Renormalizable noncommutative QFT II Kyoto, 22 nd February 2011

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Introduction

- Motivation: Improve QFT in 4 dimensions
- Renormalizable QFT (not summmable)
- add "gravity" effects or
- Quantize Space-Time
- Renormalizable Noncommutative QFT IR/UV mixing
  - Main Result H G + R Wulkenhaar ...
  - use renormalized pertubation theory
  - needs regularization - renormalization
  - use Renormalization group methods
  - Taming the Landau Ghost (Borel) summable ?
- almost Solvable, nontrivial !
- Fermions - Spectral triple

Introduction

- Classical field theories for fundamental interactions (electroweak, strong, gravitational) are of geometrical origin
- Quantum field theory for standard model (electroweak+strong) is renormalisable
- Gravity is not renormalisable

Renormalisation group interpretation

- space-time being smooth manifold $\Rightarrow$ gravity scaled away
- weakness of gravity determines Planck scale where geometry is something different

promising approach: noncommutative geometry
unifies standard model with gravity as classical field theories

Can we make sense of renormalisation in NCG?

First step: construct quantum field theories on simple noncommutative geometries, e.g. the Moyal space

Moyal space

algebra of rapidly decaying functions over D-dimensional Euclidean space with $\star$-product

$$(a \star b)(x) = \int d^D y d^D k a(x + \tfrac{1}{2}\Theta \cdot k) b(x+y)\, e^{iky}$$

where $\Theta = -\Theta^T \in M_D(\mathbb{R})$

- $\star$-product is associative, noncommutative, and most importantly: non-local
- construction of field theories with non-local interaction
- This non-locality has serious consequences for the renormalisation of the resulting quantum field theory

nc Scalar field model

Operator formulation e g : $D = 2$ $[c\hat{T}, \hat{X}] = i\Theta$

ϕ^4 on nc $\mathbb{R}^4$, $[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}$ antisymmetric, $u_p = e^{ip\hat{x}}$ or equivalently star product,

$\partial_\mu u_p = ip_\mu u_p = i[\tilde{x}_\mu, u_p]$ $\tilde{x}_\mu := (\theta^{-1})_{\mu\nu}\hat{x}^\nu$

$$\Phi = \int dp e^{ip\hat{x}} \Phi_p$$

ϕ^4 action

$$S = \frac{1}{2} Tr(-[\tilde{x}_\mu, \Phi][\tilde{x}^\mu, \Phi] + m^2\Phi\Phi + \frac{\lambda}{2}\Phi^4)$$

yields Schrödinger equation:

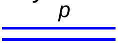
$$[\tilde{x}_\mu, [\tilde{x}^\mu, \Phi]] + m^2\Phi + \lambda\Phi^3 = E\Phi$$

UV/IR-mixing

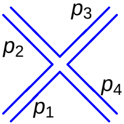
- naïve ϕ^4-action (ϕ-real, Euclidean space) on Moyal plane

$$S = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \star \partial^\mu \phi + \frac{m^2}{2} \phi \star \phi + \frac{\lambda}{4} \phi \star \phi \star \phi \star \phi \right)(x)$$

- Feynman rules:



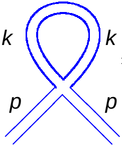
$$= \frac{1}{p^2 + m^2}$$



$$= \frac{\lambda}{4} \exp \left(-\frac{\mathrm{i}}{2} \sum_{i<j} p_i^\mu p_j^\nu \theta_{\mu\nu} \right)$$

- cyclic order of vertex momenta is essential
  $\Rightarrow$ ribbon graphs

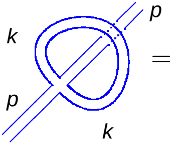
- one-loop two-point function, *planar contribution*:



$$= \frac{\lambda}{6} \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 + m^2}$$

to be treated by usual regularisation methods, can be put to 0

- planar nonregular contribution:



$$= \frac{\lambda}{12} \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik\cdot\Theta\cdot p}}{k^2+m^2} \sim (\Theta p)^{-2}$$

