Non-commutative Geometries via Modular Theory

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RIMS International Conference on
Noncommutative Geometry and Physics

Research Institute for Mathematical Sciences, Kyoto, Japan
09 November 2010

Abstract 1

We make use of Tomita-Takesaki modular theory in order to reconstruct non-commutative spectral geometries (formally similar to spectral triples) from suitable states over (categories of) operator algebras and further elaborate on the utility of such a formalism in an algebraic theory of quantum gravity, where space-time is spectrally reconstructed a posteriori from (partial) observables and states in a covariant quantum theory. Some relations with A.Carey-J.Phillips-A.Rennie modular spectral triples and with (loop) quantum gravity will be described.

Abstract 2

This is an ongoing joint research with

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MSC-2000: 46L87, 46L51, 46L10, 46M15, 18F99, 58B34, 81R60, 81T05, 83C65.

Keywords: Modular Theory, Non-commutative Geometry, Spectral Triple, Category Theory, Quantum Physics, Space-Time.

This research work is totally supported by my pocket ;-)

Outline

- Mathematical Preliminaries
  - Review of Tomita-Takesaki Modular Theory
  - Review of A. Connes' Non-commutative Geometry
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- Motivations from Physics
  - Spectral and Quantum Space-time
  - Modular Theory in Quantum Gravity

* Modular Algebraic Quantum Gravity

  * 1) Construction of Modular Spectral Geometries

  * 2) Physical Meaning of Modular Spectral Geometries

  * 5) Connections with Loop Quantum Gravity

  ? 6) Foundations of Quantum Physics

Ideology

* *space-time should be spectrally reconstructed a posteriori from a basic operational theory of observables and states;*
* *A.Connes' non-commutative geometry provides the natural environment where to attempt an implementation of the spectral reconstruction of space-time;*
* *Tomita-Takesaki modular theory should be the main tool to achieve the previous goals, associating to operational data, spectral non-commutative geometries.*
* *. . . (relationalism via categorical covariance).*

- Tomita-Takesaki Modular Theory

Modular Theory

Operator Algebras 1 – C*-algebras

A complex unital **algebra** $\mathcal{A}$ is a vector space over $\mathbb{C}$ with an associative unital bilinear multiplication.[1]

An **involution** on $\mathcal{A}$ is a conjugate linear map $* : \mathcal{A} \to \mathcal{A}$ such that $(a^*)^* = a$ and $(ab)^* = b^* a^*$, for all $a, b \in \mathcal{A}$.

An involutive complex unital algebra is $\mathcal{A}$ called a **C*-algebra** if $\mathcal{A}$ is a Banach space with a norm $a \mapsto \|a\|$ such that $\|ab\| \leq \|a\| \cdot \|b\|$ and $\|a^* a\| = \|a\|^2$, for all $a, b \in \mathcal{A}$.

Notable examples are the algebras of continuous complex valued functions $C(X; \mathbb{C})$ on a compact topological space with the "sup norm" and the algebras of linear bounded operators $\mathcal{B}(\mathcal{H})$ on a given Hilbert space $\mathcal{H}$.

[1] $\mathcal{A}$ is **Abelian** (commutative) if $ab = ba$, for all $a, b \in \mathcal{A}$.

Operator Algebras 2 – Von Neumann Algebras

A **von Neumann algebra** $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ is a C*-algebra acting on the Hilbert space $\mathcal{H}$ that is closed under the **weak-operator topology**: $A_n \xrightarrow{n\to\infty} A$ iff $\langle \xi \mid A_n\eta\rangle \xrightarrow{n\to\infty} \langle \xi \mid A\eta\rangle$, $\forall \xi, \eta \in \mathcal{H}$, or equivalently under the σ-**weak topology**:
$A_n \xrightarrow{n\to\infty} A$ iff for all sequences $(\xi_k), (\zeta_k)$ in $\mathcal{H}$ such that $\sum_{k=1}^{+\infty} \|\xi_k\|^2 < +\infty$ and $\sum_{k=1}^{+\infty} \|\zeta_k\|^2 < +\infty$ we have $\sum_{k=1}^{+\infty} \langle \xi_k \mid A_n\zeta_k\rangle \xrightarrow{n\to+\infty} \sum_{k=1}^{+\infty} \langle \xi_k \mid A\zeta_k\rangle$.

The **pre-dual** $\mathcal{M}_*$ of a von Neumann algebra $\mathcal{M}$ is the set of all σ-weakly continuous functionals on $\mathcal{M}$. It is a Banach subspace of the dual $\mathcal{M}^*$. The von Neumann algebra $\mathcal{M}$ is always the dual of $\mathcal{M}_*$. By a theorem of S.Sakai, a C*-algebra $\mathcal{A}$ is isomorphic to a von Neumann algebra $\mathcal{M}$ if and only if it is a dual of a Banach space.

Gel'fand Naĭmark Segal Representation

A **state** ω over a unital C*-algebra $\mathcal{A}$ is a linear function $\omega : \mathcal{A} \to \mathbb{C}$ that is positive $\omega(x^*x) \geq 0$ for all $x \in \mathcal{A}$ and normalized $\omega(1_{\mathcal{A}}) = 1$.

To every state ω over a unital C*-algebra $\mathcal{A}$ we can associate its **Gel'fand-Naĭmark-Segal representation** i.e. a unital $*$-homomorphism $\pi_\omega : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\omega)$ over a Hilbert space $\mathcal{H}_\omega$ with a norm one vector ξ_ω such that $\omega(x) = \langle \xi_\omega \mid \pi_\omega(x)\xi_\omega \rangle$ for all $x \in \mathcal{A}$.

Dynamical Systems 1

A **C*-dynamical system** $(\mathcal{A}, \alpha)$ is a C*-algebra $\mathcal{A}$ equipped with a group homomorphism $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ that is strongly continuous i.e. $g \mapsto \|\alpha_g(x)\|$ is a continuous map for all $x \in \mathcal{A}$.

Similarly a **von Neumann dynamical system** $(\mathcal{M}, \alpha)$ is a von Neumann algebra acting on the Hilbert space $\mathcal{H}$ equipped with a group homomorphism $\alpha : G \to \mathrm{Aut}(\mathcal{M})$ that is weakly continuous i.e. $g \mapsto \langle \xi \mid \alpha_g(x)\eta \rangle$ is continuous for all $x \in \mathcal{M}$ and all $\xi, \eta \in \mathcal{H}$.

Dynamical Systems 2

For a **one-parameter** (C* or von Neumann) dynamical system $(\mathcal{A}, \alpha)$, with $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$, an element $x \in \mathcal{A}$ is α-**analytic** if there exists a holomorphic extension of the map $t \mapsto \alpha_t(x)$ to an open horizontal strip $\{z \in \mathbb{C} \mid |\operatorname{Im} z| < r\}$, with $r > 0$, in the complex plane.

The set of α-analytic elements is always α-invariant (i.e. for all x analytic, $\alpha(x)$ is analytic) $*$-subalgebra of $\mathcal{A}$ that is norm dense in the C* case and weakly dense in the von Neumann case.

Kubo Martin Schwinger States

A state ω on a one-parameter C*-dynamical system $(\mathcal{A}, \alpha)$ is a (α, β)-**KMS state**, for $\beta \in \mathbb{R}$, if for all pairs of elements x, y in a norm dense α-invariant $*$-subalgebra of α-analytic elements of $\mathcal{A}$ we have $\omega(x\alpha_{i\beta}(y)) = \omega(yx)$.

In the case of a von Neumann dynamical system $(\mathcal{M}, \alpha)$, a (α, β)-KMS state must be normal[2] and should satisfy the above property for all pairs of elements in a weakly dense α-invariant $*$-subalgebra of α-analytic elements of $\mathcal{M}$.

[2] ω is **faithful** if $\omega(x) = 0 \Rightarrow x = 0$; it is **normal** if for every increasing bounded net of positive elements $x_\lambda \to x$, we have $\omega(x_\lambda) \to \omega(x)$.

Modular Theory 1

The **modular theory** of von Neumann algebras has been created by M.Tomita in 1967 and perfectioned by M.Takesaki around 1970. It is a very deep theory that, to every von Neumann algebra $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ acting on a Hilbert space $\mathcal{H}$, and to every vector $\xi \in \mathcal{H}$ that is cyclic i.e.

$$\overline{(\mathcal{M}\xi)} = \mathcal{H}$$

and separating i.e. for $A \in \mathcal{M}$,

$$A\xi = 0 \Rightarrow A = 0,$$

associates:

Modular Theory 2

1) a one parameter unitary group $t \mapsto \Delta_\xi^{it} \in \mathcal{B}(\mathcal{H})$
2) and a conjugate-linear isometry $J_\xi : \mathcal{H} \to \mathcal{H}$ such that:

$$\Delta_\xi^{it} \mathcal{M} \Delta_\xi^{-it} = \mathcal{M}, \quad \forall t \in \mathbb{R},$$
$$J_\xi \mathcal{M} J_\xi = \mathcal{M}',$$
$$J_\xi \circ J_\xi = \mathrm{Id}_{\mathcal{H}}, \quad J_\xi \circ \Delta_\xi = \Delta_\xi^{-1} \circ J_\xi,$$

where the **commutant** $\mathcal{M}'$ of $\mathcal{M}$ is defined by:

$$\mathcal{M}' := \{A' \in \mathcal{B}(\mathcal{H}) \mid [A', A]_- = 0, \ \forall A \in \mathcal{M}\}.$$

Modular Theory 3

More generally, given

- a von Neumann algebra $\mathcal{M}$ and
- a faithful normal state[3] (more generally for a faithful normal semi-finite weight) ω on the algebra $\mathcal{M}$,

the modular theory allows to create:

- a one parameter group of $*$-automorphisms of the algebra $\mathcal{M}$,

$$\sigma^\omega : t \mapsto \sigma_t^\omega \in \mathrm{Aut}(\mathcal{M}), \quad \text{with} \quad t \in \mathbb{R},$$

such that:

[3] ω is faithful if $\omega(x) = 0 \Rightarrow x = 0$; it is normal if for every increasing bounded net of positive elements $x_\lambda \to x$, we have $\omega(x_\lambda) \to \omega(x)$.

Modular Theory 4

- in the Gel'fand-Naĭmark-Segal representation π_ω induced by the weight ω, on the Hilbert space $\mathcal{H}_\omega$, the automorphism group σ^ω is implemented by a unitary one parameter group $t \mapsto \Delta_\omega^{it} \in \mathcal{B}(\mathcal{H})$:
$$\pi_\omega(\sigma_t^\omega(x)) = \Delta_\omega^{it}\pi_\omega(x)\Delta_\omega^{-it}, \quad \forall x \in \mathcal{M}, \quad \forall t \in \mathbb{R};$$
- there is a conjugate-linear isometry $J_\omega : \mathcal{H} \to \mathcal{H}$, with $J_\omega^2 = \mathrm{Id}_{\mathcal{H}_\omega}$ and $J_\omega\Delta_\omega = \Delta_\omega^{-1}J_\omega$, whose adjoint action implements a conjugate-linear $*$-isomorphism $\gamma_\omega : \pi_\omega(\mathcal{M}) \to \pi_\omega(\mathcal{M})'$, between $\pi_\omega(\mathcal{M})$ and its commutant:
$$\gamma_\omega(\pi_\omega(x)) = J_\omega\pi_\omega(x)J_\omega, \quad \forall x \in \mathcal{M}.$$

Modular Theory 5

The operators J_ω and Δ_ω are called respectively the **modular conjugation operator** and the **modular operator** induced by the state (weight) ω.
We will call "**modular generator**" the operator

$$K_\omega := \log \Delta_\omega$$

i.e the self-adjoint generator of the unitary one parameter group

$$t \mapsto \Delta^{it} = e^{iK_\omega t}.$$

Modular Theory 6

The modular automorphism group associated with ω is the only one parameter automorphism group that satisfies the Kubo-Martin-Schwinger **KMS-condition** with respect to the state ω, at inverse temperature $\beta = -1$, i.e.

$$\omega(\sigma_t^\omega(x)) = \omega(x), \quad \forall x \in \mathcal{M}$$

and for all $x, y \in \mathcal{M}$, there exists a function $F_{x,y} : \mathbb{R} \times [0, \beta] \to \mathbb{C}$ such that:

$$\begin{gathered} F_{x,y} \quad \text{is holomorphic on } \mathbb{R}\times]0, \beta[, \\ F_{x,y} \quad \text{is bounded continuous on } \mathbb{R} \times [0, \beta], \\ F_{x,y}(t) = \omega(\sigma_t^\omega(y)x), \quad t \in \mathbb{R}, \\ F_{x,y}(i\beta + t) = \omega(x\sigma_t^\omega(y)), \quad t \in \mathbb{R}. \end{gathered}$$

nc-geometry

Modular Theory 7

A **weight** ω on a C*-algebra $\mathcal{A}$ is a map $\omega : \mathcal{A}_+ \to [0, +\infty]$ such that $\omega(x+y) = \omega(x) + \omega(y)$ and $\omega(\alpha x) = \alpha\omega(x)$, for all $x, y \in \mathcal{A}_+$ and $\alpha \in \mathbb{R}_+$. A **trace** is a weight that, for all $x \in \mathcal{A}$, satisfies $\omega(x^* x) = \omega(x x^*)$.

