The Gysin Sequence for Quantum Lens Spaces

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The Classical Gysin Sequence

- Projective spaces $\mathbb{CP}^n = S^{2n+1}/\mathrm{U}(1)$ and lens spaces $L^{(n,r)} = S^{2n+1}/\mathbb{Z}_r$ give principal bundles $S^{2n+1} \to L^{(n,r)} \xrightarrow{\pi} \mathbb{CP}^n$.

- This yields an exact sequence in topological K-theory,

$$0 \to K^1(L^{(n,r)}) \xrightarrow{\delta} K^0(\mathbb{CP}^n) \xrightarrow{\alpha} K^0(\mathbb{CP}^n) \xrightarrow{\pi^*} K^0(L^{(n,r)}) \to 0,$$

  the **classical Gysin sequence**,

- where δ is a "connecting homomorphism" and α is multiplication by the Euler class $\chi(\mathcal{O}(-r)) := 1 - [\mathcal{O}(-r)]$.

- From this we find that $K^1(L^{(n,r)}) = \mathrm{Ker}(\alpha)$ and $K^0(L^{(n,r)}) = \mathrm{Coker}(\alpha)$.

Quantum Projective Spaces

- The algebra of coordinate functions $\mathcal{A}(S_q^{2n+1})$ on quantum odd-dimensional spheres (à la Vaksman-Soibelman) is generated by $\{z_i, z_i^*\}_{i=0,\dots,n}$ subject to the relations $z_i z_j = q^{-1} z_j z_i$ *etc.* and the sphere relation

$$z_0 z_0^* + z_1 z_1^* + \ldots + z_n z_n^* = 1.$$

- There is a $\mathsf{U}(1)$-action $z_j \mapsto \lambda z_j$ for $\lambda \in \mathsf{U}(1)$; define the invariant subalgebra

$$\mathcal{A}(\mathbb{CP}_q^n) := \mathcal{A}(S_q^{2n+1})^{\mathsf{U}(1)}$$

  to be the coordinate algebra of the quantum projective space (generated by functions $z_i z_j^*$, $i, j = 0, 1, \ldots, n$). The fibration $S_q^{2n+1} \to \mathbb{CP}_q^n$ is a **quantum principal bundle** with structure group $\mathsf{U}(1)$.

- Define $C(S_q^{2n+1})$ and $C(\mathbb{CP}_q^n)$ to be the universal enveloping C^*-algebras generated by these coordinate algebras: the C^*-algebras of *continuous* functions.

K-Theory of Quantum Projective Spaces

Prop (d'Andrea-Landi '10)

There exist projections $P_N \in \mathsf{M}_{d_N}(\mathcal{A}(\mathbb{CP}^n_q))$, $N = 0, 1, \ldots, n$, such that $[P_N]$ generate the theory $K_0(C(\mathbb{CP}^n_q))$ (algebraic generators!).

- Indeed we have $\mathcal{A}(\mathbb{CP}^n_q)$-bimodules $\mathcal{L}_N := P_N \mathcal{A}(\mathbb{CP}^n_q)^{d_N}$ for all $N \in \mathbb{Z}$ obeying

$$\mathcal{L}_N \otimes_{\mathcal{A}(\mathbb{CP}^n_q)} \mathcal{L}_M \simeq \mathcal{L}_{N+M}, \qquad \mathcal{L}_N^{\otimes M} \simeq \mathcal{L}_{NM}.$$

- Chern numbers coming from pairing with K-homology:

$$c_0(\mathcal{L}_N) = 1, \qquad c_1(\mathcal{L}_N) = -N, \qquad N \in \mathbb{Z},$$

i.e. $\mathcal{L}_N$ is the space of sections of a line bundle over $\mathbb{CP}^n_q$ with winding number $-N$.

Theorem (d'Andrea-Landi '10)

With the Euler class $u = \chi(\mathcal{L}_{-1}) := 1 - [\mathcal{L}_{-1}]$ we have

$$K_0(C(\mathbb{CP}^n_q)) \simeq \mathbb{Z}[u]/\langle u^{n+1} \rangle \simeq \mathbb{Z}^{n+1}.$$

Quantum Lens Spaces

Definition

For $r \geq 2$ we define the algebra

$$\mathcal{A}(L_q^{n,r}) := \bigoplus_{N \in \mathbb{Z}} \mathcal{L}_{rN}$$

of coordinate functions on the quantum lens space $L_q^{n,r}$. Write $C(L_q^{n,r})$ for the universal enveloping C^*-algebra.

This is the $*$-algebra of all elements of $\mathcal{A}(S_q^{2n+1})$ invariant under the $\mathbb{Z}_r$-action $z_j \mapsto e^{2\pi i/r} z_j$, $j = 0, 1, \ldots n$. Thus we have algebra inclusions

$$\mathcal{A}(\mathbb{CP}_q^n) \hookrightarrow \mathcal{A}(L_q^{n,r}) \hookrightarrow \mathcal{A}(S_q^{2n+1}).$$

Proposition

The algebra inclusion $j : \mathcal{A}(\mathbb{CP}_q^n) \hookrightarrow \mathcal{A}(L_q^{n,r})$ is a quantum principal bundle with structure group $\widetilde{\mathrm{U}}(1) = \mathrm{U}(1)/\mathbb{Z}_r$ under the action $\sigma : \widetilde{\mathrm{U}}(1) \to \mathrm{Aut}(\mathcal{A}(L_q^{n,r}))$.

