DQ and Weight Homogeneous Poisson structures

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Outline.

1. DQ.
2. Weight homogeneous stuff.
3. Extensions and applications.
4. Example.

Deformation Quantization.

M. Kontsevich proved the following :

Theorem

Let $(\mathbb{R}^k, \pi)$ be a Poisson manifold and $F, G \in C^\infty(\mathbb{R}^k)$. The operator

$$F *_K G := F \cdot G + \sum_{n=1}^{\infty} \hbar^n \left(\frac{1}{n!} \sum_{\Gamma \in \mathbf{Q_{n,2}}} \omega_\Gamma B_{\Gamma,\pi}(F, G) \right)$$

defines an associative product
$C^\infty(\mathbb{R}^k))[[\hbar]] \times C^\infty(\mathbb{R}^k)[[\hbar]] \longrightarrow C^\infty(\mathbb{R}^k)[[\hbar]]$. *The map* $[*] \mapsto [\pi]$ *is a one-to-one correspondence.*

The set $\mathbf{Q_{n,2}}$ is a special family of graphs Γ. Each Γ gives rise to a bidifferential operator $B_{\Gamma,\pi}(F, G) = \sum_{R,S} b_i^{RS} \partial_R(F) \partial_S(G)$ on $C^\infty(\mathbb{R}^k) \times C^\infty(\mathbb{R}^k)$. The coefficient $\omega_\Gamma \in \mathbb{R}$ is calculated by integrating a differential form Ω_Γ also encoded in Γ.

Deformation Quantization.

Theorem (Kontsevich)

Let $\mathcal{U} : \mathcal{T}_{poly}(\mathbb{R}^k) \longrightarrow \mathcal{D}_{poly}(\mathbb{R}^k)$ *be the map defined by its Taylor coefficients*

$$\mathcal{U}_n := \sum_{\overline{m}\geq 0} \left(\sum_{\Gamma \in \mathbf{Q}_{n,\overline{m}}} \omega_\Gamma B_\Gamma \right).$$

Then $\mathcal{U}$ *is an* $L_\infty -$*morhism and a quasi-isomorphism.*

Cattaneo-Felder Generalization.

A coisotropic submanifold $C \subset X$ is a submanifold such that the ideal $I(C) \subset C^\infty(X)$ of functions vanishing on C, is a Poisson subalgebra of $C^\infty(X)$.

The Relative Formality Theorem proves an $L_\infty-$ quasi-isomorphism from $\mathcal{T}(X, C) = lim_{\leftarrow} \mathcal{T}(X)/I(C)^n \mathcal{T}(X)$, the DGLA of multivector fields in an infinitesimal neighbourghood of C to $\tilde{\mathcal{D}}(\mathcal{A}) = \oplus_n \tilde{\mathcal{D}}^n(\mathcal{A})$ where $\tilde{\mathcal{D}}^n(\mathcal{A}) = \prod_{p+q-1=n} \mathrm{Hom}^p(\otimes^q \mathcal{A}, \mathcal{A})$, $\mathcal{A} = \Gamma(C, \wedge T_X)$.

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Weight homogeneous vector fields.

• Consider $\mathbb{R}^k$ with coordinate functions $x_1, \ldots, x_k$. A $k-$ tuple of positive integers $\overline{\omega} := (\overline{\omega}_1, \ldots, \overline{\omega}_k)$ is called a *weight vector*.

• An $F \in C^\infty(\mathbb{R}^k)$ is called *weight homogeneous* with respect to $\overline{\omega}$ if there is an $r \in \mathbb{N}$ such that
$F(\lambda^{\overline{\omega}_1} x_1, \ldots, \lambda^{\overline{\omega}_k} x_k) = \lambda^r F(x_1, \ldots, x_k), \forall \lambda \in \mathbb{R}$.

• The number r is called the *weight* of F and we will write $\overline{\omega}(F) = r$.

• A $p-$ vector field on $\mathbb{R}^k$ is *weight homogeneous* if applying it to weight homogeneous functions $F_1, \ldots, F_p \in C^\infty(\mathbb{R}^k)$ we get a weight homogeneous smooth function.

• If $P, F_1, \ldots, F_p$ are weight homogeneous of weights $\overline{\omega}(P), \overline{\omega}(F_1), \ldots, \overline{\omega}(F_p)$ then $P(F_1, \ldots, F_p)$ is weight homogeneous of weight $\overline{\omega}(P) + \sum_{i=1}^{n} \overline{\omega}(F_i)$ (or $P(F_1, \ldots, F_p) = 0$).