The usual **GNS-representation** associated to states admits a similar formulation in the case of weights. To every weight ω on the C*-algebra $\mathcal{A}$ there is a triple $(\mathcal{H}_\omega, \pi_\omega, \eta_\omega)$, where $\mathcal{H}_\omega$ is a Hilbert space, π_ω is a $*$-representation of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H}_\omega)$ and $\eta_\omega : \mathfrak{L}_\omega \to \mathcal{H}_\omega$ is a linear map with dense image defined on the left ideal $\mathfrak{L}_\omega := \{x \in \mathcal{A} \mid \omega(x^* x) < +\infty\}$, such that $\pi_\omega(x)\eta_\omega(z) = \eta_\omega(xz)$ and $\omega(y^* xz) = \langle \eta_\omega(y) \mid \pi_\omega(x)\eta_\omega(z)\rangle_{\mathcal{H}_\omega}$ for all $x \in \mathcal{A}$ and all $y, z \in \mathfrak{L}_\omega$.

Modular Theory 8

A weight is **faithful** if $\omega(x) = 0$ implies $x = 0$. A weight on a von Neumann algebra $\mathcal{M}$ is **normal** if for every increasing bounded net in $\mathcal{M}_+$ with $x_\lambda \to x \in \mathcal{M}_+$ we have $\omega(x_\lambda) \to \omega(x)$ and it is **semi-finite** if the linear span of the cone $\mathfrak{M}_{\omega+} := \{x \in \mathcal{M}_+ \ \ \omega(x) < +\infty\}$ is dense in the σ-weak operator topology in $\mathcal{M}$.

Modular Theory 9

Tomita-Takesaki modular theory can be extended to the case of normal semi-finite faithful weights on a von Neumann algebra and the formulation is essentially identical to the one already described in case of states.

A von Neumann algebra $\mathcal{M}$ is **semi-finite** if and only if it admits a normal semi-finite faithful trace τ. In this case for every normal semi-finite faithful weight ω, the modular automorphism group $t \mapsto \sigma_t^\omega$ is inner i.e. there exists a positive invertible operator h affiliated[4] to $\mathcal{M}$ such that $\sigma_t^\omega(x) = h^{it} x h^{-it}$ for all $t \in \mathbb{R}$ and $x \in \mathcal{M}$.

[4] This means that all the spectral projections of h are contained in $\mathcal{M}$.

Connes-Radon-Nikodym Theorem – Part 1

Let ϕ be a normal semi-finite faithful weight on the von Neumann algebra $\mathcal{M}$. For every other normal semi-finite faithful weight ψ on $\mathcal{M}$, there exists a strongly continuous family $t \mapsto u_t$ of unitaries in $\mathcal{M}$ such that for all $x \in \mathcal{M}$ and all $t, s \in \mathbb{R}$:

$$\sigma_t^{\psi}(x) = u_t \sigma_t^{\phi}(x) u_t^*, \quad u_{t+s} = u_t \sigma_t^{\phi}(u_s).$$

Furthermore, defining $\sigma_t^{\psi,\phi}(x) := u_t \sigma_t^{\phi}(x) = \sigma_t^{\psi}(x) u_t$, there exists a unique such family, denoted by $t \mapsto (D\phi : D\phi)_t$ for all $t \in \mathbb{R}$, and called the **Connes-Radon-Nikodym derivative** *of ψ with respect to ϕ, that satisfies the following variant of the KMS-condition: there exists a bounded continuous function on $\mathbb{R} \times [0, 1]$ analytic on $\mathbb{R} \times]0, 1[$ such that for all $x, y \in \mathfrak{L}_{\phi} \cap \mathfrak{L}_{\psi}^*$ and for all $t \in \mathbb{R}$, $f(t) = \psi(\sigma_t^{\psi,\phi}(x) y)$ and $f(t + i) = \phi(y \sigma_t^{\psi,\phi}(x))$.*

Connes-Radon-Nikodym Theorem – Part 2

If $t \mapsto u_t$ is a strongly continuous family of unitaries in $\mathcal{M}$ such that $u_{t+s} = u_t \sigma_t^\phi(u_s)$, for all $t, s \in \mathbb{R}$, there exists a unique normal semi-finite faithful weight ψ on $\mathcal{M}$ such that $(D\psi : D\phi)_t = u_t$, for all $t \in \mathbb{R}$.

The Connes-Radon-Nikodym derivatives satisfy the following properties for all normal semi-finite faithful weights $\omega_1, \omega_2, \omega_3$ on $\mathcal{M}$ and for all $t \in \mathbb{R}$:

$$(D\omega_1 : D\omega_2)_t \cdot (D\omega_2 : D\omega_3)_t = (D\omega_1 : D\omega_3)_t,$$
$$(D\omega_1 : D\omega_2)_t^* = (D\omega_2 : D\omega_1)_t.$$

Connes Spatial Derivative Theorem

Let $\mathcal{M}$ be a von Neumann algebra on the Hilbert space $\mathcal{H}$ and $\mathcal{M}'$ its commutant. For any normal semi-finite faithful weight ω on $\mathcal{M}$ and any normal semi-finite faithful weight ω' on $\mathcal{M}'$, there exists a positive operator $\Delta(\omega|\omega')$, the **Connes' spatial derivative** *of ω with respect to ω' such that: $\Delta(\omega'|\omega) = \Delta(\omega|\omega')^{-1}$ and $\forall t \in \mathbb{R}$,*

$$\sigma_t^{\omega}(x) = \Delta(\omega|\omega')^{it} x \Delta^{-it}(\omega|\omega') \quad \forall x \in \mathcal{M},$$
$$\sigma_{-t}^{\omega'}(y) = \Delta(\omega|\omega')^{it} y \Delta(\omega|\omega')^{-it} \quad \forall y \in \mathcal{M}'.$$

Furthermore, if ω_1 and ω_2 are normal semi-finite faithful weights on $\mathcal{M}$ we also have

$$\Delta(\omega_2|\omega')^{it} = (D\omega_2 : D\omega_1)_t \Delta(\omega_1|\omega')^{it}, \quad \forall t \in \mathbb{R}.$$

Conditional Expectations

A **conditional expectation** $\Phi : \mathcal{A} \to \mathcal{B}$ from a unital C*-algebra $\mathcal{A}$ onto a unital C*-subalgebra $\mathcal{B}$ is a completely positive map[5] such that $\Phi(b) = b$, $\Phi(x + y) = \Phi(x) + \Phi(y)$, $\Phi(b_1 x b_2) = b_1 \Phi(x) b_2$, for all $b, b_1, b_2 \in \mathcal{B}$ and all $x, y \in \mathcal{A}$.

By a theorem of J.Tomiyama, Φ is a conditional expectation if and only if Φ is a projection of norm one onto a subalgebra.

A conditional expectation is a generalization of the notion of state that appears as long as we allow values to be taken in an arbitrary C*-algebra in place of the usual complex numbers $\mathbb{C}$.

[5] This means that for all $n \in \mathbb{N}$, $\Phi^{(n)} : \mathbb{M}_n(\mathcal{A}) \to \mathbb{M}_n(\mathcal{B})$ is positive, where $\mathbb{M}_n(\mathcal{A})$ denotes the unital C*-algebra of $n \times n$ $\mathcal{A}$-valued matrices and $\Phi^{(n)}$ is obtained applying Φ to every entry.

Takesaki Conditional Expectation Theorem 1

Conditional expectations and modular theory are related by this result by M.Takesaki.

Theorem

Let $\mathcal{N}$ be a von Neumann subalgebra of the von Neumann algebra $\mathcal{M}$ and let ω be a normal semi-finite faithful weight on the von Neumann algebra $\mathcal{M}$ such that $\omega|_{\mathcal{N}}$ is semi-finite. The von Neumann algebra $\mathcal{N}$ is **modularly stable** *i.e. $\sigma_t^\omega(\mathcal{N}) = \mathcal{N}$ for all $t \in \mathbb{R}$, if and only if there exists a conditional expectation $\Phi : \mathcal{M} \to \mathcal{N}$ onto $\mathcal{N}$ such that $\omega \circ \Phi = \omega$.*

Such conditional expectation is unique and normal.

Operator Valued Weights

In the same way as weights are an "unbounded" version of states, we also have an "unbounded" version of conditional expectations. Here the role of real $\mathbb{R}$ or positive real numbers $\mathbb{R}_+$ as possible values of a state, respectively weight, is taken by a von Neumann algebra $\mathcal{N}$ and its positive part $\mathcal{N}_+$ and the set $\widehat{\mathbb{R}}_+ := [0, +\infty]$ of extended positive reals is replaced by $\widehat{\mathcal{N}}_+$, the **extended positive cone** of $\mathcal{N}$, defined as the set of lower semi-continuous maps $m : \mathcal{M}_{*+} \to [0, +\infty]$ such that $m(\phi + \psi) = m(\phi) + m(\psi)$, $m(\alpha\phi) = \alpha m(\phi)$, $\forall \phi, \psi \in \mathcal{M}_{*+}$, $\alpha \in \mathbb{R}_+$.

An **operator valued weight** from the von Neumann algebra $\mathcal{M}$ to the von Neumann algebra $\mathcal{N}$ is a map $\Phi : \mathcal{M}_+ \to \widehat{\mathcal{N}}_+$ taking values in the extended positive cone of $\mathcal{N}$ such that: $\Phi(x + y) = \Phi(x) + \Phi(y)$, $\Phi(\alpha x) = \alpha\Phi(x)$ and $\Phi(u^* x u) = \Phi(x)$, $\forall x, y \in \mathcal{M}_+$, $\alpha \in \mathbb{R}_+$, $u \in \mathcal{N}$.

Takesaki Operator Valued Weight Theorem

With these definitions, Takesaki's conditional expectation theorem can be generalized as follows.

Theorem

The existence of a normal semi-finite faithful operator valued weight Φ *onto a subalgebra* $\mathcal{N}$ *of the von Neumann algebra* $\mathcal{M}$ *is equivalent to the existence of a pair of normal semi-finite faithful weights* ω *on* $\mathcal{N}$ *and* $\widetilde{\omega}$ *on* $\mathcal{M}$ *such that* $\sigma_t^{\omega}(x) = \sigma_t^{\widetilde{\omega}}(x)$ *for all* $x \in \mathcal{N}$*. There is a unique such* Φ *with the property* $\widetilde{\omega} = \Phi \circ \omega$*.*

Modular Theory for Operator Valued Weights

- For every normal semi-finite faithful operator valued weight Θ from $\mathcal{M}$ to $\mathcal{N}$, and for every normal semi-finite faithful weight ϕ on $\mathcal{N}$, the restriction of the modular one-parameter group $t \mapsto \sigma_t^{\phi\circ\Theta}$ to the relative commutant $\mathcal{M} \cap \mathcal{N}'$ is independent from the choice of ϕ and defines the **modular group** $t \mapsto \sigma_t^{\Theta}$ **of the operator valued weight**.
- Similarly, for every pair of normal semi-finite faithful operator valued weights Θ, Ξ from $\mathcal{M}$ to $\mathcal{N}$, for every normal semi-finite faithful weight ϕ on $\mathcal{N}$, $[D\phi \circ \Theta : D\phi \circ \Xi]_t$ is independent of the choice of ϕ and defines the **cocycle derivative of** Θ **relative to** Ξ.
- To every normal semi-finite faithful weight Θ from $\mathcal{M}$ to $\mathcal{N}$ corresponds a unique normal semi-finite faithful weight Θ' from $\mathcal{N}'$ to $\mathcal{M}'$ such that $\sigma_t^{\Theta} = \sigma_{-t}^{\Theta'}$. Under such bijective correspondence $[D\Theta : D\Xi]_t = [D\Xi' : D\Theta']_{-t}$.

