Formal and non-formal deformation quantizations of the complex unit ball in $\mathbb{C}^n$

Stéphane Korvers (UCL, F.R.S.-FNRS Research Fellow)
Joint work with Professor Pierre Bieliavsky (UCL)

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Formal and non-formal deformation quantizations

↙

On a symplectic manifold M :

? $K_\nu(-,-,-)$? explicit 3-point kernel such that the formula

$$(f *_\nu g)(x) = \int_{M\times M} K_\nu(x,y,z)\, f(y)\, g(z)\, dy\, dz$$

↙ (Liouville)

- defines an associative product on an « interesting » space of functions ($\ni f, g$).
- admits an asymptotic expansion : $f *_\nu g = fg + \sum_{k=1}^{+\infty} \nu^k C_k(f,g)$

Let $\mathbb{D}$ be a homogeneous complex bounded domain in $\mathbb{C}^n$.

General problem

Can we determine explicitly all $\mathrm{Aut}\,(\mathbb{D})$-invariant, both formal and non-formal deformation quantizations of $\mathbb{D}$?

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Structural point

(Pyatetskii-Shapiro theory)

- $\exists \tilde{\mathbb{S}} \subset \mathrm{Aut}(\mathbb{D})$ solvable Lie group acting simply transitively on $\mathbb{D}$;
- $\tilde{\mathbb{S}} = (\ldots(\mathbb{S}_N \ltimes \mathbb{S}_{N-1}) \ltimes \ldots \ltimes \mathbb{S}_2) \ltimes \mathbb{S}_1$ where :
  (1) $\mathbb{S}_j$ is the Iwasawa group of $G_j = SU(1, n_j)$,
  (2) $\mathbb{S}_j$ acts simply transitively on the complex unit ball in $\mathbb{C}^{n_j}$.

=> Look at a our problem for $\mathbb{D}_n$ the unit ball in $\mathbb{C}^n$, $n \in \mathbb{N} \setminus \{0\}$

$\mathbb{D}_n \simeq \mathbb{S} =$ Iwasawa group of $G := SU(1, n) = \mathrm{Aut}(\mathbb{D}_n)$

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Explicit resolution

- The resolution is associated with the determination of a $\mathbb{S}$-equivariant convolution operator that intertwines the $\mathbb{S}$-invariant deformation theory (Bieliavsky, Gayral, ...) with the G-invariant one.
- The kernel of this operator is described by a hierarchy of PDE's, but ...

It is not so easy ...

Here is <u>one</u> of the equation for $n > 1$: $\Box_{(a,\vec{v},\xi)}\,\vartheta \;=\; i\,\xi\,e^{-2a}\,\vartheta$ where

$$
\begin{aligned}
\Box_{(a,\vec{v},\xi)} \quad = \quad & \frac{i\,\xi\,e^{2a}}{4}\,\left[\left[\left(1+\sqrt{1-\nu^2\xi^2}\right)(\,\vec{v}\,|\,\vec{v}\,)\,+\,2\,\kappa\right]^2+\,4\,(n+3)\,\nu^2\right]\,\mathrm{Id} \\
& + 4\,i\,\nu^2\,\xi\,e^{2a}\,\partial_a \\
& - 3\,i\,\nu^2\,\xi\,e^{2a}\,\Theta\,+\;e^{2a}\,\left[\left[1+\sqrt{1-\nu^2\xi^2}-\nu^2\xi^2\right](\,\vec{v}\,|\,\vec{v}\,)\,+\,2\,\sqrt{1-\nu^2\xi^2}\;\kappa\right]\,\Xi \\
& - 4\,i\,e^{2a}\,\left[2-3\,\nu^2\xi^2\right]\,\partial_\xi \\
& + i\,\nu^2\,\xi\,e^{2a}\,\partial_a^2 \\
& + \frac{i\,e^{2a}}{2\,\xi}\,\left[\nu^2\xi^2\,(\,\vec{v}\,|\,\vec{v}\,)\,-\,2\left(-1+\sqrt{1-\nu^2\xi^2}\right)\,\kappa\right]\,\Delta \\
& + \frac{i}{\xi}\,e^{2a}\left[2\left(-1+\sqrt{1-\nu^2\xi^2}\right)+\nu^2\xi^2\right]\left(\Theta^2-\Theta\right)\,+\,i\,\nu^2\,\xi\,e^{2a}\left(\Xi^2+\Theta\right) \\
& - 4\,i\,\xi\,e^{2a}\,\left[1-\nu^2\xi^2\right]\,\partial_\xi^2 \\
& - \frac{2\,i}{\xi}\,e^{2a}\,\left[-1+\sqrt{1-\nu^2\xi^2}+\nu^2\xi^2\right]\,\Theta\,\partial_a \\
& - 4\,i\,e^{2a}\,\left[1-\nu^2\xi^2\right]\,\partial_a\,\partial_\xi \\
& - 4\,i\,e^{2a}\,\left[-1+\sqrt{1-\nu^2\xi^2}+\nu^2\xi^2\right]\,\Theta\,\partial_\xi \\
& + \frac{1}{\xi^2}\,e^{2a}\,\left[-1+\sqrt{1-\nu^2\xi^2}+\nu^2\xi^2\right]\,\Xi\,\Delta \\
& - \frac{i}{4\,\xi^3}\,e^{2a}\,\left[2\left(-1+\sqrt{1-\nu^2\xi^2}\right)+\nu^2\xi^2\right]\,\Delta^2
\end{aligned}
$$

- These PDE's were explicitly written and solved
  (1) for $n = 1$: Bieliavsky, Detournay, Spindel (2009)
  (2) for $n > 1$:

Theorem [Bieliavsky - K., 2013]

For each G-invariant deformation theory on $\mathbb{D}_n$, there exists $g \in \mathcal{D}'(\mathbb{R})[[\nu]]$ (with a possible reparameterization of ν), such that the convolution operator with kernel

$$\begin{aligned}\mathcal{V}(a,r,z) &= \int_{-\infty}^{+\infty} d\xi\ \nu^2 \operatorname{sign}(\xi)\, e^{-2a+i\xi z} \int_{-\infty}^{+\infty} d\gamma\ \left(\gamma^2+1\right)^{\frac{n-3}{2}} \\ &\quad g\left(\frac{-4\nu^2 \operatorname{sign}(\xi)\, e^{-2a}}{\gamma^2+1}\left(1 - \cosh^2\left(\frac{\operatorname{arcsinh}(i\nu\xi)}{2}\right)(\gamma^2+1)\right)\right) \\ &\quad \exp\left(-\frac{\kappa}{\nu}\operatorname{arccotan}(\gamma) + \frac{\gamma}{\nu}\left(\frac{e^{-2a}}{\gamma^2+1} + \cosh^2\left(\frac{\operatorname{arcsinh}(i\nu\xi)}{2}\right) r^2\right)\right)\end{aligned}$$

is an intertwiner with the $\mathbb{S}$-invariant deformation theory.


Thank you !!