A (nonlinear) noncommutative sigma model

$$f : A \longrightarrow A_\Theta$$

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References

[MR]

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Motivation

In classical sigma-models in string theory, the fields are maps $g\colon \Sigma \to X$, where Σ is closed and 2-dimensional, representing a *string worldsheet*, and the target space X is 10-dimensional space-time.
The leading terms in the action are

$$S(g) = \int_\Sigma \|\nabla g(x)\|^2 d\sigma(x) + \int_{\widehat{\Sigma}} \widehat{g}^*(H), \qquad (1)$$

where σ is volume measure on Σ and the 2nd term is the Wess-Zumino term.

Without the WZ term, the critical points of the action are just harmonic maps $\Sigma \to X$. T-duality considerations suggested that very often one should consider spacetimes which are *noncommutative* spaces. For example, “bundles” of noncommutative tori over some base space, such as the C^*-algebra of the discrete Heisenberg group. as discussed in earlier lectures. What should replace maps $g\colon \Sigma \to X$ and the action (**??**) when X becomes noncommutative?

Motivation

It's natural to start with the simplest interesting case, where X is a noncommutative 2-torus (or rotation algebra) $A = A_\Theta$. We are primarily interested in the case where Θ is irrational.

Naively, since a map $g\colon \Sigma \to X$ is equivalent to a C^*-algebra morphism $C_0(X) \to C(\Sigma)$, one's first guess would be to consider $*$-homomorphisms $A \to C(\Sigma)$, where Σ is still an ordinary 2-manifold.

But if A is simple, there are no non-trivial such maps. Hence we are led to consider a sigma-model based on $*$-homomorphisms between A and noncommutative tori.

$$A \to A_\theta$$

Sigma model: the general case

Recall that a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is given by an involutive unital algebra $\mathcal{A}$ represented as bounded operators on a Hilbert space $\mathcal{H}$ and a self-adjoint operator D with compact resolvent such that the commutators $[D, a]$ are bounded for all $a \in \mathcal{A}$.

A spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is said to be *even* if the Hilbert space $\mathcal{H}$ is endowed with a $\mathbb{Z}_2$-grading γ which commutes with all $a \in \mathcal{A}$ and anti-commutes with D. Suppose in addition that $(\mathcal{A}, \mathcal{H}, D)$ is $(2, \infty)$-summable, which means (assuming for simplicity that D has no nullspace) that $\mathrm{Tr}_\omega(a|D|^{-2}) < \infty$, where Tr_ω denotes the Dixmier trace.

Now

$$\psi_2(a_0, a_1, a_2) = \mathrm{Tr}((1+\gamma)a_0[D, a_1][D, a_2])$$

defines a positive Hochschild 2-cocycle on $\mathcal{A}$, where $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the grading operator on $\mathcal{H}$, and where Tr denotes the Dixmier trace composed with D^{-2}. The positivity of ψ_2 means that $\langle a_0 \otimes a_1, b_0 \otimes b_1 \rangle = \psi_2(b_0^* a_0, a_1, b_1^*)$ defines a positive sesquilinear form on $\mathcal{A} \otimes \mathcal{A}$.

Although we consider the canonical trace Tr instead of the above trace, all the properties go through with either choice. Using the Dixmier trace Tr_ω composed with D^{-2} has the advantage of **scale invariance**, i.e., it is invariant under the replacement of D by λD for any nonzero $\lambda \in \mathbb{C}$, which becomes relevant when one varies the metric, although for special classes of metrics, the scale invariance can be obtained by other means also.

We now give a prescription for energy functionals in the sigma-model consisting of homomorphisms $\varphi\colon \mathcal{B} \longrightarrow \mathcal{A}$, from a smooth subalgebra of a C^*-algebra $\mathcal{B}$ with target the given even $(2, \infty)$-summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$.

Observing that $\varphi^*(\psi_2)$ is a positive Hochschild 2-cocycle on $\mathcal{B}$, we need to choose a formal "metric" on $\mathcal{B}$, which is a positive element $G \in \Omega^2(\mathcal{B})$ in the space of universal 2-forms on $\mathcal{B}$. Then evaluation

$$\mathcal{L}_{G,D}(\varphi) = \varphi^*(\psi_2)(G) \geq 0$$

defines a general sigma-model action.

Summarizing, the data for a general sigma-model action

1. A $(2, \infty)$-summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$;
2. A positive element $G \in \Omega^2(\mathcal{B})$ in the space of universal 2-forms on $\mathcal{B}$, known as a metric on $\mathcal{B}$.