• The weighted Euler vector field $E_{\overline{\omega}} = \sum_{i=1}^{k} \overline{\omega}_i x_i \frac{\partial}{\partial x_i}$, traces the weight of homogeneous elements ; $\mathcal{L}_{E_{\overline{\omega}}}(F) = \overline{\omega}(F) F$, $\mathcal{L}_{E_{\overline{\omega}}}(P) = \overline{\omega}(P) P$.

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Weight homogeneous Poisson structures.

As a bivector π, a Poisson structure on $\mathbb{R}^k$ is weight homogeneous iff the functions $\{x_i, x_j\}$ are weight homogeneous of weight $\overline{\omega}(\pi) + \overline{\omega}_i + \overline{\omega}_j$.

Standard examples include
• Ordinary polynomial Poisson structures (quadratic, cubic, etc) ; take $\overline{\omega} = (1, \dots, 1)$.
• Transverse Poisson structures to adjoint orbits (a nilpotent orbit in a semi-simple Lie algebra).
• Graded symplectic forms are weight homogeneous elements of $C^\infty(T[1]M)$.

Can be extended to Nambu-Poisson structures.

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Weight homogeneous Formality.

Let π be a weight homogeneous Poisson structure on $\mathbb{R}^k$ and F_i, G be weight homogeneous smooth functions.
A polydifferential operator is called weight homogeneous iff $\overline{\omega}(B) = \overline{\omega}(B(F_1, \dots, F_s)) - \sum_{i=1}^{s} \overline{\omega}(F_i)$.

• If $F, G \in C^\infty(\mathbb{R}^k)$ are weight homogeneous, then the terms in the Taylor expansion of $F * G$ are weight homogeneous.

$$\overline{\omega}(B_\Gamma^\pi) = -t \cdot \overline{\omega}(\pi), \quad \forall \Gamma \in Q_{t,2}$$

• If $\psi_1, \dots, \psi_l$ are weight homogeneous skew-symmetric multivector fields, the same is true for the terms in the Taylor expansion of the Formality morphism

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Weight homogeneous linear Poisson structures.

Let's look at Lie algebras for a moment.

• Taking a Lie subalgebra $\mathfrak{m} \subset \mathfrak{g}$ and a character χ of $\mathfrak{m}$ one can apply the Relative Formality Theorem for $X = \mathfrak{g}^*$, $C = \mathfrak{m}_\chi^\perp$. There is an extensive study of the construction, the algebraic properties and the relations to harmonic analysis of Lie groups, of the **reduction algebra** $H^0(\mathfrak{m}_\chi^\perp, \mathrm{d})$ (Cattaneo-Torossian).

• The differential $\mathrm{d} : \; S(\mathfrak{g}/\mathfrak{m}) \longrightarrow S(\mathfrak{g}/\mathfrak{m}) \otimes \mathfrak{m}^*$ is written as $\mathrm{d} = \sum_{i=1}^{\infty} \mathrm{d}^{(i)}$ where $\mathrm{d}^{(i)} = \sum_{\Gamma \in \mathcal{B}_i \cup \mathcal{BW}_i} \omega_\Gamma B_\Gamma$.

• If π is weight homogeneous, then
$\Gamma \in \mathcal{B}_t \Rightarrow \overline{\omega}(B_\Gamma) = -t\overline{\omega}(\pi) - \overline{\omega}(L(e_\infty))$ and
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A variant of the generalized Duflo Isomorphism.

Theorem (B) For every Lie algebra $\mathfrak{g}$, ($\mathfrak{m}, \chi$ as before), there is a non-canonical associative algebra isomorphism

$$\mathcal{Q}_\hbar : \; H^0_{(\hbar)}(\mathfrak{m}_\chi^\perp, \mathrm{d}^{(\hbar)}) \xrightarrow{\sim} \left(U_{(\hbar)}(\mathfrak{g})/U_{(\hbar)}(\mathfrak{g})\mathfrak{m}_\chi\right)^{\mathfrak{m}}$$

It combines what you would expect from the generalized Duflo Isomorphism ($\overline{\beta}_{\mathfrak{q},(\hbar)} \circ \partial_{q^{\frac{1}{2}}_{(\hbar)}}$), with Kontsevich's graphs and operators.

Technical difficulty : The deformation parameter $\hbar$ is important. $\hbar = 1$ is not the same as no $\hbar$ at all.
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Applications for representation theory.