Connes-Takesaki Duality and Falcone Takesaki Non-commutative Flow of Weights 1

Let $\mathcal{M}$ be a von Neumann algebra. There exist a canonical one-parameter W-dynamical system, the* **Falcone- Takesaki non-commutative flow of weights** *of $\mathcal{M}$, $(\widetilde{\mathcal{M}}, \theta)$ and a canonical normal $*$-morphism $\iota : \mathcal{M} \to \widetilde{\mathcal{M}}$ such that:*

- *the image of the canonical isomorphism coincides with the fixed points algebra of the dynamical system i.e. $\iota(\mathcal{M}) = \widetilde{\mathcal{M}}^\theta$,*
- *for every faithful semi-finite normal weight ϕ on $\mathcal{M}$ there is a canonical isomorphism of the W*-dynamical system $(\widetilde{\mathcal{M}}, \theta)$ with the W*-dynamical system $(W^*(\mathcal{M}, \sigma^\phi), \widehat{\sigma}^\phi)$ induced by the dual action of σ^ϕ on the W*-covariance algebra of $(\mathcal{M}, \sigma^\phi)$,*

Connes-Takesaki Duality and Falcone Takesaki Non-commutative Flow of Weights 2

- *there is a canonical operator valued weight Θ from $\widetilde{\mathcal{M}}$ onto $\iota(\mathcal{M})$, given for all $x \in \widetilde{\mathcal{M}}_+$ by $\Theta(x) = \int \theta_t(x)\, dt$, such that, for every faithful semi-finite normal weight ϕ on $\mathcal{M}$, the dual faithful semi-finite normal weight $\widetilde{\phi} := \phi \circ \Theta$ on $\widetilde{\mathcal{M}}$ induces an inner modular automorphism group i.e. $\sigma_t^{\widetilde{\phi}} = \mathrm{Ad}_{e^{ik_\phi t}}$ with generator k_ϕ affiliated to $\widetilde{\mathcal{M}}$,*
- *there is a canonical faithful semi-finite normal trace τ on $\widetilde{\mathcal{M}}$ that is rescaling the one-parameter group θ i.e. $\tau \circ \theta_t = e^{-t}\tau$, for all $t \in \mathbb{R}$ and for all faithful semi-finite normal weights ϕ on $\mathcal{M}$ we have that $\tau(x) = \widetilde{\phi}(e^{-k_\phi/2} x e^{-k_\phi/2})$, $\forall x \in \widetilde{\mathcal{M}}_+$,*

Connes-Takesaki Duality and Falcone Takesaki Non-commutative Flow of Weights 3

- *for all faithful semi-finite normal weights ϕ on $\mathcal{M}$, we have that the W*-dynamical system $(W^*(\widetilde{\mathcal{M}}, \theta), \widehat{\theta})$ induced by the dual action of θ on the W*-covariance algebra $W^*(\widetilde{\mathcal{M}}, \theta)$ of $(\widetilde{\mathcal{M}}, \theta)$ is canonically isomorphic with the W*-dynamical system $(\mathcal{M} \otimes \mathcal{B}(L^2(\mathbb{R})), \sigma^\phi \otimes \rho)$, where $\rho_t := \mathrm{Ad}_{\lambda_{-t}}$ with $(\lambda_t \xi)(s) := \xi(s - t)$ the usual left regular action of $\mathbb{R}$ on $L^2(\mathbb{R})$.*

- Non-commutative Geometry

Non-commutative Geometry 1

Non-commutative geometry, elaborated by A.Connes starting approximately from 1980, is the name of a very fast developing mathematical theory that is making use of operator algebras to find algebraic generalizations of most of the structures currently available in mathematics: measurable, topological, differential, metric etc.

The fundamental idea, implicitly used in A.Connes' non-commutative geometry is a powerful extension of R.Decartes' analytic geometry:

Non-commutative Geometry 2

- to "trade" "geometrical spaces" X of points with their Abelian algebras of (say complex valued) functions $f : X \to \mathbb{C}$,
- to "translate" the geometrical properties of spaces into algebraic properties of the associated (commutative) algebras
- to "spectrally reconstruct", in the commutative case, the original geometric space X as a derived entity (the spectrum of the algebra)
- to define a non-commutative space (topological, measurable, differential, metric, ...) as the "dual" of a non-commutative algebra that satisfies the suitable algebraic axioms

The existence of dualities between categories of "geometrical spaces" and categories "constructed from Abelian algebras" is the starting point of any generalization of geometry to the non-commutative situation.

Non-commutative Geometry 3

Here are some examples:

- **Hilbert:** between algebraic sets and finitely generated algebras over an algebraically closed field.
- **Stone:** between totally disconnected compact Hausdorff topological spaces and Boolean algebras.
- **Gel'fand-Naĭmark:** between the category of continuous maps of compact Hausdorff topological spaces and the category of unital involutive homomorphisms of unital commutative C*-algebras.
- **Halmos-von Neumann:** between the category of measure spaces and the category of commutative von Neumann algebras.

Non-commutative Geometry 4

- **Serre-Swan (equivalence):** between the category of finite-dimensional locally trivial vector bundles over a compact Hausdorff topological space and the category of finite projective modules over a commutative unital C*-algebra.
- **Takahashi:** between the category of Hilbert bundles on (different) compact Hausdorff spaces and the category of Hilbert C*-modules over (different) commutative unital C*-algebras.

For "suitable categories of manifolds" the most appropriate candidate objects for a duality are A.Connes' spectral triples ...and possibly some variants of them ...

Connes Spectral Triples 1

A (compact) **spectral triple** $(\mathcal{A}, \mathcal{H}, D)$ is given by:

- a unital pre-C*-algebra $\mathcal{A}$ (usually closed under holomorphic functional calculus);
- a (faithful) representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ of $\mathcal{A}$ on the Hilbert space $\mathcal{H}$;
- a one-parameter group of of unitaries whose generator D, the Dirac operator, is such that
  - the domain $\mathrm{Dom}(D)$ is invariant under all the operators $\pi(a)$, with $a \in \mathcal{A}$,
  - all the commutators $[D, \pi(a)]_- := D \circ \pi(a) - \pi(a) \circ D$, defined on $\mathrm{Dom}(D)$, can be extended to bounded linear operators on $\mathcal{H}$,
  - the resolvent $(D - \mu I)^{-1}$ is compact for all $\mu \notin \mathrm{Sp}(D)$.

Several additional technical conditions should be imposed on a spectral triple in order to formulate "reconstruction results".

Other Spectral Geometries
Examples and Connes' Reconstruction Theorem

Connes Spectral Triples 2

- A spectral triple is called **even** if there exists a **grading operator**, i.e. a bounded self-adjoint operator $\Gamma \in \mathcal{B}(\mathcal{H})$ such that:

$$\Gamma^2 = \mathrm{Id}_{\mathcal{H}}; \quad [\Gamma, \pi(a)]_- = 0, \ \forall a \in \mathcal{A}; \quad [\Gamma, D]_+ = 0,$$

where $[x, y]_+ := xy + yx$ is the anticommutator of x, y.
A spectral triple that is not even is called **odd**.

Connes Spectral Triples 3

- A spectral triple is n-**dimensional** iff there exists an integer n such that the Dixmier trace of $|D|^{-n}$ is finite nonzero.
- A spectral triple is θ-**summable** if $\exp(-tD^2)$ is a trace-class operator for all $t > 0$.
- A spectral triple is **finite** if $\mathcal{H}_\infty := \cap_{k=1}^{\infty} \mathrm{Dom}\, D^k$ is a finite projective $\mathcal{A}$-bimodule and **absolutely continuous** if there exists an Hermitian form $(\xi, \eta) \mapsto (\xi \mid \eta)$ on $\mathcal{H}_\infty$ such that, for all $a \in \mathcal{A}$, $\langle \xi \mid \pi(a)\eta \rangle$ is the Dixmier trace of $\pi(a)(\xi \mid \eta)|D|^{-n}$.

Connes Spectral Triples 4

- A spectral triple is **regular** if the function

$$\Xi_x : t \mapsto \exp(it|D|)x\exp(-it|D|)$$

is regular, i.e. $\Xi_x \in \mathrm{C}^\infty(\mathbb{R}, \mathcal{B}(\mathcal{H}))$,[6]
for every $x \in \Omega_D(\mathcal{A})$, where[7]

$$\Omega_D(\mathcal{A}) := \operatorname{span}\{\pi(a_0)[D,\pi(a_1)]_- \cdots [D,\pi(a_n)]_- \mid \ n \in \mathbb{N}, \\ a_0, \dots, a_n \in \mathcal{A}\} .$$

[6] This condition is equivalent to $\pi(a), [D,\pi(a)]_- \in \cap_{m=1}^\infty \mathrm{Dom}\,\delta^m$, for all $a \in \mathcal{A}$, where δ is the derivation given by $\delta(x) := [|D|, x]_-$.

[7] We assume that for $n = 0$ the term in the formula simply reduces to $\pi(a_0)$.

Connes Spectral Triples 5

- An n-dimensional spectral triple is said to be **orientable** if in the non-commutative Clifford algebra $\Omega_D(\mathcal{A})$ there is a volume element $\sum_{j=1}^{m} \pi(a_0^{(j)})[D, \pi(a_1^{(j)})]_- \cdots [D, \pi(a_n^{(j)})]_-$ that coincides with the grading operator Γ in the even case or the identity operator in the odd case.[8]

[8] In the following, in order to simplify the discussion, we will always refer to a "grading operator" Γ that actually coincides with the grading operator in the even case and that is by definition the identity operator in the odd case.

Connes Spectral Triples 6

- A spectral triple is said to have **real structure** if there exists an antiunitary operator $J : \mathcal{H} \to \mathcal{H}$ such that:

$$[\pi(a), J\pi(b^*)J^{-1}]_- = 0, \quad \forall a, b \in \mathcal{A},$$
$$[\,[D, \pi(a)]_-, J\pi(b^*)J^{-1}]_- = 0, \quad \forall a, b \in \mathcal{A},$$
$$J^2 = \pm \mathrm{Id}_{\mathcal{H}}, \quad [J, D]_\pm = 0 \quad \text{and, in the even case,} \quad [J, \Gamma]_\pm = 0,$$

where the choice of $\pm$ in the last three formulas depends on the "dimension" n of the spectral triple modulo 8 in accordance to the following table:

n	0	1	2	3	4	5	6	7
$J^2 = \pm \mathrm{Id}_{\mathcal{H}}$	$+$	$+$	$-$	$-$	$-$	$-$	$+$	$+$
$[J, D]_\pm = 0$	$-$	$+$	$-$	$-$	$-$	$+$	$-$	$-$
$[J, \Gamma]_\pm = 0$	$-$		$+$		$-$		$+$	

Connes Spectral Triples 7

- a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ satisfies **Poincaré duality** if the C*-module completion of $\mathcal{H}_\infty$ is a Morita equivalence bimodule between (the norm completions of) $\mathcal{A}$ and $\Omega_D(\mathcal{A})$.[9]
- A spectral triple will be called **Abelian** or commutative whenever $\mathcal{A}$ is Abelian.
- A spectral triple is **irreducible** if there is no non-trivial closed subspace in $\mathcal{H}$ that is invariant for $\pi(\mathcal{A}), D, J, \Gamma$.

[9] A.Connes formulated this requirement in a different form: "A real spectral triple is said to satisfy **Poincaré duality** if its fundamental class in the KR-homology of $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ induces (via Kasparov intersection product) an isomorphism between the K-theory $K_\bullet(\mathcal{A})$ and the K-homology $K^\bullet(\mathcal{A})$ of $\mathcal{A}$". See A.Rennie-J.Varilly `arXiv:math.OA/0610418`, `arXiv:math.OA/0703719` for some details on the equivalence of these statements.

