^2x_8^3x_9^2 + 108x_3x_5^2x_6^3x_7x_8x_9^2\\
&-216x_3x_5^2x_6^2x_7x_8^2x_9^2 + 108x_3x_5^2x_6x_7x_8^3x_9^2 + 432x_3x_5x_6^3x_7x_8x_9^3\\
&-1080x_3x_5x_6^2x_7x_8^2x_9^3 + 864x_3x_5x_6x_7x_8^3x_9^3 - 216x_3x_5x_7x_8^4x_9^3\\
&-144x_4x_6x_7^3x_8^3x_9^2 - 324x_3x_4x_5x_6^2x_7x_8x_9^2 + 432x_3x_4x_5x_6x_7x_8^2x_9^2\\
&-108x_3x_4x_5x_7x_8^3x_9^2 - 1296x_3x_4x_6^2x_7x_8x_9^3 + 2160x_3x_4x_6x_7x_8^2x_9^3\\
&-864x_3x_4x_7x_8^3x_9^3 + 108x_5x_6^2x_7^2x_8^2x_9^2 - 72x_5x_6x_7^2x_8^3x_9^2 + 1080x_6^2x_7^2x_8^2x_9^3\\
&-1440x_6x_7^2x_8^3x_9^3 + 432x_7^2x_8^4x_9^3 + 216x_2x_3x_7^2x_8^2x_9^2 - 648x_3x_4^2x_6x_7x_8x_9^2
\end{aligned}
$$

The Poisson manifold (F_n, π_n)—Cont'd

$$
\begin{aligned}
&+ 216x_3x_4^2x_7x_8^2x_9^2 - 36x_3x_5^2x_6^3x_9^2 + 108x_3x_5^2x_6^2x_8x_9^2 - 108x_3x_5^2x_6x_8^2x_9^2 \\
&+ 36x_3x_5^2x_8^3x_9^2 - 72x_3x_5x_6^3x_9^3 + 216x_3x_5x_6^2x_8x_9^3 - 216x_3x_5x_6x_8^2x_9^3 \\
&+ 72x_3x_5x_8^3x_9^3 + 432x_4x_6x_7^2x_8^2x_9^2 - 72x_4x_7^2x_8^3x_9^2 + 108x_3x_4x_5x_6^2x_9^2 \\
&- 216x_3x_4x_5x_6x_8x_9^2 + 108x_3x_4x_5x_8^2x_9^2 + 216x_3x_4x_6^2x_9^3 - 432x_3x_4x_6x_8x_9^3 \\
&+ 216x_3x_4x_8^2x_9^3 - 108x_5x_6^2x_7x_8x_9^2 + 144x_5x_6x_7x_8^2x_9^2 - 36x_5x_7x_8^3x_9^2 \\
&- 432x_6^2x_7x_8x_9^3 + 720x_6x_7x_8^2x_9^3 - 288x_7x_8^3x_9^3 - 216x_2x_3x_7x_8x_9^2 \\
&+ 108x_3x_4^2x_5x_6x_9 + 216x_3x_4^2x_6x_9^2 - 108x_3x_4^2x_8x_9^2 - 432x_4x_6x_7x_8x_9^2 \\
&+ 144x_4x_7x_8^2x_9^2 + 36x_3x_4^3x_9 + 36x_5x_6^2x_9^2 - 72x_5x_6x_8x_9^2 + 36x_5x_8^2x_9^2 \\
&+ 72x_6^2x_9^3 - 144x_6x_8x_9^3 + 72x_8^2x_9^3 + 72x_2x_3x_5x_9 + 72x_2x_3x_9^2 \\
&+ 72x_4x_5x_6x_9 + 144x_4x_6x_9^2 - 72x_4x_8x_9^2 + 36x_4^2x_9
\end{aligned}
$$

