Multiplicative integrability of Poisson symmetric spaces: $\mathbb{CP}^n$

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KWZ program

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Karasev-Weinstein-Zakrzewski

Apply geometric quantization to $\mathcal{G}$ and compare the outcome with deformation quantization of (M, π).

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If the obstruction is not present (meaning of the word *integrable*) then the groupoid as also a symplectic manifold *compatible* with the Lie groupoid structure.

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3. Bohr-Sömmerfeld condition identifying $\mathcal{G}^{BS}/\mathcal{F}$;
4. (Twisted) convolution C^*–algebra $C^*(\mathcal{G}^{BS}/\mathcal{F}; \sigma_0)$.

Motivating example

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Outcome

Quantum torus $p \star q = e^{\hbar} q \star p$.

Multiplicative polarization

A groupoid polarization $\mathcal{F} \subseteq T^{\mathbb{C}}\mathcal{G}$ is *multiplicative* (Hawkins JSG 2008) if, letting

$$\mathcal{F}_2 = (\mathcal{F} \times \mathcal{F}) \cap T^{\mathbb{C}}\mathcal{G}_2$$

then

$$m_*(\mathcal{F}_2(\gamma, \eta)) = \mathcal{F}(m(\gamma, \eta))$$

for any composable pair $(\gamma, \eta) \in \mathcal{G}_2$.

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Problem: there are topological obstructions to the existence of real multiplicative polarizations

$\mathbb{CP}^1$-obstruction

Let π be **any** integrable Poisson structure on $\mathbb{CP}^1$, then there are no real multiplicative polarizations on its symplectic groupoid (linked to non existence of rank 1 foliations on $\mathbb{CP}^1$).

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Bruhat-Poisson structure on $\mathbb{CP}^1$:

$$\pi_B = \begin{cases} -\imath(1+|z|^2)\partial_z \wedge \partial_{\overline{z}} & \text{on } \mathbb{CP}^1 \setminus [1,0] \\ \\ -\imath|w|^2(1+|w|^2)\partial_w \wedge \partial_{\overline{w}} & \text{on } \mathbb{CP}^1 \setminus [0,1] \end{cases}$$

Still possibile to perform KWZ procedure with a **singular** multiplicative polarization (Bonechi, C., Staffolani, Tarlini JGP 2012).

Loosening requirements

What do we really need for a C^*–groupoid convolution algebra?

- $\mathcal{G} \to \mathcal{G}_{\mathcal{F}}$ **Lagrangian fibration of topological groupoids;**
- $\mathcal{G}_{\mathcal{F}}^{bs}$ Bohr–Sömmerfeld subgroupoid carrying a left Haar measure;
- the prequantization cocycle descending to $\mathcal{G}_{\mathcal{F}}^{bs}$;
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coincide with the modular function of the quasi invariant measure on the base space, implement **KMS condition.**

integrable

A family $F = \{f_1, \dots f_n\}$ of functions, $N = \frac{1}{2}\dim \mathcal{G}$, is an integrable system if are in involution $\{f_i, f_j\} = 0$ and $df_1 \wedge \dots df_N \neq 0$ on a dense open subset of M.

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The integrable system is called *multiplicative* if the distribution $\mathcal{F} = \langle X_{f_1}, \dots X_{f_N} \rangle$ is multiplicative;

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modular

The integrable system is called *modular* if the modular function f_V is in involution with all f_i's.

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1. $\mathcal{G}_F(M)$ is a topological groupoid and $\mathcal{G} \to \mathcal{G}_F(M)$ a topological groupoid epimorphism;
2. For each pair l_1, l_2 of composable leaves $m : l_1 \times l_2 \to l_1 l_2$ induces a surjective map in homology ($\Rightarrow$ subgroupoid $\mathcal{G}^{bs}_{\mathcal{F}}$).

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3. $\mathcal{G}_F^{bs}(M)$ admits a left Haar system (guaranteed if it is étale).

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Non symplectic are all quotient by coisotropic subgroups:

$$U_t(n) = \sigma_t S(U(1) \times U(n))\sigma_t^{-1} \subseteq SU(n+1)$$

where

$$\sigma_t = \begin{pmatrix} \sqrt{1-t} & 0 & \sqrt{t} \\ 0 & \mathrm{id}_{n-1} & 0 \\ -\sqrt{t} & 0 & \sqrt{1-t} \end{pmatrix}$$

Some equivalences. In fact:

$$\psi : \mathbb{CP}^n \to \mathbb{CP}^n; \qquad \psi(\pi_t) = -\pi_{1-t}$$

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Poisson pencil

Let π_λ be the Fubini-Study bivector. Then $[\pi_\lambda, \pi_0] = 0$ (Koroshkin-Radul-Rubtsov CMP '93) and $\pi_t = \pi_0 + t\pi_\lambda$.