Connes Spectral Triples 8

Given an orientable compact Riemannian spin m-dimensional differentiable manifold M, with a given complex spinor bundle $S(M)$, a given spinorial charge conjugation C_M and a given volume form μ_M, define by

- $\mathcal{A}_M := C^\infty(M;\mathbb{C})$ the algebra of complex valued regular functions on the differentiable manifold M,
- $\mathcal{H}_M := \mathrm{L}^2(M;S(M))$ the Hilbert space of "square integrable" sections of the given spinor bundle $S(M)$ of the manifold M i.e. the completion of the space $\Gamma^\infty(M;S(M))$ of smooth sections of the spinor bundle $S(M)$ equipped with the inner product $\langle\sigma \mid \tau\rangle := \int_M \langle\sigma(p) \mid \tau(p)\rangle_p \, \mathrm{d}\mu_M$, where $\langle \mid \rangle_p$, with $p \in M$, is the unique inner product on $S_p(M)$ compatible with the Clifford action and the Clifford product.

Connes Spectral Triples 9

- D_M the Atiyah-Singer Dirac operator i.e. the closure of the operator that on $\Gamma^\infty(M; S(M))$ is obtained by "contracting" with the Clifford multiplication, the unique spinorial covariant derivative $\nabla^{S(M)}$ (induced on $\Gamma^\infty(M; S(M))$ by the Levi-Civita covariant derivative of M);
- J_M the unique antilinear unitary extension $J_M : \mathcal{H}_M \to \mathcal{H}_M$ of the operator determined by the spinorial charge conjugation C_M by $(J_M\sigma)(p) := C_M(\sigma(p))$ for $\sigma \in \Gamma^\infty(M; S(M))$, $p \in M$;
- Γ_M the unique unitary extension on $\mathcal{H}_M$ of the operator given by fiberwise grading on $S_p(M)$, with $p \in M$.[10]

[10] The grading is actually the identity in odd dimension.

Connes, Rennie-Varilly Theorem 1

Theorem (Connes)

The data $(\mathcal{A}_M, \mathcal{H}_M, D_M)$ define an Abelian regular finite m-dimensional spectral triple that is real, with real structure J_M, orientable, with grading Γ_M, and that satisfies Poincaré duality.

Connes, Rennie-Varilly Theorem 2

Theorem (Connes, Rennie-Varilly)

Let $(\mathcal{A}, \mathcal{H}, D)$ be an irreducible Abelian real (with real structure J and grading Γ) strongly regular[11] *m-dimensional finite absolutely continuous orientable spectral triple, with totally antisymmetric Hochschild cocycle in the last m entries and satisfying Poincaré duality.*
The spectrum of (the norm closure of) $\mathcal{A}$ can be endowed, with the structure of an m-dimensional connected compact spin Riemannian manifold M with an irreducible complex spinor bundle $S(M)$, a charge conjugation J_M and a grading Γ_M such that:
$\mathcal{A} \simeq C^\infty(M; \mathbb{C})$, $\mathcal{H} \simeq L^2(M, S(M))$, $D \simeq D_M$, $J \simeq J_M$, $\Gamma \simeq \Gamma_M$.

[11] In the sense described in A.Connes `arXiv:0810.2088v1`.

Antonescu Christensen Triples for AF C*-algebras

Among the spectral triples that are "purely quantal", we mention the following construction.[12]

Given a filtration of unital finite dimensional C*-algebras $\mathcal{A}_0 := \mathbb{C}1_{\mathcal{A}} \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_n \subset \mathcal{A}_{n+1} \subset \cdots$ and a faithful state ω on the inductive limit of the filtration $\mathcal{A} := (\cup_{n=1}^{+\infty} \mathcal{A}_n)^-$ with GNS representation $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$, denote by $P_n \in \mathcal{B}(\mathcal{H}_\omega)$ the othogonal projection onto $\pi_\omega(\mathcal{A}_n)\xi_\omega$, by $E_n := P_n - P_{n-1}$ (we assume $E_0 := P_0$) and by θ_n the continuous projection of $\mathcal{A}$ onto $\mathcal{A}_n$ (that satisfies $\theta_n(a)\xi_\omega = P_n a \xi_\omega$). For any sequence (β_n) such that $\sum_{n=1}^{+\infty} \beta_n < +\infty$ and any sequence (γ_n) such that $\|\theta_n(a) - \theta_{n-1}(a)\| \leq \gamma_n \|E_n a \xi_\omega\|$ for all $a \in \mathcal{A}$, there is a family of spectral triples $(\mathcal{A}, \mathcal{H}_\omega, D_{(\alpha_n)})$, indexed by a sequence of positive real numbers $(\alpha_n) := (\gamma_n/\beta_n)$, with $D_{(\alpha_n)} := \sum_{n=1}^{+\infty} \alpha_n E_n$.

[12] E.Christensen, C.Ivan, `arXiv:math.OA/0309044`.

Other Spectral Geometries 1

Several other variants for the axioms of spectral triples have been considered or proposed:

- **non-compact spectral triples**,[13] see the talk by A.Carey ...
- **spectral triples for quantum groups**,[14]
- **Lorentzian spectral triples**,[15]

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[13] V.Gayral, J.M.Gracia-Bondia, B.Iochum, T.Schüker, J.C.Varilly, `hep-th/0307241`.

[14] L.Dabrowski, G.Landi, A.Sitarz, W.van Suijlekom, J.Varilly, `math.QA/0411609`.

[15] A.Strohmaier, `math-ph/0110001`;
M.Paschke, A.Sitarz `math-ph/0611029`;
M.Borris, R.Verch `arXiv:0812.0786`;
A.Rennie, in preparation.

Other Spectral Geometries 2

- **non-commutative Riemannian manifolds** and **non-commutative phase-spaces**[16]; **non-commutative Riemannian geometries**[17] (Here the basic "commutative example" is the triple $(C^\infty(M), L^2(\Lambda_\bullet(M)), d + d*)$ for a Riemannian manifold M),
- **"von Neumann" semi-finite and modular spectral triples.**[18],[19] that are of paramount importance here.

[16] J.Fröhlich, O.Grandjean, A.Recknagel, hep-th/9612205; mat-ph/9807006.
[17] S.Lord, Riemannian Geometries, math-ph/0010037;
S.Lord, A.Rennie, J.Varilly, in preparation.
[18] M-T.Benameur, T.Fack, math.KT/0012233.
[19] See for example: M-T.Benameur, A.Carey, J.Phillips, A.Rennie, F.Sukochev, K.Wojciechowski, math.OA/0512454, A.Carey, J.Phillips, A.Rennie, arXiv:0707.3853, arXiv:0801.4605, Semi-finite Non-commutative Geometry and Some Applications ESI preprint 2061 (2008).

- Modular Non-commutative Geometry

Semi-finite Spectral Triples

A **semi-finite spectral triple** $(\mathcal{A}, \mathcal{H}, D)$ relative to a normal semi-finite faithful trace τ on a semi-finite von Neumann algebra $\mathcal{N}$, is given by:

- a faithful representation $\pi : \mathcal{A} \to \mathcal{N} \subset \mathcal{B}(\mathcal{H})$ of a unital $*$-algebra $\mathcal{A}$ inside a semi-finite von Neumann algebra $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ acting on the Hilbert space $\mathcal{H}$,
- a (non-necessarily bounded) self-adjoint operator D on the Hilbert space $\mathcal{H}$ such that
  - the domain $\mathrm{Dom}(D) \subset \mathcal{H}$ of D is invariant under all the elements $\pi(x) \in \pi(\mathcal{A})$,
  - the operators $[D, \pi(x)]_-$ defined on $\mathrm{Dom}(D)$ can be extended to bounded operators in the von Neumann algebra $\mathcal{N}$,
  - for all $\mu \notin \mathrm{Sp}(D)$, the resolvent $(D - \mu I)^{-1}$ is a τ-compact operator in $\mathcal{N}$ i.e. it is in the norm closure of the ideal generated by all the projections $p = p^2 = p^* \in \mathcal{N}$ with $\tau(p) < +\infty$.

Modular Spectral Triples (after Carey-Phillips-Rennie)

A **modular spectral triple** $(\mathcal{A}, \mathcal{H}_\omega, D)$ relative to a semi-finite von Neumann algebra $\mathcal{N}$ and a faithful α-KMS-state ω on the $*$-algebra $\mathcal{A}$ is given by:

- a faithful representation of $\mathcal{A}$ in $\mathcal{N} \subset \mathcal{B}(\mathcal{H}_\omega)$ where $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ is the GNS-representation of $(\mathcal{A}, \omega)$;
- a faithful normal semi-finite weight ϕ on $\mathcal{N}$ whose modular authomorphism group σ^ϕ is inner in $\mathcal{N}$ and such that $\sigma^\phi(\pi_\omega(x)) = \pi_\omega(\alpha(x))$ for all $x \in \mathcal{A}$;
- ϕ restrict to a faithful semi-finite trace $\tau := \phi|_{\mathcal{N}^{\sigma^\phi}}$ on the fixed point algebra $\mathcal{N}^{\sigma^\phi} \subset \mathcal{N}$;
- a self-adjoint operator D on $\mathcal{H}_\omega$ (with domain invariant under all the $\pi_\omega(x)$, for $x \in \mathcal{A}$) such that for all $x \in \mathcal{A}$, $[\mathcal{D}, \pi_\tau(x)]_-$ extends to a bounded operator in $\mathcal{N}$ and for μ in the resolvent set of D, for all $f \in \pi_\omega(\mathcal{A})^{\sigma^\phi}$, $f(D - \mu I)^{-1}$ is a τ-compact operator relative to the semi-finite trace τ on $\mathcal{N}^{\sigma^\phi}$.

▸ Motivations from Physics

Modular Theory and Antonescu Christensen AF Triples 1

Let $\mathcal{A}$ be an AF C*-algebra, acting on the Hilbert space $\mathcal{H}$, $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots \subset \mathcal{A}_n \subset \mathcal{A}_{n+1} \subset \cdots \subset \mathcal{A}$ a filtration of $\mathcal{A}$ by unital inclusions of finite dimensional C*-algebras.
Let $\xi \in \mathcal{H}$ be a cyclic and separating vector for the von Neumann algebra $\mathcal{R} := \mathcal{A}''$ and denote by Δ_ξ, J_ξ and $(\sigma^\xi_t)_{t\in\mathbb{R}}$ the modular/conjugation operators and modular group relative to the pair $(\mathcal{R}, \xi)$. For any $x \in \mathcal{R}$, define[20] $\pi_n(x) \in \mathcal{A}_n$ by $\pi_n(x)\xi = P_n x\xi$.

Proposition

The following conditions are equivalent:

a) $\pi_n : \mathcal{R} \to \mathcal{A}_n$ *is a* $\langle \xi \mid \cdot\xi \rangle$*-invariant conditional expectation.*

b) $\sigma^\xi_t(\mathcal{A}_n) = \mathcal{A}_n$, $t \in \mathbb{R}$.

Furthermore, $\Delta^{it}_\xi P_n = P_n \Delta^{it}_\xi$, $t \in \mathbb{R}$ *and* $J_\xi P_n = P_n J_\xi$.

[20] See R.Longo, Pac. J. Math 75 (1978) 199-205.

Modular Theory and Antonescu Christensen AF Triples 2

Proposition

If the filtration of the AF C-algebra is modularly stable, for any choice of the sequence* (α_n) *satisfying the* Antonescu-Christensen conditions, *the corresponding Dirac operator is a modular invariant, i.e.:*

$$\Delta_\xi^{it} D \Delta_\xi^{-it} = D, \quad \forall t \in \mathbb{R}, \quad \textit{and furthermore}, \quad J_\xi D = D J_\xi .$$

Therefore, in the non-tracial case, spectral triples associated to filtrations that are stable under the modular group have a non-trivial group of automorphisms

Modular Theory and Antonescu Christensen AF Triples 3

We now provide some explicit example of AF-algebras whose filtration is stable under a non-trivial modular group.

Example (Power's factors - part 1)

For every $n \in \mathbb{N}_0$, let $\mathcal{A}_n := \mathbb{M}_2(\mathbb{C}) \otimes \cdots \otimes \mathbb{M}_2(\mathbb{C})$ be the tensor product of n copies of $\mathbb{M}_2(\mathbb{C})$ and $\mathcal{A}_0 := \mathbb{C}$.
Define the unital inclusion $\iota_n : \mathcal{A}_n \to \mathcal{A}_{n+1}$ by $\iota_n : x \mapsto x \otimes 1_{\mathbb{M}_2(\mathbb{C})}$.
For every $n \in \mathbb{N}$, let $\phi_n : \mathbb{M}_2(\mathbb{C}) \to \mathbb{C}$ be a faithful state and consider the (automatically faithful) infinite tensor product state $\omega := \phi_1 \otimes \cdots \otimes \phi_n \otimes \cdots$ on the inductive limit C*-algebra $\mathcal{A} := \otimes_{j=1}^{\infty} \mathbb{M}_2(\mathbb{C})$. Let $\mathcal{M}$ be the von Neumann algebra obtained as the weak closure of $\mathcal{A}$ in the GNS-representation of ω. We continue to denote with the same symbol the (automatically faithful) normal extension of ω to $\mathcal{M}$.