The Poisson manifold (F_n, π_n)—Cont'd

$$
\begin{aligned}
&\{x_9, \{x_1, x_5\}\} \\
&= -36x_3x_5^2x_6^3x_7^3x_8^3x_9^2 + 108x_3x_4x_5x_6^2x_7^3x_8^3x_9^2 + 108x_3x_4^2x_6x_7^3x_8^3x_9^2 \\
&+ 108x_3x_5^2x_6^3x_7^2x_8^2x_9^2 - 108x_3x_5^2x_6^2x_7^2x_8^3x_9^2 - 324x_3x_4x_5x_6^2x_7^2x_8^2x_9^2 \\
&+ 216x_3x_4x_5x_6x_7^2x_8^3x_9^2 + 36x_2x_3x_7^3x_8^3x_9^2 - 324x_3x_4^2x_6x_7^2x_8^2x_9^2 \\
&+ 108x_3x_4^2x_7^2x_8^3x_9^2 - 108x_3x_5^2x_6^3x_7x_8x_9^2 + 216x_3x_5^2x_6^2x_7x_8^2x_9^2 \\
&- 108x_3x_5^2x_6x_7x_8^3x_9^2 + 72x_4x_6x_7^3x_8^3x_9^2 + 324x_3x_4x_5x_6^2x_7x_8x_9^2 \\
&- 432x_3x_4x_5x_6x_7x_8^2x_9^2 + 108x_3x_4x_5x_7x_8^3x_9^2 - 108x_2x_3x_7^2x_8^2x_9^2 \\
&+ 324x_3x_4^2x_6x_7x_8x_9^2 - 216x_3x_4^2x_7x_8^2x_9^2 + 36x_3x_5^2x_6^3x_9^2 - 108x_3x_5^2x_6^2x_8x_9^2 \\
&+ 108x_3x_5^2x_6x_8^2x_9^2 - 36x_3x_5^2x_8^3x_9^2 - 216x_4x_6x_7^2x_8^2x_9^2 + 72x_4x_7^2x_8^3x_9^2 \\
&- 108x_3x_4x_5x_6^2x_9^2 + 216x_3x_4x_5x_6x_8x_9^2 - 108x_3x_4x_5x_8^2x_9^2 \\
&+ 108x_2x_3x_7x_8x_9^2 - 216x_3x_4^2x_5x_6x_9 - 108x_3x_4^2x_6x_9^2 + 108x_3x_4^2x_8x_9^2 \\
&+ 216x_4x_6x_7x_8x_9^2 - 144x_4x_7x_8^2x_9^2 - 72x_3x_4^3x_9 - 72x_2x_3x_5x_9 \\
&- 36x_2x_3x_9^2 - 72x_4x_5x_6x_9 - 72x_4x_6x_9^2 + 72x_4x_8x_9^2 - 36x_4^2x_9
\end{aligned}
$$

The Poisson manifold (F_n, π_n)—Cont'd

$$
\begin{aligned}
&\{x_5, \{x_9, x_1\}\} \\
&= 72x_3x_5x_6^3x_7^6x_8^6x_9^3 - 216x_3x_4x_6^2x_7^6x_8^6x_9^3 - 432x_3x_5x_6^3x_7^5x_8^5x_9^3 \\
&+ 216x_3x_5x_6^2x_7^5x_8^6x_9^3 + 1296x_3x_4x_6^2x_7^5x_8^5x_9^3 - 432x_3x_4x_6x_7^5x_8^6x_9^3 \\
&- 72x_6^2x_7^6x_8^6x_9^3 + 1080x_3x_5x_6^3x_7^4x_8^4x_9^3 - 1080x_3x_5x_6^2x_7^4x_8^5x_9^3 \\
&+ 216x_3x_5x_6x_7^4x_8^6x_9^3 - 3240x_3x_4x_6^2x_7^4x_8^4x_9^3 + 2160x_3x_4x_6x_7^4x_8^5x_9^3 \\
&- 216x_3x_4x_7^4x_8^6x_9^3 + 432x_6^2x_7^5x_8^5x_9^3 - 144x_6x_7^5x_8^6x_9^3 \\
&- 1440x_3x_5x_6^3x_7^3x_8^3x_9^3 + 2160x_3x_5x_6^2x_7^3x_8^4x_9^3 - 864x_3x_5x_6x_7^3x_8^5x_9^3 \\
&+ 72x_3x_5x_7^3x_8^6x_9^3 + 4320x_3x_4x_6^2x_7^3x_8^3x_9^3 - 4320x_3x_4x_6x_7^3x_8^4x_9^3 \\
&+ 864x_3x_4x_7^3x_8^5x_9^3 - 1080x_6^2x_7^4x_8^4x_9^3 + 720x_6x_7^4x_8^5x_9^3 - 72x_7^4x_8^6x_9^3 \\
&+ 108x_3x_4^2x_6x_7^3x_8^3x_9^2 + 1080x_3x_5x_6^3x_7^2x_8^2x_9^3 - 2160x_3x_5x_6^2x_7^2x_8^3x_9^3 \\
&+ 1296x_3x_5x_6x_7^2x_8^4x_9^3 - 216x_3x_5x_7^2x_8^5x_9^3 - 3240x_3x_4x_6^2x_7^2x_8^2x_9^3 \\
&+ 4320x_3x_4x_6x_7^2x_8^3x_9^3 - 1296x_3x_4x_7^2x_8^4x_9^3 + 36x_5x_6^2x_7^3x_8^3x_9^2 \\
&+ 1440x_6^2x_7^3x_8^3x_9^3 - 1440x_6x_7^3x_8^4x_9^3 + 288x_7^3x_8^5x_9^3 + 36x_2x_3x_7^3x_8^3x_9^2
\end{aligned}
$$