Consider a unital C^*-algebra generated by the n unitaries $\{U_j : i = 1, \dots n\}$, with finitely many relations, and let $\mathcal{B}$ be a suitable subalgebra consisting of rapidly vanishing series whose terms are (noncommutative) monomials in the U_i's. Then a choice of metric $G \in \Omega^2(\mathcal{B})$ is given by

$$G = \sum_{j,k=1}^{n} G_{jk} (dU_j)^* dU_k,$$

where the matrix (G_{jk}) is symmetric, real-valued, and positive definite. Then we compute the energy functional in this case,

$$\mathcal{L}_{G,D}(\varphi) = \varphi^*(\psi_2)(G) = \sum_{j,k=1}^{n} G_{jk} \operatorname{Tr}((1+\gamma)[D, \varphi(U_j)^*][D, \varphi(U_k)]) \geq 0.$$

The *Euler-Lagrange equations* for φ to be a critical point of $\mathcal{L}_{G,D}$ can be derived, but since the equations are long, we omit them.

We next give several examples of this sigma-model energy functional. In all of these cases, the target algebra will be A_θ^∞.

The first example is the Dąbrowski-Krajewski-Landi model, consisting of non-unital $*$-homomorphisms $\varphi\colon \mathbb{C} \longrightarrow A_\theta^\infty$. Note that $\varphi(1) = e$ is a projection in the noncommutative torus A_θ, and for any $(2, \infty)$-summable spectral triple $(A_\theta^\infty, \mathcal{H}, D)$ on the noncommutative torus, our sigma-model energy functional is

$$\mathcal{L}_D(\varphi) = \operatorname{Tr}\left[(1+\gamma)[D, e][D, e]\right].$$

Choose the even spectral triple given by $\mathcal{H} = L^2(A_\theta) \otimes \mathbb{C}^2$ consisting of the Hilbert space closure of A_θ in the canonical scalar product coming from the trace, tensored with the 2-dimensional representation space of spinors.

Let $D = \gamma_1 \delta_1 + \gamma_2 \delta_2$ be the Dirac operator, where

$$\gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

are the Pauli matrices, we calculate that

$$\mathcal{L}_D(\varphi) = \sum_{j=1}^{2} \mathrm{Tr}\left[(\delta_j e)^2\right],$$

recovering the action in [DKL] and the Euler-Lagrange equation $(\Delta e)e = e(\Delta e)$ there.

Next, we consider the model due to Rosenberg.
It consists of a unital $*$-homomorphisms $\varphi\colon C(S^1) \longrightarrow A_\theta^\infty$.
Let U be the unitary given by multiplication by the coordinate function z on S^1 (considered as the unit circle $\mathbb{T}$ in $\mathbb{C}$). The metric $G \in \Omega^2(C(S^1))$ given by $dU^* dU$.

Then $\varphi(U)$ is a unitary in the noncommutative torus A_θ, and for any $(2,\infty)$-summable spectral triple $(A_\theta^\infty, \mathcal{H}, D)$ on the noncommutative torus, our sigma-model energy functional is

$$\mathcal{L}_D(\varphi) = \operatorname{Tr}\left[(1+\gamma)[D, \varphi(U)^*][D, \varphi(U)]\right].$$

Choosing the particular spectral triple on the noncommutative torus as above, we calculate that

$$\mathcal{L}_D(\varphi) = \sum_{j=1}^{2} \operatorname{Tr}\left[(\delta_j(\varphi(U)))^* \delta_j(\varphi(U))\right],$$

recovering the action and the Euler-Lagrange equation

$$\varphi(U)^* \Delta(\varphi(U)) + (\delta_1(\varphi(U)))^* \delta_1(\varphi(U)) + (\delta_2(\varphi(U)))^* \delta_2(\varphi(U)) = 0$$

The final example is the one treated in this talk. For any (smooth) homomorphism $\varphi\colon A_\Theta \longrightarrow A_\theta$ and any $(2,\infty)$-summable spectral triple $(A_\theta^\infty, \mathcal{H}, D)$, and any positive element $G \in \Omega^2(\mathcal{A}_\Theta)$ (or metric on $\mathcal{A}_\Theta$) given by

$$G = \sum_{j,k=1}^{2} G_{ij}(dU_j)^* dU_k,$$

the energy of φ is

$$\mathcal{L}_{G,D}(\varphi) = \varphi^*(\psi_2)(G) = \sum_{j,k=1}^{2} G_{jk}\,\mathrm{Tr}((1+\gamma)[D, \varphi(U_j)^*][D, \varphi(U_k)]) \geq 0.$$

where U, V are the canonical generators of A_Θ.