- non-planar graphs finite (noncommutativity as a regulator) but $\sim p^{-2}$ for small momenta (renormalisation not possible)

$\Rightarrow$ leads to non-integrable integrals when inserted as subgraph into bigger graphs: UV/IR-mixing

The UV/IR-mixing problem and its solution

- *observation*: euclidean quantum field theories on Moyal space suffer from UV/IR mixing problem which destroys renormalisability if quadratic divergences are present

Theorem

The quantum field theory defined by the action

$$\boxed{S = \int d^4x \Big(\frac{1}{2}\phi \star (\Delta + \Omega^2 \tilde{x}^2 + \mu^2)\phi + \frac{\lambda}{4!}\phi \star \phi \star \phi \star \phi\Big)(x)}$$

with $\tilde{x} = 2\Theta^{-1} \cdot x$, ϕ – real, Euclidean metric is perturbatively renormalisable to all orders in λ.

The additional oscillator potential $\Omega^2 \tilde{x}^2$

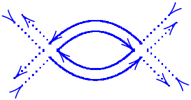
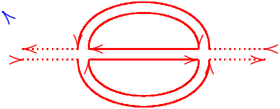
- implements mixing between large and small distance scales
- results from the renormalisation proof

Intuitive remarks

Langmann-Szabo duality

$$\left.\begin{array}{rcl}\tilde{x} & \longmapsto & p \\ \phi(x) & \longmapsto & \hat{\phi}(p)\end{array}\right\} + \text{Fourier transformation}$$

- leaves $\int d^4x\,(\phi\star\phi\star\phi\star\phi)(x)$ and $\int d^4x\,(\phi\star\phi)(x)$ invariant
- transforms $\int d^4x\,(\phi\star\Delta\phi)(x)$ into $\int d^4x\,(\phi\star\tilde{x}^2\phi)(x)$

- with  also its LS-dual is divergent
- also the LS-dual of  is divergent

renormalisation requires $\int d^4x\,(\phi\star\tilde{x}^2\phi)(x)$ in initial action

History of the renormalisation proof

- exact renormalisation group equation in matrix base

  H. G., R.Wulkenhaar

  - simple interaction, complicated propagator
  - power-counting from decay rate and ribbon graph topology

- multi-scale analysis in matrix base

  V. Rivasseau, F. Vignes-Tourneret, R.Wulkenhaar

  - rigorous bounds for the propagator (requires large Ω)

- multi-scale analysis in position space

  R. Gurau, J. Magnen, V. Rivasseau, F. Vignes-Tourneret

  - simple propagator (Mehler kernel), oscillating vertex
  - distinction between sum and difference of propagator ends
  - Schwinger parametric representation

    R. Gurau, V. Rivasseau, T. Krajewski,...

  - reduction to Symanzik type hyperbolic polynomials

The matrix base of the Moyal plane

- central observation (in 2D):
  $f_{00} := 2e^{-\frac{1}{\theta}(x_1^2+x_2^2)} \quad \Rightarrow \quad f_{00} \star f_{00} = f_{00}$
- left and right creation operators:
  $$f_{mn}(x_1,x_2) = \frac{(x_1+ix_2)^{\star m}}{\sqrt{m!(2\theta)^m}} \star \left(2e^{-\frac{1}{\theta}(x_1^2+x_2^2)}\right) \star \frac{(x_1-ix_2)^{\star n}}{\sqrt{n!(2\theta)^n}}$$
  $$f_{mn}(\rho,\varphi) = 2(-1)^m \sqrt{\frac{m!}{n!}} e^{i\varphi(n-m)} \left(\sqrt{\frac{2}{\theta}}\rho\right)^{n-m} e^{-\frac{\rho^2}{\theta}} L_m^{n-m}\left(\frac{2}{\theta}\rho^2\right)$$
- satisfies: $(f_{mn} \star f_{kl})(x) = \delta_{nk} f_{ml}(x)$
  $$\int d^2x\, f_{mn}(x) = \delta_{mn}$$
- Fourier transformation has the same structure