Standard $\mathbb{CP}^n$: symplectic foliation

Projecting the chain of Poisson subgroups

$$SU(1) \subseteq SU(2) \subseteq \ldots \subseteq SU(n)$$

one gets the chain of Poisson submanifolds

$$\{*\} \subseteq \mathbb{CP}^1 \subseteq \ldots \subseteq \mathbb{CP}^{n-1}$$

In homogeneous coordinates

$$P_k = \{[X_1, \ldots, X_k, 0, \ldots, 0]\}$$

is a Poisson submanifold. All symplectic leaves are contractible and symplectomorphic to standard $\mathbb{C}^k$.

Non standard $\mathbb{CP}^n$: symplectic foliation

Let

$$P_k(t) = \left\{ F_{k,t} = t \sum_{i=1}^{k} |X_i|^2 - (1-t) \sum_{i=k+1}^{n} |X_i|^2 = 0 \right\}$$

Then $\bigcup_{i=1}^{n} P_i(t)$ is the singular part; complement has $n+1$ connected contractible leaves $\simeq \mathbb{C}^n$.

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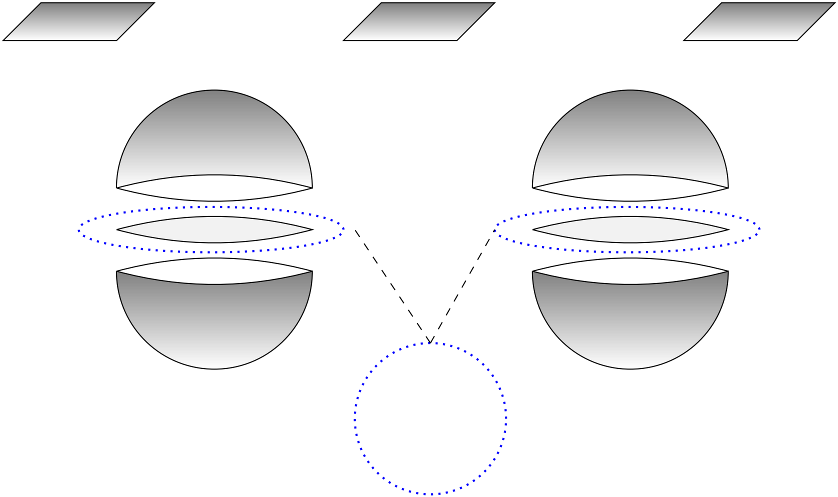
Scheme of the singular part for $\mathbb{CP}^3$:

$$\begin{array}{ccccccccc}
\mathbb{S}^5 & & & & \frac{\mathbb{S}^3 \times \mathbb{S}^3}{\mathbb{S}^1} & & & & \mathbb{S}^5 \\
& \nwarrow & & \nearrow & & \nwarrow & & \nearrow & \\
& & \mathbb{S}^3 & & & & \mathbb{S}^3 & & \\
& & & \nwarrow & & \nearrow & & & \\
& & & & \mathbb{S}^1 & & & &
\end{array}$$

symplectic foliation of $\mathbb{CP}^2_t$



The symplectic groupoid

$$\mathcal{G}(\mathbb{CP}^n, \pi_t) = \{[g\gamma] \, : g \in SU(n+1), \gamma \in SB(n+1, \mathbb{C}), {}^g\gamma \in U_t(n)^\perp\}$$

is a fibre bundle over $\mathbb{CP}^n$ with contractible fibre $U_t(n)^\perp$.

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It carries a hamiltonian $\mathbb{T}^n$–action with momentum map

$$h([g\gamma]) = \log p_{A_{n+1}}(\gamma)$$

The Cartan $\mathbb{T}^n \subseteq SU(n+1)$ acts on $(\mathbb{CP}^n, \pi_\lambda)$ with moment map

$$c : \mathbb{CP}^n \to \mathfrak{t}_n^*; \qquad \mathrm{Im}\, c = \Delta_n$$

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Suitable basis H_k of $\mathfrak{t}_n$ such that

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2. $\sigma_{H_k} = \{b_k, --\}$, with $b_k = \log |c_k - 1 + t|$.

Summarizing actions

- Hamiltonian $\mathbb{T}^n$–action on $\mathbb{CP}^n$ with momentum map $c : \mathbb{CP}^n \to \mathbb{R}^n$;
- Hamiltonian $\mathbb{T}^n$–action on $\mathcal{G}(\mathbb{CP}^n, \pi_t)$ with momentum map $h : \mathbb{CP}^n \to \mathbb{R}^n$ by groupoid 1–cocycles;

Let us consider

$$\mathcal{F} = \{l^* c_i, h_i \ \ldots i = 1, \ldots, n\}$$

Theorem

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Aim: prove this integrable system is well behaved.