Modular Theory and Antonescu Christensen AF Triples 4

Example (Powers Factors - continuation)

If σ^ω denotes the modular group of $\mathcal{M}$, we have that $\sigma^\omega_t(x_1 \otimes \cdots \otimes x_n \otimes 1 \otimes \cdots) = \sigma^{\phi_1}_t(x_1) \otimes \cdots \otimes \sigma^{\phi_n}_t(x_n) \otimes 1 \otimes \cdots$.
The choice of

$$\phi_j(x) := \operatorname{tr} \begin{bmatrix} \lambda & 0 \\ 0 & 1-\lambda \end{bmatrix} x, \quad j = 1, \dots, n, \quad \lambda \in]0, 1/2[,$$

gives rise to the so called Power's factor that is a factor of type III_μ with $\mu := \lambda/(1-\lambda)$.

Corollary

The Antonescu-Christensen Dirac operators associated to the natural filtration of the Powers factors are modular invariant.

Modular Theory and Antonescu Christensen AF Triples 5

Since, for modular filtrations of an AF C*-algebra, the modular generator K_ω and the Dirac operators of Antonescu-Christensen commute, it is natural to ask if there are sitations where it is possible to assume $D = K_\omega$ or a proportionality between them.

Of course, it is already clear from the definition that every Antonescu-Christensen Dirac operator has a positive spectrum (since by construction $\alpha_n > 0$), and hence the previous question should be interpreted in a "loose way" allowing some freedom for some (significant) alteration of the constructions.

Tensorial Spectral Symmetrization 1

Given a "Dirac ket" spectral triple $(\mathcal{A}, \mathcal{H}, D)$, consider its associated "Dirac bra" spectral triple $(\mathcal{A}, \mathcal{H}', D')$, where:

- $\mathcal{H}'$ denotes the Hilbert space dual of $\mathcal{H}$,
- the operator D' is given by $D' := \Lambda \circ D \circ \Lambda^{-1}$, with $\Lambda : \mathcal{H} \to \mathcal{H}'$ the usual conjugate-linear Riesz isomorphism,
- the $*$-algebra $\mathcal{A}$ is represented on $\mathcal{H}'$ via the faithful representation $\pi'(x) := \Lambda \circ \pi(x) \circ \Lambda^{-1}$, for all $x \in \mathcal{A}$.

Tensorial Spectral Symmetrization 2

The tensor product Hilbert space $\mathcal{H} \otimes \mathcal{H}'$, is equipped with a conjugate-linear "flip operator" J defined on homogeneous tensors by $J(\xi \otimes \eta) := \Lambda^{-1}(\eta) \otimes \Lambda(\xi)$, and carries two commuting representations of $\mathcal{A}$ given, for all $x \in \mathcal{A}$, $\xi \in \mathcal{H}$ and $\eta \in \mathcal{H}'$ by:

$$\pi(x)(\xi \otimes \eta) := (\pi(x)\xi) \otimes \eta$$
$$\pi'(x)(\xi \otimes \eta) := \xi \otimes (\pi'(x)\eta) = J\pi(x)J(\xi \otimes \eta).$$

We can define on $\mathcal{H} \otimes \mathcal{H}'$ a new Dirac operator

$$K := D \otimes I - I \otimes D'$$

(remember that $D' := \Lambda \circ D \circ \Lambda^{-1}$) obtaining a new spectral triple

$$(\mathcal{A}, \mathcal{H} \otimes \mathcal{H}', K).$$

- Motivations from Physics

Motivations from Physics

- Which indications we have that non-commutative geometry is going to be useful in quantum gravity?
- Which indications we have that modular theory is going to be important in quantum gravity?
- Is there a direct physical link between the two?
- What is the physical meaning of modular non-commutative geometries?

Algebraic Modular Quantum Gravity

Non-commutative Space-Time 1

There are 4 main reasons to look beyond classical space-time in physics:

1) Quantum effects (Heisenberg uncertainty principle), coupled to the general relativistic effect of the stress-energy tensor on the curvature of space-time (Einstein equation), entail that at very small scales the space-time manifold structure might be "unphysical". (B.Riemann, A.Einstein, S.Doplicher-K.Fredenhagen-J.Roberts[21]).

[21] S.Doplicher, K.Fredenhagen, J.Roberts, The Structure of Spacetime at the Planck Scale and Quantum Fields, Commun. Math. Phys., **172**, 187 (1995), `hep-th/0303037`.

Non-commutative Space-Time 2

2) Modification to the short scale structure of space-time might help to resolve the problems of "ultraviolet divergences" in QFT (W.Heisenberg, H.Snyder[22] and many others) and of "singularities" in General Relativity.
3) The quest to extend Einstein's geometrical interpretation of gravity to other interactions via a Kaluza-Klein scheme.
4) Already in general relativity space-time is not a fundamental a-priori entity, but is determined a-posteriori from events/interaction that are relationally obtained from observable quantities.

[22] H.Snyder, Quantized Spacetime, Phys. Rev., **71**, 38-41 (1947).

Non-commutative Space-Time 3

There are 4 main reasons why the introduction of non-commutative space-time in physics might answer the previous requests:

1) In general relativity space-time is (in part) "dynamical" and in quantum physics dynamical degrees of freedom are described via non-commutative algebras of observables
2) In non-commutative geometry, the notion of point become "fuzzy"
3) A.Connes' view of the standard model in particle physics as a "classical" non-commutative geometry of space-time (with spectral triples)[23]
4) In non-commutative geometry, space is already recasted in a spectral form appropriate for reconstruction from observables.

[23] A.Connes, Essay on Physics and Noncommutative Geometry, in: The Interface of Mathematics and Particle Physics, ed. D.Quillen, Clarendon Press, (1990). Modular Algebraic Quantum Gravity back

Spectral Space-Time 1

By "spectral space-time" we mean the idea that space-time (commutative or not) has to be "reconstructed a posteriori", in a spectral way, from other operationally defined degrees of freedom (geometrical or not). The origin of this "pregeometrical philosophy" is not clear:

- Space-time as a "relational" a posteriori entity originate from G.W. Leibnitz, G. Berkeley, E. Mach.

Spectral Space-Time 2

- Pregeometrical speculations date as back as Pythagoras, but in their modern form, they start with J.A. Wheeler's "pregeometry"[24],[25] and "it from bit"[26] proposals.
- R. Geroch [27] has been the first to suggest a "shift" from space-time to algebras of functions over it, in order to address the problems of singularities in general relativity.

[24] J.A. Wheeler, Pregeometry: Motivations and Prospects, in: Quantum Theory and Gravitation, ed. A. Marlov, Academic Press (1980).

[25] D. Meschini, M. Lehto, J. Piilonen, Geometry, Pregeometry and Beyond, Stud. Hist. Philos. Mod. Phys., **36**, 435-464 (2005), `gr-qc/0411053`.

[26] J.A. Wheeler, It from Bit, Sakharov Memorial Lectures on Physics, vol. 2, Nova Science (1992).

[27] R. Geroch, Einstein Algebras, Commun. Math. Phys., **26**, 271-275 (1972).

Spectral Space-Time 3

- R. Feynman-F. Dyson proof of Maxwell equations, from non-relativistic QM of a free particle, indicates that essential information about the underlying space-time is already contained in the algebra of observables of the system[28].
- The "reconstruction" of (classical Minkowski) space-time from suitable states over the observable algebra in algebraic quantum field theory has been considered by S. Doplicher[29], A. Ocneanu[30], U. Bannier[31].

[28] An argument recently revised and extended to non-commutative configuration spaces by T. Kopf-M. Paschke `arXiv:math-ph/0301040,0708.0388`

[29] S. Doplicher, private conversation, Rome, April 1995.

[30] As reported in: A. Jadczyk, Algebras Symmetries, Spaces, in: Quantum Groups, H. D. Doebner, J. D. Hennig ed., Springer (1990).

[31] U. Bannier, Intrinsic Algebraic Characterization of Space-Time Structure, Int. J. Theor. Phys., **33**, 1797-1809 (1994).

Spectral Space-Time 4

- Extremely important rigorous results on the "reconstruction of classical Minkowski space-time" from the vacuum state in algebraic quantum field theory, via Tomita-Takesaki modular theory, have been obtained in the "geometric modular action" program by D. Buchholz-S. Summers[32],[33],[34].

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[32] D. Buchholz, O. Dreyer, M. Florig, S. Summers, Geometric Modular Action and Spacetime Symmetry Groups, Rev. Math. Phys., 12, 475-560 (2000), `math-ph/9805026`.

[33] S. Summers, Yet More Ado About Nothing: The Remarkable Relativistic Vacuum State, `arXiv:0802.1854v1`

[34] S. J. Summers, R. K. White, On Deriving Space-Time from Quantum Observables and States, `hep-th/0304179`.

Spectral Space-Time 5

- Tomita-Takesaki modular theory is also used in the "modular localization program" by R. Brunetti-D. Guido-R. Longo[35]. In this context a reconstruction of space-time has been conjectured by N. Pinamonti[36].

[35] R. Brunetti, D. Guido, R. Longo, Modular Localization and Wigner Particles, `arXiv:math-ph/0203021`.

[36] N. Pinamonti, On Localization of Position Operators in Möbius covariant Theories, `math-ph/0610070`.

Spectral Space-Time 6

- That non-commutative geometry provides a suitable environment for the implementation of spectral reconstruction of space-time from states and observables in quantum physics has been my research motivation since 1990 and it is still an open work in progress[37].
- The idea that space-time might be spectrally reconstructed, via non-commutative geometry, from Tomita-Takesaki modular theory applied to the algebra of physical observables was elaborated in 1995 by myself and independently by R. Longo. Since then, this conjecture is the main subject and goal of our investigation[38].

[37] P.B., Hypercovariant Theories and Spectral Space-time (2001).

[38] P.B., Modular Spectral Triples in Non-commutative Geometry and Physics, Research Report, Thai Research Fund, (2005).

Quantum Gravity via NCG 1

It is often claimed that NCG provides the right mathematics (a kind of quantum version of Riemannian geometry) for a mathematically sound theory of quantum gravity, [39],[40].

Among the available approaches to quantum gravity via NCG:

- J. Madore's "derivation based approach"[41];
- S. Majid's "quantum group approach"[42].

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[39] L. Smolin, Three Roads to Quantum Gravity, Weidenfeld & Nicolson (2000).

[40] P. Martinetti, What Kind of Noncommutative Geometry for Quantum Gravity?, Mod. Phys. Lett., **A20**, 1315 (2005), `gr-qc/0501022`.

[41] J. Madore, An Introduction to Non-commutative Geometry and its Physical Applications, Cambridge University Press (1999).

[42] S. Majid, Hopf Algebras for Physics at the Planck Scale, J. Classical and Quantum Gravity, **5**, 1587-1606 (1988). S. Majid, Algebraic Approach to Quantum Gravity I,II,III, `arXiv:hep-th/0604130`, `arXiv:hep-th/0604182`, `http://philsci-archive.pitt.edu/archive/00003345`.

Quantum Gravity via NCG 2

Current applications of NCG to quantum gravity have been limited to some example or to attempts to make use of its mathematical framework "inside" some already established theories such as "strings" or "loops". Among these, we mention:

- the interesting examples studied by C. Rovelli[43] and F. Besard[44];
- the applications to string theory in the work by A. Connes-M. Douglas-A. Schwarz[45] and more recently in the works by V.Mathai and collaborators on T-duality,[46]

[43] C. Rovelli, Spectral Noncommutative Geometry and Quantization: a Simple Example, Phys. Rev. Lett., **83**, 1079-1083 (1999), gr-qc/9904029.

[44] F. Besnard, Canonical Quantization and Spectral Action, a Nice Example, gr-qc/0702049.

[45] A. Connes, M. Douglas, A. Schwarz, Noncommutative Geometry and Matrix Theory: Compactification on Tori, hep-th/9711162.