Extension to four dimensions

non-vanishing components: $\theta = \Theta_{12} = -\Theta_{21} = \Theta_{34} = -\Theta_{43}$
double indices

non-local $\star$-product becomes simple *matrix product*

$$S[\phi] = \sum_{m,n,k,l \in \mathbb{N}^2} \left(\frac{1}{2} \phi_{mn} \Delta_{mn;kl} \phi_{kl} + \frac{\lambda}{4!} \phi_{mn} \phi_{nk} \phi_{kl} \phi_{lm} \right)$$

important: $\Delta_{mn;kl} = 0$ unless $m - l = n - k$
$SO(2) \times SO(2)$ angular momentum conservation

- diagonalisation of Δ yields recursion relation for Meixner polynomials
- closed formula for propagator $G = (\Delta)^{-1}$
- $G_{\substack{m\ m\\0\ 0};\substack{m\ m\\0\ 0}} \sim \frac{\theta/8}{\sqrt{\frac{4}{\pi}(m+1) + \Omega^2 (m+1)^2}}$
- $G_{\substack{m_1\ m_1\\m_2\ m_2};\substack{0\ 0\\0\ 0}} = \frac{\theta}{2(1+\Omega)^2 (m_1+m_2+1)} \left(\frac{1-\Omega}{1+\Omega} \right)^{m_1+m_2}$

RG FLOW

Wilson RG-Flow
divide covariance for free Euclidean scalar field into slices

$$\Phi_m = \sum_{j=0}^{m} \phi_j, \quad C_j = \int_{M^{-2j}}^{M^{-2(j-1)}} d\alpha \frac{e^{-m^2\alpha - x^2/4\alpha}}{\alpha^{D/2}}$$

integrate out degrees of freedom

$$Z_{m-1}(\Phi_{m-1}) = \int d\mu_m(\phi_m) e^{-S_m(\phi_m + \Phi_{m-1})}$$

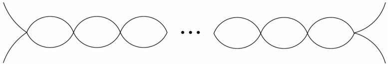
$$Z_{m-1}(\Phi_{m-1}) = e^{-S_{m-1}(\Phi_{m-1})}$$

Landau Ghost

- superficial degree of divergence for Feynman graph G

$$D = 4 \quad \omega(G) = 4 - N(G)$$

- BPHZ Theorem: renormalizability
- but: certain chain of finite subgraphs with m bubbles grows like



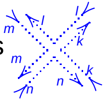
$$\int \frac{d^4q}{\left(q^2+m^2\right)^3} \left(\log|q|\right)^m \simeq C^m m!$$

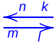
- not Borel summable

$$\lambda_j \simeq \frac{\lambda_0}{1-\beta\lambda_0 j}$$

- sign of β positive: Landau ghost, triviality

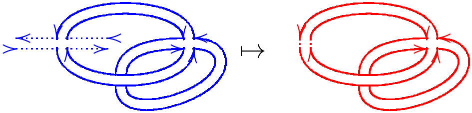
Ribbon graphs

Feynman graphs are ribbon graphs with V vertices 

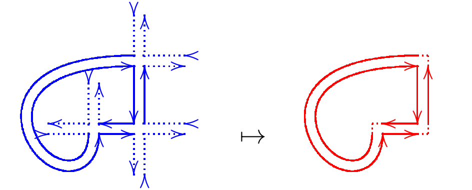
and I edges 

$= G_{mn;kl}$ and N external legs

- leads to F faces, B of them with external legs
- ribbon graph can be drawn on Riemann surface of genus $g = 1 - \frac{1}{2}(F - I + V)$ with B holes