Towards an Exact Sequence in K-Theory

- For each $\mathcal{L}_N$ we define its pull back $\widetilde{\mathcal{L}}_N := \mathcal{L}_N \otimes_{\mathcal{A}(\mathbb{CP}^n_q)} \mathcal{A}(L^{n,r}_q)$.
- This induces an obvious map in K-theory:

$$j_* : K_0(C(\mathbb{CP}^n_q)) \to K_0(C((L^{n,r}_q)), \qquad j_*([\mathcal{L}_N]) := \widetilde{\mathcal{L}}_N.$$

- Note that $\mathcal{L}_N$ is not free when $N \neq 0$ but

$$\widetilde{\mathcal{L}}_{-r} = \mathcal{L}_{-r} \otimes_{\mathcal{L}_0} \mathcal{A}(L^{n,r}_q) \simeq \mathcal{A}(L^{n,r}_q) = \widetilde{\mathcal{L}}_0,$$

  so $\widetilde{\mathcal{L}}_{-r}$ is free. Thus for example $\widetilde{\mathcal{L}}^{\otimes r}_{-1} \simeq \widetilde{\mathcal{L}}_0$ and so $\widetilde{\mathcal{L}}_{-1}$ is torsion.

- There is also a natural map

$$\alpha : K_0(C(\mathbb{CP}^n_q) \to K_0(C(\mathbb{CP}^n_q)$$

  given by multiplying by the Euler class $\chi(\mathcal{L}_{-r}) := 1 - [\mathcal{L}_{-r}]$.

- So we are looking for a sequence

$$0 \to K_1(C(L^{(n,r)}_q)) \to K_0(C(\mathbb{CP}^n_q)) \xrightarrow{\alpha} K_0(C(\mathbb{CP}^n_q)) \xrightarrow{j_*} K_0(C(L^{(n,r)}_q)) \to 0$$

Hilbert C^*-Modules and KK-Theory

- Let us write $A := C(L_q^{(n,r)})$ and $F := C(\mathbb{CP}_q^n)$, with F the fixed point subalgebra of A under the $\widetilde{\mathrm{U}}(1)$-action $\sigma : \widetilde{\mathrm{U}}(1) \to \mathrm{Aut}(A)$.
- There is a conditional expectation

$$\tau : A \to F, \qquad \tau(a) := \int_0^{2\pi} \sigma_t(a)\mathrm{d}t$$

and hence a Hilbert C^*-module $A \to X \leftrightharpoons F$ coming from the F-valued inner product $\langle a, b\rangle_F := \tau(a^* b)$.
- Let

$$\mathfrak{D} : \mathfrak{Dom}(\mathfrak{D}) \subseteq X \to X$$

be the infinitesimal generator of the action $\sigma : \widetilde{\mathrm{U}}(1) \to \mathrm{Aut}(A)$.

Theorem (Carey, Neshveyev, Nest, Rennie '11)

The pair $(X, \mathfrak{D})$ determines a class λ in the bivariant KK-theory $KK_1(A, F)$.

The Kasparov Product

- We use the internal Kasparov product

$$- \otimes_A - : K_1(A) \times KK_1(A, F) \to K_0(F)$$

to define a map

$$\mathsf{Ind}_{\mathfrak{D}} : K_1(A) \to K_0(F), \qquad \mathsf{Ind}_{\mathfrak{D}}(-) = - \otimes_A \lambda.$$

- Putting this all together yields a sequence

$$0 \to K_1(C(L_q^{(n,r)})) \xrightarrow{\mathsf{Ind}_{\mathfrak{D}}} K_0(C(\mathbb{CP}_q^n)) \xrightarrow{\alpha} K_0(C(\mathbb{CP}_q^n)) \xrightarrow{j_*} K_0(C(L_q^{(n,r)})) \to 0.$$

Theorem (Arici-B-Landi '14)

This **quantum Gysin sequence** is exact.

The Quantum Gysin Sequence

- The proof uses the mapping cone exact sequence

$$0 \to S(A) \xrightarrow{i} M(F,A) \xrightarrow{ev} F \to 0,$$

where $S(A) = C_0(0,1) \otimes A$ and

$$M(F,A) = \{ f \in C([0,1],A) \mid f(0) = 0,\ f(1) \in F \},$$

- together with the six-term exact sequence

$$\begin{array}{ccccc}
K_0(S(A)) & \longrightarrow & K_0(M(F,A)) & \longrightarrow & K_0(F) \\
\uparrow & & & & \downarrow \\
K_1(F) = 0 & \longleftarrow & K_1(M(F,A)) = 0 & \longleftarrow & K_1(S(A))
\end{array}$$

- to get a commutative diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & K_0(S(A)) & \xrightarrow{i_*} & K_0(M(F,A)) & \xrightarrow{\mathrm{ev}_*} & K_0(F) & \xrightarrow{\partial} & K_1(S(A)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \times[-\mathcal{L}_{-r}]} & & \downarrow{\scriptstyle Bott} & & \\
0 & \longrightarrow & K_1(A) & \xrightarrow{\mathrm{Ind}_{\mathfrak{D}}} & K_0(F) & \xrightarrow{\alpha} & K_0(F) & \xrightarrow{j_*} & K_0(A) & \longrightarrow & 0.
\end{array}$$