[46] arXiv:0708.2648, arXiv:0911.1886, arXiv:1007.4696

Quantum Gravity via NCG 3

- the links between loop quantum gravity (spin networks), quantum information and NCG described by F. Girelli-E. Livine[47].
- the intriguing interrelations with loop quantum gravity in the works by J. Aastrup-J. Grimstrum-R. Nest[48],[49],[50] and in the recent paper by D.Denicola-M.Marcolli-A.-Z.Al Yasri[51]

[47] F. Girelli, E. Livine, Reconstructing Quantum Geometry from Quantum Information: Spin Networks as Harmonic Oscillators, Class. Quant. Grav., **22**, 3295-3314 (2005), `gr-qc/0501075`.

[48] J. Aastrup, J. Grimstrup, Spectral Triples of Holonomy Loops, `arXiv:hep-th/0503246`.

[49] J. Aastrup, J. Grimstrup, Intersecting Connes Noncommutative Geometry with Quantum Gravity, `hep-th/0601127`.

[50] J. Aastrup, J. Grimstrup, R. Nest, On Spectral Triples in Quantum Gravity I-II, `arXiv:0802.1783v1`, `arXiv:0802.1784v1`.

[51] D.Denicola-M.Marcolli-A.-Z.Al Yasri Spin Foams and Non-commutative Geometry `arXiv:1005.1057`.

Quantum Gravity via NCG 4

Unfortunately, with the only notable exception of two programs partially outlined in

- M. Paschke, An Essay on the Spectral Action and its Relation to Quantum Gravity, in: Quantum Gravity, Mathematical Models and Experimental Bounds, Birkäuser (2007),
- A. Connes, M. Marcolli, Noncommutative Geometry Quantum Fields and Motives, July 2007,

a foundational approach to quantum physics based on A. Connes' NCG has never been proposed.
The obstacles are both technical and conceptual.

Modular Theory in Physics

Modular theory in physics is "**equilibrium quantum statistical mechanics**".

- R.Kubo and P.Martin-J.Schwinger introduced the KMS condition as a characterization of equilibrium states.
- R.Haag-N.Hugenoltz-M.Winnink and M.Takesaki reformulated the KMS condition in algebraic quantum mechanics and related it with Tomita modular theory.

Modular Theory in QG 1

There are some important areas of research that are somehow connected to the problems of quantum gravity and that seem to suggest a more prominent role of Tomita-Takesaki modular theory in quantum physics (and in particular in the physics of gravity):

- Since the work of J.Bekenstein on **black holes entropy**, S.Hawking on **black holes radiation** and W.Unruh on **vacuum thermalization** for accelarated observers, it has been conjectured the existence of a deep connection between gravity (equivalence principle), thermal physics (hence Tomita-Takesaki and KMS-states) and quantum field theory; this idea has not been fully exploited so far.

Modular Theory in QG 2

- Starting from the works of J.Bisognano-E.Wichmann[52], G.Sewell[53] and more recently, H.-J.Borchers[54], there is mounting evidence that Tomita-Takesaki modular theory should play a fundamental role in the "spectral reconstruction" of the space-time information from the algebraic setting of states and observables.

[52] Bisognano J., Wichmann E. (1975). On the Duality Condition for Hermitian Scalar Fields, *J. Math. Phys.* 16, 985-1007.
Bisognano J., Wichmann E. (1976). On the Duality Condition for Quantum Fields, *J. Math. Phys.* 17, 303-321.

[53] Sewell G. (1982). Quantum Fields on Manifolds: PCT and Gravitationally Induced Thermal States, *Ann. Phys.* 141, 201.

[54] Borchers H.-J. (1992). The CPT-theorem in Two-dimensional Theories of Local Observables, *Comm. Math. Phys.* 143, 315-322.

Modular Theory in QG 3

- Some of the most interesting results in this direction have been obtained so far in H.Araki-R.Haag-D.Kastler **algebraic quantum field theory**:
  - in the theory of "half-sided modular inclusions" and modular intersections (see H.-J.Borchers[55] and references therein, H.Araki-L.Zsido[56]);

[55] Borchers H.-J. (2000). On Revolutionizing Quantum Field Theory, *J. Math. Phys.* 41, 3604-3673.

[56] Araki, H., Zsido, L. (2005). Extension of the Structure Theorem of Borchers and its Application to Half-sided Modular Inclusions, *Rev. Math. Phys.* 17, n. 5, 491-543.

Modular Theory in QG 4

- - in the "geometric modular action" program (see D.Buchholz-S.J.Summers[57], D.Buchholz-M.Florig-S.J.Summers[58], D.Buchholz-O.Dreyer-M.Florig-S.J.Summers[59], S.Summers-R.White[60];

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[57] Buchholz D., Summers S.J. (1993). An Algebraic Characterization of Vacuum States in Minkowski Space, *Commun. Math. Phys.* 155, 449-458. Buchholz D., Summers S.J., An Algebraic Characterization of Vacuum States in Minkowski Space. III. Reflection Maps. `arXiv:math-ph/0309023`.

[58] Buchholz D., Florig M., Summers S.J. (1999). An Algebraic Characterization of Vacuum States in Minkowski Space II: Continuity Aspects, *Lett. Math. Phys.* 49, 337-350.

[59] Buchholz D., Dreyer O., Florig M., Summers S.J. (2000). Geometric Modular Action and Spacetime Symmetry Groups, *Rev. Math. Phys.* 12, 475-560.

[60] Summers S., White R., On Deriving Space-Time from Quantum Observables and States, `arXiv:hep-th/0304179`.

Modular Theory in QG 5

- - in "modular nuclearity" (see for details R.Haag[61] and, for recent applications to the "form factor program", D.Buchholz-G.Lechner[62]);

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[61] Haag R. (1996). *Local Quantum Physics*, Springer.

[62] Buchholz D., Lechner G., Modular Nuclearity and Localization, `arXiv:math-ph/0402072`.
Lechner G., Polarization-Free Quantum Fields and Interaction `arXiv:hep-th/0303062`.
On the Existence of Local Observables in Theories With a Factorizing S-Matrix, `arXiv:math-ph/0405062`.
Towards the construction of quantum field theories from a factorizing S-matrix `arXiv:hep-th/0502184`.
Construction of Quantum Field Theories with Factorizing S-Matrices, `arXiv:math-ph/0601022`.
On the Construction of Quantum Field Theories with Factorizing S-Matrices, `arXiv:math-ph/0611050`.

Modular Theory in QG 6

- - in the "modular localization program" (see B.Schroer-H.-W.Wiesbrock[63], R.Brunetti-D.Guido-R.Longo[64], J.Mund-B.Schroer-J.Yngvanson[65] and N.Pinamonti[66].

[63] Schroer B., (1997). Wigner Representation Theory of the Poincaré Group, Localization, Statistics and the S-Matrix, *Nucl. Phys.* B499, 519-546. Schroer B. (1999). Modular Wedge Localization and the $d = 1 + 1$ Form Factor Program, *Ann. Phys.* 275, 190-223. Schroer B., Wiesbrock H.-W. (2000). Modular Theory and Geometry, *Rev. Math. Phys.* 12, 139-158. Schroer B., Wiesbrock H.-W. (2000). Modular Constructions of Quantum Field Theories with Interactions, *Rev. Math. Phys.* 12, 301-326.

[64] Brunetti R., Guido D., Longo R., Modular Localization and Wigner Particles, `arXiv:math-ph/0203021`.

[65] Mund J., Schroer B., Yngvason J., String-localized Quantum Fields and Modular Localization, `arXiv:math-ph/0511042`.

[66] Pinamonti N., On Localization and Position Operators in Möbius Covariant Theories, `arXiv:math-ph/0610070`, 25 October 2006.

Modular Theory in QG 7

- Starting with the construction of cyclic cocycles from supersymmetric quantum field theories by A.Jaffe-A.Lesniewski-K.Osterwalder[67], there has always been a constant interest in the possible deep structural relationship between supersymmetry, modular theory of type III von Neumann algebras and non-commutative geometry (see D. Kastler[68] and A.Jaffe-O.Stoytchev[69]).

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[67] Jaffe A., Lesniewski A., Osterwalder K. (1988). Quantum K-theory I: The Chern Character, *Commun. Math. Phys.* 118, 1-14.
Jaffe A., Lesniewski A., Osterwalder K. (1989). On Super-KMS Functionals and Entire Cyclic Cohomology, *K-theory* 2, 675-682.

[68] Kastler D. (1989). Cyclic Cocycles from Graded KMS functionals, *Commun. Math. Phys.* 121(2), 345-350.

[69] Jaffe A., Stoytchev O. (1991). The Modular Group and Super-KMS Functionals, in: *Differential geometric methods in theoretical physics*, Lecture Notes in Phys. 375, 382-384, Springer.

Modular Theory in QG 8

- Some deep results by R.Longo[70] established a bridge between the theory of superselections sectors and cyclic cocycles obtained by super-KMS states. The recent work by D.Buchholz-H.Grundling[71] opens finally a way to construct super-KMS functionals (and probably spectral triples) in algebraic quantum field theory.

[70] Longo R., Notes for a Quantum Index Theorem, `arXiv:math/0003082`.
[71] Buchholz D., Grundling H., Algebraic Supersymmetry: a Case Study, `arXiv:math-ph/0604044`.

Modular Theory in QG 9

- In the context of C.Rovelli "**thermal time hypothesis**"[72] in quantum gravity, A.Connes-C.Rovelli[73] (see also P.Martinetti-C.Rovelli[74] and P.Martinetti[75]) have been using Tomita-Takesaki modular theory in order to induce a macroscopic time evolution for a relativistic quantum system.

[72] Rovelli C. (2004). *Quantum Gravity*, Cambridge University Press.

[73] Connes A., Rovelli C. (1994). Von Neumann Algebra Automorphisms and Time-Thermodynamic Relation in General Covariant Quantum Theories, *Class. Quant. Grav.* 11, 2899-2918.

[74] Martinetti P., Rovelli C. (2003). Diamond's Temperature: Unruh Effect for Bounded Trajectories and Thermal Time Hypothesis, *Class. Quant. Grav.* 20, 4919-4932.

[75] Martinetti P., A Brief Remark on Unruh Effect and Causality, `arXiv:gr-qc/0401116`.

Modular Theory in QG 10

- A.Connes-M.Marcolli[76] with the "cooling procedure" are proposing to examine the operator algebra of observables of a quantum gravitational system, via modular theory, at "different temperatures" in order to extract by "symmetry breaking" an emerging geometry.
- This point of view is further elaborated in the recent work by D.Denicola-M.Marcolli-A.Z.al Yasri[77] where it is applied to specific algebras obtained by the kinematic of spin foams.

back

[76] Connes A., Marcolli M., Noncommutative Geometry, Quantum Fields and Motives, (preliminary version) July 2007.

[77] Spin Foams and Non-commutative Geometry arXiv:1005.1057.

Modular Theory in QG 11

- The idea that space-time might be spectrally reconstructed, via non-commutative geometry, from Tomita-Takesaki modular theory applied to the algebra of physical observables was elaborated in 1995 by one of the authors (P.B.) and independently (motivated by the possibility to obtain cyclic cocycles in algebraic quantum field theory from modular theory) by R.Longo. Since then this conjecture is still the main subject and motivation of our investigation[78].

[78] Bertozzini P. (2001). Spectral Space-Time and Hypercovariant Theories, unpublished.
Bertozzini P., Conti R., Lewkeeratiyutkul W. (2005). *Modular Spectral Triples in Non-commutative Geometry and Physics*, Research Report, Thai Research Fund, Bangkok.

Modular Theory in QG 12

Similar speculations on the interplay between modular theory and (some aspects of) space-time geometry have been suggested by S.Lord [79] and by M.Paschke-R.Verch[80].

- Roberto Conti has raised the somehow puzzling question whether it is possible to reinterpret the one parameter group of modular automorphisms as a renormalization (semi-)group in physics. The connection with P.Cartier's idea of a "universal Galois group"[81], currently developed by A.Connes-M.Marcolli, is extremely intriguing.

[79] Lord S., Riemannian Geometries, `arXiv:math-ph/0010037`.

[80] Paschke M., Verch R., Local Covariant Quantum Field Theory over Spectral Geometries, `arXiv:gr-qc/0405057`.