More explicity, let $g = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \in M_2(\mathbb{R})$ be a symmetric real-valued positive definite matrix. Then one can consider the 2-dimensional complexified Clifford algebra, with self-adjoint generators $\gamma_\mu \in M_2(\mathbb{C})$ and relations

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = g^{\mu\nu}, \qquad \mu, \nu = 1, 2,$$

where $(g^{\mu\nu})$ denotes the matrix g^{-1}. Then with $\mathcal{H}$ as before, define $D = \sum_{\mu=1}^{2} \gamma_\mu \delta_\mu$. The energy in this more general case is

$$\mathcal{L}_{G,D}(\varphi) = \varphi^*(\psi_2)(G) = \sum_{j,k=1}^{2} \sum_{\mu,\nu=1}^{2} G_{jk} g^{\mu\nu} \operatorname{Tr}(\delta_\mu(\varphi(U_j))^* \delta_\nu(\varphi(U_k)) \geq 0. \tag{2}$$

In this case, the trace Tr is either the Dixmier trace composed with D^{-2}, or the canonical trace on A_θ multiplied by the factor $\sqrt{\det(g)}$, to make the energy scale invariant.

There is a rather large literature on “noncommutative Wess-Zumino theory” or “noncommutative WZW theory”.

Most of this literature seems to deal with the Wess-Zumino-Witten model (where spacetime is a compact group) or with the Moyal product, but we have been unable to find anything that applies to our situation where both spacetime and the worldsheet are represented by noncommutative C^*-algebras (or dense subalgebras thereof). For that reason, we will attempt here to reformulate the theory from scratch.

The classical Wess-Zumino term is associated to a closed 3-form H with integral periods on X (the spacetime manifold). If Σ^2 is the boundary of a 3-manifold W^3, and if $\varphi\colon \Sigma \to X$ extends to $\widetilde{\varphi}\colon W \to X$, the Wess-Zumino term is

$$\mathcal{L}_{WZ}(\varphi) = \int_W (\widetilde{\varphi})^*(H).$$

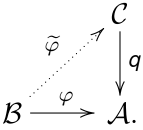
The fact that H has integral periods guarantees that $e^{2\pi i \mathcal{L}_{WZ}(\varphi)}$ is well-defined, i.e., independent of the choice of W and the extension $\widetilde{\varphi}$ of φ.

To generalize this to the noncommutative world, we need to dualize all spaces and maps. We replace X by $\mathcal{B}$ (which in the classical case would be $C_0(X)$), Σ by $\mathcal{A}$, and W by $\mathcal{C}$.

Since H classically was a cochain on X (for de Rham cohomology), it becomes a **degree 3 cyclic cycle** on $\mathcal{B}$.
The integral period condition can be replaced by requiring

$$\langle H, u \rangle \in \mathbb{Z} \tag{3}$$

for all classes $u \in K^1(\mathcal{B})$ in K-homology.
The inclusion $\Sigma \hookrightarrow W$ dualizes to a map $q\colon \mathcal{C} \to \mathcal{A}$, and we suppose $\varphi\colon \mathcal{B} \to \mathcal{A}$ has a factorization



The noncommutative Wess-Zumino term then becomes

$$\mathcal{L}_{WZ}(\varphi) = \langle \widetilde{\varphi}_*(H), [\mathcal{C}] \rangle,$$

with $[\mathcal{C}]$ a cyclic cochain (corresponding to integration over W.)

The integral period condition is relevant for the same reason as in the classical case—if we have another “boundary” map $q'\colon \mathcal{C}' \to \mathcal{A}$ and corresponding $\widetilde{\varphi}'\colon \mathcal{B} \to \mathcal{C}'$, and if $\mathcal{C} \oplus_{\mathcal{A}} \mathcal{C}'$ is “closed,” so that $[\mathcal{C}] - [\mathcal{C}']$ corresponds to a class $u \in K^1(\mathcal{C} \oplus_{\mathcal{A}} \mathcal{C}')$, then

$$\langle \widetilde{\varphi}_*(H), [\mathcal{C}] \rangle - \langle \widetilde{\varphi}'_*(H), [\mathcal{C}'] \rangle = \langle H, (\widetilde{\varphi} \oplus \widetilde{\varphi}')^*(u) \rangle \in \mathbb{Z},$$

- thus $e^{2\pi i \mathcal{L}_{WZ}(\varphi)}$ is the same when computed via $[\mathcal{C}]$ or via $[\mathcal{C}']$.