$F = 1 \quad g = 1$
$I = 3 \quad B = 1$
$V = 2 \quad N = 2$

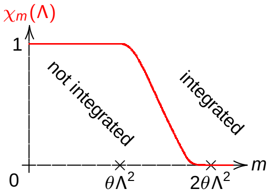


$L = 2 \quad g = 0$
$I = 3 \quad B = 2$
$V = 3 \quad N = 6$

First proof: exact renormalisation group equations

QFT defined via partition function $Z[J] = \int \mathcal{D}[\phi]\, e^{-S[\phi]-\mathrm{tr}(\phi J)}$

- Wilson's strategy: integration of field modes ϕ_{mn} with indices $\geq \theta\Lambda^2$ yields effective action $L[\phi, \Lambda]$
- variation of cut-off function $\chi(\Lambda)$ with Λ modifies effective action:



exact renormalisation group equation [Polchinski equation]

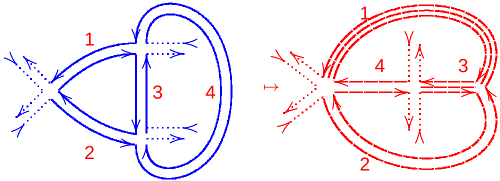
$$\Lambda \frac{\partial L[\phi,\Lambda]}{\partial \Lambda} = \sum_{m,n,k,l} \frac{1}{2} Q_{mn;kl}(\Lambda) \left(\frac{\partial L[\phi,\Lambda]}{\partial \phi_{mn}} \frac{\partial L[\phi,\Lambda]}{\partial \phi_{kl}} - \frac{\partial^2 L[\phi,\Lambda]}{\partial \phi_{mn}\, \partial \phi_{kl}} \right)$$

with $Q_{mn;kl}(\Lambda) = \Lambda \frac{\partial (G_{mn;kl}\, \chi_{mn;kl}(\Lambda))}{\partial \Lambda}$

- renormalisation = proof that there exists a regular solution which depends on only a finite number of initial data

Second proof: multi-scale analysis

- propagator cut into slices: $G_{mn;kl} = \sum_{i=1}^{\infty} G^i_{mn;kl}$
  estimations:
  $$0 \leq G^i_{mn;kl} \leq K_1 M^{-i} e^{-c_1 M^{-i}(\|m\|+\|n\|+\|k\|+\|l\|)} \delta_{m-l,-(k-n)}$$
  $$\sum_l \Big(\max_{n(l),k(l)} G^i_{mn;kl} \Big) \leq K_2 M^{-i} e^{-c_2 M^{-i}\|m\|}$$
- induces scale attribution $i_\delta \in \mathbb{N}^+$ for each edge δ of the graph
- $SO(2) \times SO(2)$ symmetry implemented by dual graphs (vertices $\Leftrightarrow$ faces)



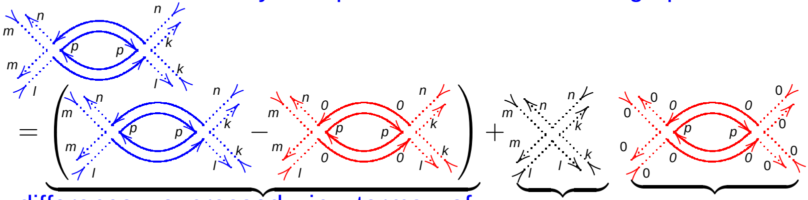
  - index-difference (= angular momentum) conserved at propagators and vertices
  - power-counting degree of divergence of graphs $2\,\#(\text{inner vertices}) - \#(\text{edges})$
    $= 2(F-B) - I = 4-4g-2V+I - 2B = (2-\frac{N}{2}) - 2(2g+B-1)$

Conclusion

All non-planar graphs and all planar graphs with $\geq$ 6 external legs are convergent

Renormalisation

Problem: infinitely many planar 2- and 4-leg graphs diverge
Solution: discrete Taylor expansion about reference graphs:



difference expressed in terms of $|G_{np;pn} - G_{0p;p0}| \leq K_3 M^{-i} \frac{\|n\|}{M^i} e^{-c_3\|p\|}$