[81] Cartier P. (2001). A Mad Day's Work: from Grothendieck to Connes and Kontsevich, The Evolution of Concepts of Space and Symmetry, *Bull. Amer. Math. Soc.* 38 n. 4, 389-408.

- Modular Algebraic Quantum Gravity

Modular Algebraic Quantum Gravity 1

We propose[82] a "thermal" reconstruction of "quantum realities" (quantum space-time-matter), via Tomita-Takesaki modular theory, starting from suitable "event states" on "categories" of abstract operator algebras describing "partial physical observables".

The fundamental input of the project is the recognition that Tomita-Takesaki modular theory (the "heart" of equilibrium quantum statistical mechanics) can be reinterpreted as a way to associate non-commutative spectral geometries (axiomatically similar to A.Connes' spectral triples) to appropriate states over the algebras of observables of a physical system.

[82] P.B., R.Conti, W.Lewkeeratiyutkul, Modular Algebraic Quantum Gravity, work in progress.

Modular Algebraic Quantum Gravity 2

In this way, for every "observer" (specified by an algebra of "partial observables"), a different quantum geometry naturally emerges, induced by each possible "covariant kinematic" (specified by an "event state"). These "virtual realities" interrelate with each other via a "categorical covariance principle" replacing the usual diffeomorphisms group of general relativity.

In our opinion this provides a new solid approach to the formulation of an algebraic (modular) theory of quantum gravity, and to the foundations of quantum physics, in which (quantum) space-time is reconstructed a posteriori.

Construction of Modular Spectral Geometries 1

- We make use of Tomita-Takesaki modular theory of operator algebras to associate non-commutative geometrical objects (only formally similar to A. Connes' spectral-triples) to suitable states over C*-algebras.
- In the same direction we also stress the close connection of these "spectral geometries" to the modular spectral triples introduced by A.Carey, J.Phillips, A.Rennie, F.Sukochev.

Construction of Modular Spectral Geometries 2

- Let ω be a faithful KMS-state over the C*-algebra $\mathcal{A}$. By definition there exists a unique one parameter automorphism group $t \mapsto \sigma_t^\omega$ that satisfies the KMS-condition at inverse temperature $\beta = 1$.
- We consider the GNS-representation $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ induced by the state ω and note that ξ_ω is cyclic and separating for the von Neumann algebra $\pi_\omega(\mathcal{A})'' \subset \mathcal{B}(\mathcal{H}_\omega)$.
- By Tomita-Takesaki theorem, there is a unique one-parameter unitary group $t \mapsto \Delta_\omega^{it}$ such that

$$\pi_\omega(\sigma_t^\omega(x)) = \Delta_\omega^{it}\pi_\omega(x)\Delta_\omega^{-it}, \quad \forall t \in \mathbb{R}.$$

  Let $K_\omega := \log \Delta_\omega$ be the modular generator and J_ω the modular involution determined by ξ_ω on $\mathcal{H}_\omega$.

Construction of Modular Spectral Geometries 3

- Define $\mathcal{A}_\omega := \{x \in \mathcal{A} \mid [K_\omega, \pi_\omega(x)]_- \in \pi_\omega(\mathcal{A})''\}$ and note that $\mathcal{A}_\omega$ is a $*$-algebra and that $\mathcal{A}_\omega \xi_\omega$ is a core for the operator K_ω.
- By Tomita-Takesaki we have $[[K_\omega, \pi_\omega(x)]_-, J_\omega \pi_\omega(y) J_\omega]_- = 0$
- We define the **modular spectral geometry** associated to the pair $(\mathcal{A}, \omega)$ to be:

$$(\mathcal{A}_\omega, \mathcal{H}_\omega, \xi_\omega, K_\omega, J_\omega).$$

When $\mathcal{N} := \pi_\omega(\mathcal{A})''$ is a semifinite von Neumann algebra, taking $D := K_\omega$, this structure seems strictly related to the notion of modular spectral triple introduced by A.Carey-J.Phillips-A.Rennie.

Construction of Modular Spectral Geometries 4

Despite the superficial similarity between modular spectral geometries and real spectral triples, there are some radical differences:

- The modular generator K_ω has a spectrum $\mathrm{Sp}(K_\omega)$ that is a symmetric set under reflection in $\mathbb{R}$ and can often be continuous, a situation that reminds of "propagation" and that does not have much in common with the usual first-order elliptic Dirac operators that appear in the definition of A.Connes' spectral triples.
- There is no grading anticommuting with K_ω; although a natural grading is clearly present via the spectral decomposition of K_ω into a positive and negative component, such grading always commutes with the modular generator.

Construction of Modular Spectral Geometries 5

- The resolvent properties of K_ω do not seem to fit immediately with the requirements of index theory, although we expect that in the case of periodic modular flows the index theory developed by A.Carey-J.Phillips-A.Rennie-F.Suchocev for modular spectral triples will apply.
- Contrary to the situation typical of A.Connes spectral triples, the $*$-algebra $\mathcal{A}_\omega$ is stable under the one-parameter group generated by K_ω that coincides with $t \mapsto \sigma^\omega_t$.

Construction of Modular Spectral Geometries 6

The attribute "geometry" attached to such an algebraic gadget is justified by presence of natural structures related to differentiability and integral calculus:

- there is an intrinsic notion of smoothness provided by the modularly stable filtration $\mathcal{A}^{\infty}_{\omega} \subset \cdots \mathcal{A}^{n+1}_{\omega} \subset \mathcal{A}^{n}_{\omega} \subset \cdots \mathcal{A}^{0}_{\omega} \subset \mathcal{A}$ of $*$-algebras given, for $r \in \mathbb{N} \cup \{+\infty\}$, by $\mathcal{A}^{r}_{\omega} := \{x \in \mathcal{A} \mid [t \mapsto \sigma^{\omega}_{t}(x)] \in C^{r}(\mathbb{R}; \mathcal{A})\}$, with $C^{r}(\mathbb{R}; \mathcal{A})$ denoting the family of $\mathcal{A}$-valued r-times continuously differentiable functions.
  Via the faithful representation π_{ω} we get a filtration $\pi_{\omega}(\mathcal{A})^{\infty} \subset \cdots \pi_{\omega}(\mathcal{A})^{n+1} \subset \pi_{\omega}(\mathcal{A})^{n} \subset \cdots \pi_{\omega}(\mathcal{A})^{0} \subset \pi_{\omega}(\mathcal{A})$ of smooth operators on the Hilbert space $\mathcal{H}_{\omega}$, where $\pi_{\omega}(\mathcal{A})^{r} := \{x \in \mathcal{A} \mid [t \mapsto \Delta^{it}_{\omega}\pi_{\omega}(x)\Delta^{-it}_{\omega}] \in C^{r}(\mathbb{R}; \mathcal{B}(\mathcal{H}_{\omega}))\}$.

Construction of Modular Spectral Geometries 7

- Defining $\delta^m_\omega(x) := [K_\omega, \pi_\omega(x)]_-$, the operator K_ω seems to satisfy a variant of A.Connes' regularity condition: $\pi_\omega(x), [K_\omega, \pi_\omega(x)]_- \in \cap_{m=1}^{+\infty} \mathrm{Dom}(\delta^m_\omega)$.
- There is already a perfectly natural notion of integration available via the β-KMS state ω so that we can define $\int x \, \mathrm{d}\omega := \omega(x)$, for all $x \in \mathcal{A}$ and more generally, for all $x \in \pi_\omega(\mathcal{A})''$, $\int x \, \mathrm{d}\omega := \langle \xi_\omega \mid x\xi_\omega \rangle$.
- Note that from the above definition, for the purpose of integration, the order-one operator K_ω plays a role similar to the Laplacian D^2 in the case of spectral triples.

Construction of Modular Spectral Geometries 8

By A.Connes-M.Takesaki duality and T.Falcone-M.Takesaki non-commutative flow of weights, we obtain some relation between modular spectral geometries and a semi-finite version of them resulting in structures resembling "modular spectral triples" by A.Carey and collaborators and that deserve a careful study in order to identify their physical significance.

- Define $\mathcal{M}_\omega := \pi_\omega(\mathcal{A})''$. Note that $\mathcal{M}_\omega$ is canonically embedded in a semi-finite von Neumann algebra $\widetilde{\mathcal{M}}_\omega$, isomorphic to the crossed product $\mathcal{M}_\omega \rtimes_{\sigma^\omega} \mathbb{R}$, and hence $\mathcal{M}_\omega$ is identified with the algebra of fixed points $\widetilde{\mathcal{M}}_\omega^{\widehat{\sigma}^\omega}$ for the dual action $s \mapsto \widehat{\sigma}^\omega_s$ on $\widetilde{\mathcal{M}}_\omega$.

Construction of Modular Spectral Geometries 9

- There is an operator-valued weight Ξ_ω from $\widetilde{\mathcal{M}}_\omega$ to $\mathcal{M}_\omega$ under which the weight ω get lifted to $\widetilde{\mathcal{M}}_\omega$ as $\widetilde{\omega} := \omega \circ \Xi_\omega$ and $\mathcal{M}_\omega$ inside $\widetilde{\mathcal{M}}_\omega$ is modularly stable under the modular group $t \mapsto \sigma_t^{\widetilde{\omega}}$ of $\widetilde{\omega}$ on $\widetilde{\mathcal{M}}_\omega$.
- The modular group $t \mapsto \sigma_t^{\widetilde{\omega}}$ is inner in $\widetilde{\mathcal{M}}_\omega$ with generator k_ω affiliated to $\widetilde{\mathcal{M}}_\omega$.
- There is a natural trace on $\widetilde{\mathcal{M}}_\omega$ defined by $\tau_\omega(z) := \widetilde{\omega}(\omega^{-1/2} z \omega^{-1/2})$, for all $z \in \widetilde{\mathcal{M}}_\omega$, where $\omega^{-1} := \exp(-k_\omega)$. Furthemore, the natural trace rescales under the action of $s \mapsto \widehat{\sigma}_s^\omega$ i..e. $\tau_\omega \circ \widehat{\sigma}_s^\omega = e^{-s}\tau$.

Relation with Modular Spectral Triples 1

- Consider the von Neumann algebras, on the Hilbert space $\mathcal{H}_\omega$, $\mathcal{M}^\omega := \{x \in \mathcal{M}_\omega \mid \sigma^\omega_t(x) = x, \ \forall t \in \mathbb{R}\}$, usually called the centralizer of ω, and $\mathcal{N}_\omega := \{\pi_\omega(x)\Delta^{it}_\omega \mid x \in \mathcal{A}, \ t \in \mathbb{R}\}'' = \mathcal{M}_\omega \vee \{\Delta^{it}_\omega \mid t \in \mathbb{R}\}''$.
  We have $\mathcal{M}^\omega \subset \mathcal{M}_\omega \subset \mathcal{N}_\omega$.
  Note that $\mathcal{M}^\omega = \mathcal{M}_\omega \cap \{\Delta^{it}_\omega \mid t \in \mathbb{R}\}'$ and therefore $(\mathcal{M}^\omega)' = (\mathcal{M}_\omega)' \vee \{\Delta^{it}_\omega \mid t \in \mathbb{R}\}''$.
- Passing to the commutant von Neumann algebras we obtain $\mathcal{N}'_\omega \subset \mathcal{M}'_\omega \subset (\mathcal{M}^\omega)'$ and, since $\gamma_\omega(\mathcal{M}_\omega) = \mathcal{M}'_\omega$, one has $\gamma_\omega((\mathcal{M}^\omega)') = \mathcal{N}_\omega$.
  Being (anti-isomorphic to) the commutant of a (semi-)finite von Neumann algebra, $\mathcal{N}_\omega$ is semi-finite (see Takesaki corollary V.2.23).

Relation with Modular Spectral Triples 2

- The state ω on $\mathcal{M}_\omega$ restricts, to a trace on $\mathcal{M}^\omega$ (see Takesaki theorem VIII.2.6).
- Since $\mathcal{M}^\omega$ is modularly stable under σ^ω, by Takesaki theorem there is a unique conditional expectation $\Phi_\omega : \mathcal{M}_\omega \to \mathcal{M}^\omega$ such that $\omega = \omega|_{\mathcal{M}^\omega} \circ \Phi_\omega$ and via the conjugate-linear map γ_ω we obtain a unique conditional expectation $\Phi_{\gamma_\omega} : \mathcal{M}'_\omega \to \mathcal{N}'_\omega$ given, for all $x \in \mathcal{M}'_\omega$, by $\Phi_{\gamma_\omega} : x \mapsto \gamma_\omega \circ \Phi_\omega \circ \gamma_\omega(x)$.
- Making use of modular theory for operator valued weights, we can now associate to the conditional expectation $\Phi_{\gamma_\omega} : \mathcal{M}'_\omega \to \mathcal{N}'_\omega$ a dual faithful semi-finite normal operator valued weight $\Theta_\omega : \mathcal{N}_{\omega+} \to \widehat{\mathcal{M}}_{\omega+}$.