Now we want to apply this theory when $\mathcal{A} = A_\theta$ (or a suitable smooth subalgebra, say A_θ^∞). If we realize A_θ as the crossed product $C^\infty(S^1) \rtimes_\theta \mathbb{Z}$, we can view A_θ^∞ as the "boundary" of $\mathcal{C} = C^\infty(D^2) \rtimes_\theta \mathbb{Z}$, where D^2 denotes the unit disk in $\mathbb{C}$. The natural element $[\mathcal{C}]$ is the trace on $\mathcal{C}$ coming from normalized Lebesgue measure on D^2.

To summarize, it is possible to enhance the sigma-model action on a spacetime algebra $\mathcal{B}$ with the addition of a Wess-Zumino term $\mathcal{L}_{WZ}(\varphi)$, depending on a choice of a "flux" H.

It occurs geometrically as the foliation algebra associated to Kronecker foliations on the torus.
For each $\theta \in [0,1]$, the **noncommutative torus** A_θ is defined abstractly as the C^*-algebra generated by two unitaries U and V in an infinite dimensional Hilbert space satisfying the **Weyl commutation relation**, $UV = \exp(2\pi i\theta)VU$.
Elements in A_θ can be represented by infinite power series

$$f = \sum_{(m,n)\in\mathbb{Z}^2} a_{(n,m)}\, U^m V^n, \tag{4}$$

For each $\theta \in [0,1]$, the noncommutative torus A_θ is Morita equivalent to the foliation algebra associated to the foliation on $\mathbb{T}^2$ defined by the differential equation $dx = \theta\, dy$ on $\mathbb{T}^2$.

There is a natural smooth subalgebra A_θ^∞ called the **smooth noncommutative torus**, which is defined as those elements in A_θ that can be represented by infinite power series (**??**) with $(a_{(m,n)}) \in \mathcal{S}(\mathbb{Z}^2)$, the Schwartz space of rapidly decreasing sequences on $\mathbb{Z}^2$.

When θ is rational, A_θ is noncommutative, but is Morita equivalent to $C(\mathbb{T}^2)$. However, when θ is irrational, A_θ is a simple (i.e. highly noncommutative!). A_θ is also called a **rotation algebra** in the literature.

Maps between irrational rotation algebras: existence

We begin by classifying maps between irrational rotation algebras, using what is known about their ordered K-theory (see, e.g., Rieffel).

Theorem

Fix Θ and θ in $(0, 1)$, both irrational, and $n \in \mathbb{N}$, $n \geq 1$. There is a unital $*$-homomorphism $\varphi \colon A_\Theta \to M_n(A_\theta)$ if and only if $n\Theta = c\theta + d$ for some $c, d \in \mathbb{Z}$, $c \neq 0$. Such a $*$-homomorphism φ can be chosen to be an isomorphism onto its image if and only if $n = 1$ and $c = \pm 1$.

Maps between irrational rotation algebras: existence

This can be reformulated in the following more algebraic language. In what follows, Tr denotes the normalized trace on A_θ, extended as usual to matrices. The monoid M also appears in the theory of Hecke operators.

Lemma

*Let M be the submonoid (**not** a subgroup) of $GL(2, \mathbb{Q})$ consisting of matrices in $M_2(\mathbb{Z})$ with non-zero determinant, i.e., of integral matrices having inverses that are not necessarily integral.*
Then M is generated by $GL(2, \mathbb{Z})$ and by the matrices of the form $\begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$, $r \in \mathbb{Z} \setminus \{0\}$.

Maps between irrational rotation algebras: existence

Theorem

*Fix Θ and θ in $(0, 1)$, both irrational. Then there is a non-zero $*$-homomorphism $\varphi\colon A_\Theta \to M_n(A_\theta)$ for some n, not necessarily unital, if and only if Θ lies in the orbit of θ under the action of the monoid M of Lemma **??** on $\mathbb{R}$ by linear fractional transformations. The possibilities for $\mathrm{Tr}(\varphi(1_{A_\Theta}))$ are precisely the numbers $t = c\theta + d > 0$, $c, d \in \mathbb{Z}$ such that $t\Theta \in \mathbb{Z} + \theta\mathbb{Z}$. Once t is chosen, n can be taken to be any integer $\geq t$.*

The maps in Theorems above can always be chosen to be smooth (i.e. mapping the smooth subalgebra A_Θ^∞ to $M_n(A_\theta^\infty)$).