The topological groupoid of level sets

Let $\mathbb{R}^n$ act on $\mathbb{R}^n$ via

$$c \cdot h = (1 - t + e^{-h}(c + t - 1))$$

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and let $\mathbb{R}^n \rtimes \mathbb{R}^n\big|_{\Delta_n}$ be the action groupoid restricted to the standard simplex. Then:

$$\mathcal{G}_{\mathcal{F}}(t) = \{(c, h) \in \mathbb{R}^n \rtimes \mathbb{R}^n\big|_{\Delta_n} \,:\, c_i = c_{i+1} = 1 - t \Rightarrow h_i = h_{i+1}\}$$

is the topological groupoid of level sets.

Bohr-Sömmerfeld conditions

Level sets L_{ch} are connected with: $H^1(L_{ch}; \mathbb{Z})$ generated by hamiltonian flows of $h_j, l^* c_j$;

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BS conditions select a discret subset of lagrangians

$$\mathcal{G}^{bs}_{\mathcal{F}}(t) = \{(c,h) \in \mathcal{G}_{\mathcal{F}}(t) \; : h_k \in \hbar\mathbb{Z}, \, \log|c_k - 1 + t| \in \hbar\mathbb{Z}\}$$

This is an étale subgroupoid with a unique left Haar system.

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The modular function f_{FS} is quantized to

$$f_{FS}(c,h) = \sum_{i=1}^{n} h_i$$

The space of units is

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Groupoid orbits are labelled by $(r, s) : r + s \leq n$. Each is a transitive subgroupoid over

$$\Delta_{r,s}^{\mathbb{Z}}(t) = \left\{ (m, \infty, n) \in \overline{\mathbb{Z}}^r \times \infty \times \overline{\mathbb{Z}}^s : \begin{array}{ccc} -\frac{\log(1-t)}{\hbar} & \leq m_i & \leq m_{i+1} \\ n_i & \geq n_{i+1} & \geq -\frac{\log(t)}{\hbar} \end{array} \right\}$$

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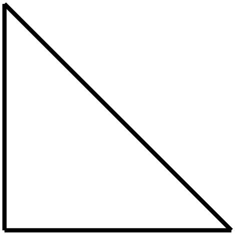
3. Groupoids thus obtained concide with:
   - Sheu for $(\mathbb{CP}^n, \pi_0)$;
   - Sheu for $\mathbb{S}^{2n-1}$ as Poisson sbmfld of $\mathbb{CP}^n, \pi_t$, $t \neq 0, 1$;
   - Sheu for $(\mathbb{CP}^1, \pi_t)$.

Example: $\mathbb{C}P^2$



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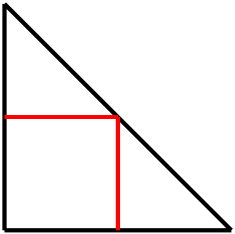

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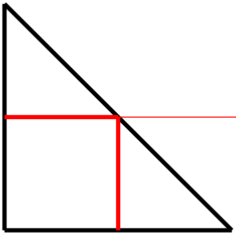

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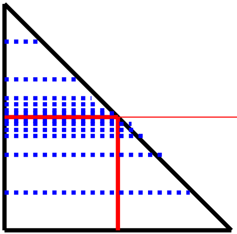

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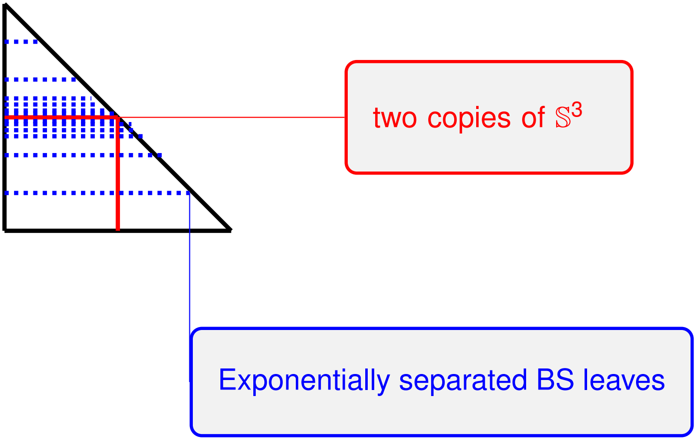

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1–cocycle $c = \log D \in Z^1(\mathcal{G};\mathbb{R})$

$A_c(t) = e^{\imath tc}$ map in $Aut(C^*\mathcal{G})$

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Modular class in $H^1_\pi(M)$

Van den Bergh bimodule

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- Functoriality: $(\mathbb{CP}^1, \pi_t)$ are all Poisson–Morita equivalent for $0 < t < 1$ and the nonstandard groupoid does not depend on t.
- What about other Hermitian symmetric spaces?

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