$=1$

put to renormalised value

- similar for all $A^{planar}_{mn;nk;kl;lm}$ $A^{planar}_{mn;nm}$ and $A^{planar}_{\substack{m^1+1 \\ m^2}\;\substack{n^1+1 \\ n^2};\substack{n^1 \\ n^2}\;\substack{m^1 \\ m^2}}$

Renormalisation of noncommutative ϕ^4_4-model to all orders

by normalisation conditions for mass, field amplitude, coupling constant and oscillator frequency

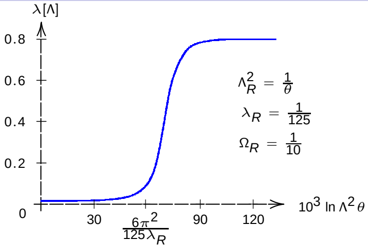
The β-function

one-loop calculation

$$\frac{\lambda[\Lambda]}{\Omega^2[\Lambda]} = \text{const}$$

$$\frac{d\lambda}{d\Lambda} = \beta_\lambda = \lambda^2 \frac{(1-\Omega^2)}{(1+\Omega^2)^3} + \mathcal{O}(\lambda^3)$$

$\lambda[\Lambda]$ diverges in commutative case



- perturbation theory remains valid at all scales!
- non-perturbative construction of the model seems possible!

How does this work?

- four-point function renormalisation with usual sign
- $\exists$ one-loop wavefunction renormalisation which compensates four-point function renormalisation for $\Omega \to 1$

The self-dual model

- $\Omega = 1$ leads to constant matrix indices for each face
- angular momentum ℓ is zero
  exponential decay in $|\ell|$ for general case
  $\Rightarrow$ self-dual model also captures general behaviour
- powerful techniques from matrix models available
  - solvable (trivial) scalar model E. Langmann, R. Szabo, K. Zarembo
  - renormalisation of ϕ^3_6 by relation to Kontsevich model
    H. Grosse, H. Steinacker

idea M. Disertori, V. Rivasseau

compute β-function for $\Omega = 1$
$\rightarrow$ model is asymptotically safe up to three loops
(cancellations established by formidable graph calculation)
$\beta = 0$ to all orders,...

Asymptotic safety to all orders

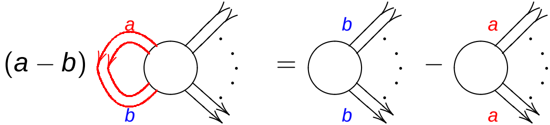
M. Disertorti, R. Gurau, J. Magnen, V. Rivasseau

Theorem

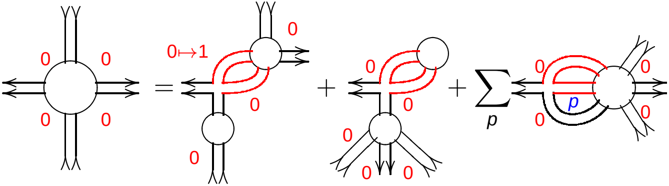
$\Gamma^4(0,0,0,0) = \lambda(1-(\partial\Sigma)(0,0))^2$ *to all orders in* λ *(up to irr.)*

where $(\partial\Sigma)(0,0) := \Sigma(1,0) - \Sigma(0,0)$ *Taylor subtraction*

Ward identity:



Dyson equation



Supersymmetric quantum mechanics

Let X be a d-dimensional smooth manifold, T^*X trivial

- $a_\mu = e^{-\omega h}\partial_\mu e^{\omega h} = \partial_\mu + W_\mu$, $a^\dagger_\mu = -e^{\omega h}\partial_\mu e^{-\omega h} = -\partial_\mu + W_\mu$
  $h \in C^\infty(X)$ Morse function, $W_\mu = \omega\partial_\mu h$
- commutation relations:
  $[a_\mu, a_\nu] = [a^\dagger_\mu, a^\dagger_\nu] = 0$, $[a_\mu, a^\dagger_\nu] = 2\omega\partial_\mu\partial_\nu h$
- d fermionic ladder operators:
  $\{b_\mu, b_\nu\} = 0$, $\{b^\dagger_\mu, b^\dagger_\nu\} = 0$, $\{b_\mu, b^\dagger_\nu\} = \delta_{\mu\nu}$
- supercharges:
  $\mathfrak{Q} := \sum_{\mu=1}^d a_\mu \otimes b^\dagger_\mu$, $\mathfrak{Q}^\dagger := \sum_{\mu=1}^d a^\dagger_\mu \otimes b_\mu$
- supersymmetry algebra:
  $\{\mathfrak{Q}, \mathfrak{Q}^\dagger\} = \mathfrak{H} = \big(-\partial_\mu\partial^\mu + \omega^2(\partial_\mu h)(\partial^\mu h)\big)\otimes 1 + \omega(\partial^\mu\partial^\nu h)\otimes[b^\dagger_\mu, b_\nu]$
  $\{\mathfrak{Q}, \mathfrak{Q}\} = \{\mathfrak{Q}^\dagger, \mathfrak{Q}^\dagger\} = 0$, $[\mathfrak{Q}, \mathfrak{H}] = [\mathfrak{Q}^\dagger, \mathfrak{H}] = 0$

cohomology of $\mathfrak{Q}$ related to Morse theory for h

Harmonic oscillator spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}_i)$

Morse function $h = \frac{1}{2}\|x\|^2$

implies constant $[a_\mu, a_\nu^\dagger] = 2\omega\delta_{\mu\nu}$

Hilbert space $\mathcal{H} = \ell^2(\mathbb{N}^d) \otimes \bigwedge(\mathbb{C}^d)$: declare ONB
$\Big\{(a_1^\dagger)^{n_1} \dots (a_d^\dagger)^{n_d} \otimes (b_1^\dagger)^{s_1} \dots (b_d^\dagger)^{s_d} |0\rangle \;:\; n_\mu \in \mathbb{N}\,, s_\mu \in \{0,1\}\Big\}$

TWO Dirac operators $\mathcal{D}_1 = \mathfrak{Q} + \mathfrak{Q}^\dagger$, $\mathcal{D}_2 = \mathrm{i}\mathfrak{Q} - \mathrm{i}\mathfrak{Q}^\dagger$

$$\mathcal{D}_1^2 = \mathcal{D}_2^2 = \mathfrak{H} = \sum_{\mu=1}^d \big(a_\mu^\dagger a_\mu \otimes 1 + 2\omega \otimes b_\mu^\dagger b_\mu\big) = 2\omega(N_b + N_f) = H \otimes 1 + \omega \otimes \Sigma$$

where

$H = -\frac{\partial^2}{\partial x_\mu \partial x^\mu} + \omega^2 x_\mu x^\mu$ – harmonic oscillator hamiltonian

$\Sigma = \sum_{\mu=1}^d [b_\mu^\dagger, b_\mu]$ – spin matrix

algebra $\mathcal{A} = \mathcal{S}(\mathbb{R}^d)$ uniquely determined by smoothness
All axioms of spectral triples satisfied, with minor adaptation

Summary

- Renormalisation is compatible with noncommutative geometry
- We can renormalise models with new types of degrees of freedom, such as dynamical matrix models
- Equivalence of renormalisation schemes is confirmed
- Important tools (multi-scale analysis) are worked out
- Construction of NCQF theories is promising
- Other models
  - Gross-Neveu model $D = 2$ F. Vignes-Tourneret
  - Degenerate Θ matrix model H. G. F. Vignes-Tourneret
    needs five relevant/marginal operators !
  - Fermions
  - induced Yang-Mills theory ? A. de Goursac, J.-C. Wallet, R.Wulkenhaar; H. G, M. Wohlgenannt