Maps between irrational rotation algebras: existence

The following improves a result of Kodaka.

Theorem

Suppose θ is irrational. Then there is a (necessarily injective) unital $*$-endomorphism $\Phi\colon A_\theta \to A_\theta$, with image $B \subsetneq A_\theta$ having non-trivial relative commutant and with a conditional expectation of index-finite type from A_θ onto B, if and only if θ is a quadratic irrational number.

The maps Φ in Theorem above can be chosen to be smooth and to induce an arbitrary group endomorphism of $K_1(A_\theta)$. But when θ is not a quadratic irrational, we do not know if A_θ has any *smooth* proper $*$-endomorphisms.

Now that we understand maps between irrational rotation algebras, we study the analogue of the action functional.

Definition

Let φ denote a unital $*$-homomorphism $A_\Theta \to A_\theta$. As before, denote the canonical generators of A_Θ and A_θ by U and V, u and v, respectively. The action $S(g)$ in our situation is

$$\begin{aligned} S(\varphi) = \mathrm{Tr}\Big(&\delta_1(\varphi(U))^*\delta_1(\varphi(U)) + \delta_2(\varphi(U))^*\delta_2(\varphi(U)) \\ &+ \delta_1(\varphi(V))^*\delta_1(\varphi(V)) + \delta_2(\varphi(V))^*\delta_2(\varphi(V))\Big). \end{aligned} \tag{5}$$

Critical points for this action are called **harmonic maps.** Here δ_1 and δ_2 are the infinitesimal generators for the “gauge action” of the group $\mathbb{T}^2$ on A_θ. More precisely, δ_1 and δ_2 are defined on the smooth subalgebra A_θ^∞ by the formulas

$$\delta_1(u) = 2\pi i u, \quad \delta_2(u) = 0, \quad \delta_1(v) = 0, \quad \delta_2(v) = 2\pi i v.$$

Note that $S(\varphi)$ in (**??**) is just the sum $E(\varphi(U)) + E(\varphi(V))$, where for a unitary $W \in A_\theta^\infty$,

$$E(W) = \mathrm{Tr}\Big(\delta_1(W)^*\delta_1(W) + \delta_2(W)^*\delta_2(W)\Big). \tag{6}$$

It was conjectured that the “special” unitaries $u^n v^m$ minimize the energy E in the connected components of $U(A_\theta^\infty)$.

The harmonic map equation for noncommutative tori

Theorem (Euler-Lagrange equations)

Let $S(\varphi)$ denote the energy functional for a unital $*$-endomorphism φ of A_θ. Then the Euler-Lagrange equations for φ to be a ***harmonic map****, that is, a critical point of $\mathcal{L}$, are:*

$$0 = \sum_{j=1}^{2} \left\{ \mathrm{Tr}\left(A\,\delta_j\left[\varphi(u)^*\delta_j(\varphi(u))\right]\right) + \mathrm{Tr}\left(B\,\delta_j\left[\varphi(v)^*\delta_j(\varphi(v))\right]\right) \right\}$$

where A, B are self-adjoint elements in A_θ, constrained to satisfy the equation,

$$A - \varphi(v)^* A \varphi(v) = B - \varphi(u)^* B \varphi(u).$$

Proof.

Consider the 1-parameter family of $*$-endomorphisms of A_θ defined by

$$\begin{aligned}\varphi_t(u) &= \varphi(u)e^{ih_1(t)} \\ &= \varphi(u)[1 + ith_1'(0) + O(t^2)], \\ \varphi_t(v) &= \varphi(v)e^{ih_2(t)} \\ &= \varphi(u)[1 + ith_2'(0) + O(t^2)],\end{aligned}$$

where $h_j(t)$, $j = 1, 2$ are 1-parameter families of self-adjoint operators with $h_1(0) = 0 = h_2(0)$. Differentiate & simplify. □

The harmonic map equation for noncommutative tori

For Θ a quadratic irrational, it was proved by [MR], as well as other interesting cases. Recently been proved by Hanfeng Li:

Theorem (Hanfeng Li)

For W in the connected component of $U(A_\theta^\infty)$ containing $u^n v^m$, $E(W) \geq E(u^n v^m) = 4\pi^2(m^2 + n^2)$, with equality if and only if $W = \lambda u^n v^m$ for some $\lambda \in \mathbb{T}$.