The Poisson manifold (F_n, π_n)—Cont'd

$$
\begin{aligned}
&- 324x_3x_4^2x_6x_7^2x_8^2x_9^2 - 432x_3x_5x_6^3x_7x_8x_9^3 + 1080x_3x_5x_6^2x_7x_8^2x_9^3 \\
&- 864x_3x_5x_6x_7x_8^3x_9^3 + 216x_3x_5x_7x_8^4x_9^3 + 72x_4x_6x_7^3x_8^3x_9^2 \\
&+ 1296x_3x_4x_6^2x_7x_8x_9^3 - 2160x_3x_4x_6x_7x_8^2x_9^3 + 864x_3x_4x_7x_8^3x_9^3 \\
&- 108x_5x_6^2x_7^2x_8^2x_9^2 + 72x_5x_6x_7^2x_8^3x_9^2 - 1080x_6^2x_7^2x_8^2x_9^3 \\
&+ 1440x_6x_7^2x_8^3x_9^3 - 432x_7^2x_8^4x_9^3 - 108x_2x_3x_7^2x_8^2x_9^2 + 324x_3x_4^2x_6x_7x_8x_9^2 \\
&+ 72x_3x_5x_6^3x_9^3 - 216x_3x_5x_6^2x_8x_9^3 + 216x_3x_5x_6x_8^2x_9^3 - 72x_3x_5x_8^3x_9^3 \\
&- 216x_4x_6x_7^2x_8^2x_9^2 - 216x_3x_4x_6^2x_9^3 + 432x_3x_4x_6x_8x_9^3 - 216x_3x_4x_8^2x_9^3 \\
&+ 108x_5x_6^2x_7x_8x_9^2 - 144x_5x_6x_7x_8^2x_9^2 + 36x_5x_7x_8^3x_9^2 + 432x_6^2x_7x_8x_9^3 \\
&- 720x_6x_7x_8^2x_9^3 + 288x_7x_8^3x_9^3 + 108x_2x_3x_7x_8x_9^2 + 108x_3x_4^2x_5x_6x_9 \\
&- 108x_3x_4^2x_6x_9^2 + 216x_4x_6x_7x_8x_9^2 + 36x_3x_4^3x_9 - 36x_5x_6^2x_9^2 + 72x_5x_6x_8x_9^2 \\
&- 36x_5x_8^2x_9^2 - 72x_6^2x_9^3 + 144x_6x_8x_9^3 - 72x_8^2x_9^3 - 36x_2x_3x_9^2 - 72x_4x_6x_9^2.
\end{aligned}
$$

One has

$$\{x_1, \{x_5, x_9\}\} + \{x_9, \{x_1, x_5\}\} + \{x_5, \{x_9, x_1\}\} = 0.$$