An basic example of weight homogeneous Poisson structure is the following :

Let $\mathfrak{g}$ be a semisimple Lie algebra, $\{e, h, f\}$ an $\mathfrak{sl}_2-$ triple. The $\mathrm{ad}h-$ action induces the decomposition $\mathfrak{g} = \bigoplus_i \mathfrak{g}(i)$, let n_x be the eigenvalue of $x \in \mathfrak{g}$.

For $\{x_1, \dots, x_k\}$ a basis of $\mathfrak{g}$, fix $\overline{\omega} = (n_1 + 2, \dots, n_k + 2)$ (Kazhdan weight).
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Let $\mathfrak{g}$ be a semisimple Lie algebra, $Q := Z_G(e,h,f)$ and $\mathfrak{q}$ be the corresponding Lie algebra. The *Slodowy slice* is $\mathcal{S} := e + \mathrm{ker}\mathrm{ad} f \subset \mathfrak{g}^*$.

Def. I : The $W-$ algebra associated to $(\mathfrak{g}, e)$ is

$$U(\mathfrak{g}, e) := \left(\mathbb{K}[G \times S][[\hbar]]\right)^G|_{\hbar=1}$$

Losev : For $V = [\mathfrak{g}, f]$ and for suitable completions $U_\hbar(\mathfrak{g})^\wedge := \mathbb{K}[\mathfrak{g}^*]^\wedge_\chi[[\hbar]], \mathbf{A}^\wedge_\hbar := \mathbb{K}[V^*]^\wedge_0[[\hbar]], \mathcal{W}^\wedge_\hbar := \mathbb{K}[S]^\wedge_\chi[[\hbar]]$ there is a $Q \times \mathbb{K}^\times -$ equivariant topological $\mathbb{K}[[\hbar]]-$ algebra isomorphism

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$W-$ algebras and their representations.

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Def. II $U(\mathfrak{g}, e) := (U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{m}_\chi)^{\mathfrak{m}}$.
• With the quantization map $\mathcal{Q}$ we have defined a new model for it.

• Let $\mathfrak{z}_\chi := \ker \mathrm{ad} e$. This space has an induced weight space decomposition. Let $x_1, \dots, x_r$ span $\mathfrak{z}_\chi$. **Premet** constructs elements Θ_i, $i = 1, \dots, r$ of the $W-$ algebra, forming a PBW basis.
• If $[x_i, x_j] = \sum_{k=1}^{r} \alpha_{ij}^k x_k$ then modulo the appropriate level of the Kazhdan filtration in the $W-$ algebra, the commutation relations between the Θ_i are

$$[\Theta_i, \Theta_j] = \sum_{k=1}^{r} \alpha_{ij}^k \Theta_k + q_{ij}(\Theta_1, \dots, \Theta_r)$$

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Computation of irred reps of $U(\mathfrak{g}, e)$.

Premet : The abelian subalgebra $U(\mathfrak{g}, e)^{\mathrm{ab}}$ of $U(\mathfrak{g}, e)$ can be used to determine the maximal spectrum and thus the 1-dimensional representations of $U(\mathfrak{g}, e)$.

$\rightarrow$ Use $\mathcal{Q}$ and the $*-$ commutator $[\cdot, \cdot]_*$ instead.

Non-trivial example : The minimal orbit of $\mathfrak{g}_2$.
$\dim(\mathfrak{g}_2) = 14, \dim(\mathfrak{m}) = 8$. We can compute all the brackets $\Theta_i * \Theta_j - \Theta_j * \Theta_i$ and compute its 1-dimensional representation. (Only using GAP4 so far).

Computation of irred reps of $U(\mathfrak{g}, e)$.

Premet : The abelian subalgebra $U(\mathfrak{g}, e)^{\mathrm{ab}}$ of $U(\mathfrak{g}, e)$ can be used to determine the maximal spectrum and thus the 1-dimensional representations of $U(\mathfrak{g}, e)$.

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Other applications.

When $\mathfrak{g}$ is nilpotent, the commutativity of $(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\chi)^{\mathfrak{h}}$ is directly related to the finite multiplicity condition in the spectral decomposition of $\mathrm{Ind}(G \uparrow H, \chi)$, the induced by H, χ unitary representation of G (think of a well-defined L^2- space on G/H with χ-invariant, compactly supported functions) :

Namely, the multiplicities $m(\tau)$ in

$$\tau_\chi \simeq \int_{\widehat{G}} m(\tau)\tau \mathrm{d}\mu(\tau) \simeq \int_{(\chi+\mathfrak{h}^\perp)/H} \tau_l \mathrm{d}\nu(l)$$

are finite iff $(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{h}_\chi)^{\mathfrak{h}}$ is commutative.

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