Corollary ("Minimal Energy Conjecture")

Suppose $\varphi\colon\ A_\theta^\infty \circlearrowleft$ is a $*$-endomorphism inducing the map on K_1 given by $\begin{pmatrix} p & q \\ r & s \end{pmatrix} \in SL(2, \mathbb{Z})$. Then $S(\varphi) \geq 4\pi^2(p^2 + q^2 + r^2 + s^2)$, with equality if and only if $\varphi(u) = \lambda u^p v^q$, $\varphi(v) = \mu u^r v^s$, $\lambda,\ \mu \in \mathbb{T}$.

The harmonic map equation for noncommutative tori

In general, we would like to understand the nature of **all critical points** of the action functional, not just the minima.

This has been only in the analysed in the special case when Θ is rational in [MR], but the irrational case is a mystery.

In the rational case, we have constructed explicit solutions to the harmonic map equation, and they turn out to be related to solutions of the equation governing a nonlinear pendulum.

A physical model

To write the partition function for the sigma model studied here, recall the expression for the energy

$$S_G(\varphi) = \sqrt{\det(g)} \sum_{i,j=1}^{n} G_{ij} g^{\mu\nu} \operatorname{Tr}(\delta_\mu(\varphi(U_i))^* \delta_\nu(\varphi(U_j)).$$

It is possible to parametrize the metrics $(g_{\mu\nu})$ by a complex parameter τ,

$$g(\tau) = (g_{\mu\nu}(\tau)) = \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}$$

where $\tau = \tau_1 + i\tau_2 \in \mathbb{C}$ is such that $\tau_2 > 0$.
Note that g is invertible with inverse given by

$$g^{-1}(\tau) = (g^{\mu\nu}(\tau)) = \tau_2^{-2} \begin{pmatrix} |\tau|^2 & -\tau_1 \\ -\tau_1 & 1 \end{pmatrix}$$

and $\sqrt{\det(g)} = \tau_2$.

A physical model

The partition function is

$$Z(G,z) = \int_{\tau\in\mathbb{C},\tau_2>0} \frac{d\tau \wedge d\bar{\tau}}{{\tau_2}^2} Z(G,\tau,z)$$

where

$$Z(G,\tau,z) = \int \mathcal{D}[\varphi] e^{-zS_{G,\tau}(\varphi)} / \int \mathcal{D}[\varphi].$$

is the renormalized integral.

This integral is much too difficult to deal with even in the commutative case, so we oversimplify by considering the semiclassical approximation, which is a sum over the critical points. Even this turns out to be highly nontrivial, and we discuss it below.

A physical model

In the special case when $\Theta = \theta$ and is not a quadratic irrational, then the semiclassical approximation to the partition function above is

$$Z(G, \tau, z) \approx \sum_{m \in M/\{\pm 1\}} \sum_{A} e^{-zS_{G,\tau}(\varphi_A)},$$

up to a normalizing factor, in the notation as explained later in this section. In this approximation,

$$Z(G, z) \approx \int_{\tau \in \mathbb{C}, \tau_2 > 0} \frac{d\tau \wedge d\bar{\tau}}{{\tau_2}^2} \sum_{m \in M/\{\pm 1\}} \sum_{A} e^{-zS_{G,\tau}(\varphi_A)}.$$

We expect $Z(G)$ and $Z(G^{-1})$ to be related as in the classical case as a manifestation of T-duality.
In the remainder we only consider contributions from the critical points (harmonic maps).

A physical model

The simplified partition function then looks like

$$Z(z) \approx \sum_{m \in M/\{\pm 1\}} \sum_{A} e^{-4\pi^2 \mathcal{D}(m,\theta)\|A\|_{HS}^2 z} . \tag{7}$$

The formula $4\pi^2\mathcal{D}(m,\theta)\|A\|_{HS}^2$ for the energy is valid not just for the automorphisms φ_A but also for the map $U \mapsto u^p v^q$, $V \mapsto u^r v^s$ with

$$A = \begin{pmatrix} p & q \\ r & s \end{pmatrix}, \quad \det A = n$$

from $A_{n\theta}$ to A_θ, which one can check to be harmonic, just as in done earlier. The associated map on K_0 corresponds to

$$m = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$$

with $\mathcal{D}(m,\theta) = 1$.
Many thanks for your attention!