Relation with Modular Spectral Triples 3

- The operator-valued weight Θ_ω can be used to lift the state ω on $\mathcal{M}_\omega$ to a faithful normal semi-finite weight ϕ_ω on $\mathcal{N}_\omega$ such that $\phi_\omega := \omega \circ \Theta_\omega$.
  By Takesaki operator valued weight theorem we have that $\sigma_t^{\phi_\omega}(x) = \sigma_t^\omega(x)$ for all $x \in \mathcal{M}_\omega$ and for all $t \in \mathbb{R}$.
- Using the definition of Θ_ω and the properties of spatial derivatives (see Connes spatial derivative theorem), we verify that the modular group induced by the weight ϕ_ω on the semi-finite von Neumann algebra $\mathcal{N}_\omega$ is given, for all $x \in \mathcal{N}_\omega$, by $\sigma^{\phi_\omega}(x) = \Delta_\omega^{it} x \Delta_\omega^{-it}$.
- The map defined for all $x \in \mathcal{N}_\omega$ by $\tau_\omega(x) := \phi_\omega(\Delta_\omega^{-1/2} x \Delta_\omega^{-1/2})$ is a faithful semi-finite normal trace (see Takesaki theorem VIII.3.14).

Relation with Modular Spectral Triples 4

$$\begin{tikzcd}
\mathcal{M}^\omega \arrow[d, "J_\omega \cdot J_\omega"'] & \mathcal{M}_\omega \arrow[l, "\Phi_\omega"'] \\
\mathcal{N}'_\omega & \mathcal{M}'_\omega \arrow[l, "\Phi_{\gamma\omega}"'] \arrow[u, "J_\omega \cdot J_\omega"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathcal{M}^\omega)'_+ \arrow[r] \arrow[d, "J_\omega \cdot J_\omega"'] & \widehat{\mathcal{M}}'_{\omega+} \\
\mathcal{N}_{\omega+} \arrow[r, "\Theta_\omega"] & \widehat{\mathcal{M}}_{\omega+} \arrow[u, "J_\omega \cdot J_\omega"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{M}^\omega \arrow[r, hook] \arrow[d, "J_\omega \cdot J_\omega"'] & \mathcal{M}_\omega \arrow[r, hook] \arrow[d, "J_\omega \cdot J_\omega"'] & \mathcal{N}_\omega \arrow[d, "J_\omega \cdot J_\omega"'] \\
\mathcal{N}'_\omega \arrow[r, hook] & \mathcal{M}'_\omega \arrow[r, hook] & (\mathcal{M}^\omega)'
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{M}^\omega \arrow[dr, "\omega|_{\mathcal{M}^\omega}"'] & \mathcal{M}_\omega \arrow[l, "\Phi_\omega"'] \arrow[d, "\omega"'] & \mathcal{N}_\omega \arrow[l, "\Theta_\omega"'] \arrow[dl, "\phi_\omega"] \\
& \mathbb{C} &
\end{tikzcd}$$

Relation with Modular Spectral Triples 5

The following results relate modular spectral geometries with modular spectral triples:

Theorem

Let $(\mathcal{A}_\omega, \mathcal{H}_\omega, \xi_\omega, K_\omega, J_\omega)$ be the modular spectral geometry associated to the pair $(\mathcal{A}, \omega)$, where ω is an α-KMS-state over the C-algebra $\mathcal{A}$. Suppose that K_ω has compact resolvent with respect to the canonical trace τ_ω on the von Neumann algebra $\mathcal{N}_\omega$ and that the extended weight $\phi_\omega : \mathcal{N}_{\omega+} \to [0, +\infty]$ is strictly semi-finite.*[83] *The data $(\mathcal{A}_\omega, \mathcal{H}_\omega, K_\omega)$ canonically provide a modular spectral triple, relative to the von Neumann algebra $\mathcal{N}_\omega$ and to the α-KMS-state ω, according to A.Carey-J.Phillips-A.Rennie.*

[83] By a strictly semi-finite weight $\phi : \mathcal{N}_+ \to [0, +\infty]$ we mean a weight ϕ whose restriction to its centralizer is semi-finite.

Relation with Modular Spectral Triples 6 – Proof 1

- $\mathcal{A}_\omega := \{x \in \mathcal{A} \mid [K_\omega, x]_- \in \mathcal{M}_\omega\}$ is a $*$-algebra that is α-invariant inside the C*-algebra $\mathcal{A}$.
- ω is an α-KMS state on $\mathcal{A}$ and $\mathcal{H}_\omega$ is the Hilbert space of the GNS-representation $(\pi_\omega, \mathcal{H}_\omega, \xi_\omega)$ of $(\mathcal{A}, \omega)$.
- The GNS representation π_ω is a covariant representation for the one-parameter group $\alpha : t \mapsto \alpha_t$ of automorphisms of $\mathcal{A}$ that is implemented on $\mathcal{H}_\omega$ by the modular one-parameter group $t \mapsto e^{iK_\omega t}$ i.e. $\pi_\omega(\alpha_t(x)) = e^{iK_\omega t}\pi_\omega(x)e^{-iK_\omega t}$, for all $x \in \mathcal{A}$ and $t \in \mathbb{R}$.
- The modular generator K_ω is affiliated to $\mathcal{N}_\omega$ and induces an inner one-parameter automorphism group of $\mathcal{N}_\omega$ that coincides with the modular group of the semi-finite normal faithful weight $\phi_\omega := \omega \circ \Theta_\omega$.

Relation with Modular Spectral Triples 7 – Proof 2

- Since $\sigma^{\phi_\omega}|_{\mathcal{M}_\omega} = \sigma^\omega$, we have $\sigma^{\phi_\omega}(\pi_\omega(x)) = \sigma^\omega(\pi_\omega(x)) = \pi_\omega(\alpha(x))$, for all $x \in \mathcal{A}$.
- Since $\Delta_\omega^{-1/2} \in \mathcal{N}_\omega$, from $\tau_\omega(z) := \phi_\omega(\Delta_\omega^{-1/2} z \Delta_\omega^{-1/2})$, we have $\phi_\omega(z) = \tau_\omega(\Delta_\omega^{1/2} z \Delta_\omega^{1/2})$, for all $z \in \mathcal{N}_\omega$.
  By assumption, $\phi_\omega|_{\mathcal{N}^{\sigma^{\phi_\omega}}}$ is a semi-finite trace.
- The operators $[K_\omega, \pi_\omega(x)]_-$, for all $x \in \mathcal{A}_\omega$ extend to operators in $\mathcal{M}_\omega \subset \mathcal{N}_\omega$. Since $\pi(\mathcal{A}_\omega)\xi_\omega$ is a core for the operator K_ω that is invariant for all the bounded operators $\pi_\omega(x)$ with $x \in \mathcal{A}_\omega$, we have that $\mathrm{Dom}(K_\omega)$ is invariant under all $\pi_\omega(x)$, for $x \in \mathcal{A}_\omega$.
- The hypothesis on the τ_ω-compactness of the resolvent of K_ω concludes the proof.

Relation with Modular Spectral Triples 8

Proposition

If $(\mathcal{A}, \mathcal{H}_\omega, D)$ is a modular spectral triple relative to a semi-finite von Neumann algebra $\mathcal{N} \subset \mathcal{B}(\mathcal{H}_\omega)$ and an α-KMS state ω on $\mathcal{A}$ whose Dirac operator D concides with the modular generator K_ω of the modular one-parameter group $t \mapsto \Delta_\omega^{it}$ on $\mathcal{H}_\omega$, the uniquely dermined data $(\mathcal{A}_\omega, \mathcal{H}_\omega, \xi_\omega, D, J_\omega)$ provide the modular spectral geometry associated to the pair $(\mathcal{A}, \alpha)$.

- Since, all the interesting examples of modular spectral triples available for now are equipped with a Dirac operator that is proportional to the modular generator K_ω, the proposition above says that modular spectral triples are "essentially" specific modular spectral geometries.

Physical Meaning of Modular SG 1

- We are trying to develope[84] an "event interpretation" of the formalism of states and observables in algebraic quantum physics that is in line with C. Isham's "history projection operator theory" [85] and/or C. Rovelli's "relativistic quantum theory"[86].
- We would like to conjecture that Tomita-Takesaki theorem plays a role of quantum Einstein equation: it associates to a state ω of the physical system a suitable non-commutative geometry.

[84] P.B., Algebraic Formalism for Rovelli Quantum Theory, in preparation.

[85] See for example C. Isham, Quantum Logic and the Histories Approach to Quantum Theory, J. Math. Phys., **35**, 2157-2185 (1994), gr-qc/9308006.

[86] C. Rovelli, Quantum Gravity, Section 5.2, Cambridge (2004).

Physical Meaning of Modular SG 2

We are aware that, in the context of Rovelli thermal time hypothesis, the modular automorphism group has been proposed as a way to recover a notion of "classical time". We point out that:

- Every self-adjoint operator (including the Dirac operator !) is the generator of a one parameter group of unitaries.
- The usual interpretation of KMS-states as equilibrium states for the modular one-parameter group and hence the association of Tomita theory with a time evolution is based on non-relativistic quantum theory. A fully covariant interpretation of the modular group might be possible, where the evolution parameter is a "scalar" moreover a covariant "thermal space-time hypothesis" should be viable.
- Tomita-Takesaki theory carry information on the deep quantum structure of the observables while the thermal time hypothesis mainly requires non-pure "statistical" states.

Physical Meaning of Modular SG 3

The data $(\mathcal{A}, \mathcal{H}_\omega, K_\omega)$ look essentially as those in Rovelli relativistic quantum theory with:

- $\mathcal{A}$ as the algebra of partial observables,
- the modular generator K_ω as the dynamical constraint,
- the fixed point algebra $\mathcal{A}^{\sigma^\omega}$ as the algebra of complete observables: $x \in \mathcal{A}^{\sigma^\omega} \Leftrightarrow [K_\omega, \pi_\omega(x)]_- = 0$.

The Hilbert space $\mathcal{H}_\omega$ is "too big" to be identified with the space of kinematical states (Rovelli 's boundary space?): the representation π_ω is not necessarily irreducible (the commutant $\pi_\omega(\mathcal{A})'$ is isomorphic to $\pi_\omega(\mathcal{A})''$). Some "kind of polarization" in $\mathcal{H}_\omega$ is needed to recover the kinematical Hilbert space.

Physical Meaning of Modular SG 4

It is not immediately clear what kind of "nc-geometry" is described by the modular spectral geometries $(\mathcal{A}_\omega, \mathcal{H}_\omega, K_\omega, J_\omega)$ reconstructed from Tomita-Takesaki, but they seem to be geometries related to the "phase space" of the physical system.

In order to obtain information on the geometry of "configuration space" it is necessary to identify "non-commutative Cartan subalgebras of coordinates" $\mathcal{B} \subset \mathcal{A}$ that allow to reconstruct the algebra of observables $\mathcal{A}$ as a "crossed product"[87].

To exploit this, a theory of crossed products of spectral triples should be developed.

Takesaki duality in modular theory should be relevant in this context.

[87] See the important results in J. Renault, Cartan Subalgebras in C*-algebras, `arXiv:0803.2284v1` and R. Exel, Noncommutative Cartan Subalgebras of C*-algebras, `arXiv:0806.4143v1`.

Categories of Modular Spectral Geometries 1

- Making contact with our current research project on "categorical non-commutative geometry" and possibly with other projects in categorical quantum gravity[88],[89], we will generalize the diffeomorphism covariance group of general relativity in a categorical context and use it to "indentify" the degrees of freedom related to the spatio-temporal structure of the physical system.

[88] J. Baez, Higher-Dimensional Algebra and Planck-Scale Physics, `arXiv:gr-qc/9902017`; J. Baez, Quantum Quandaries: a Category Theoretic Perspective, `arXiv:quant-ph/0404040`.

[89] L. Crane, Categorical Geometry and the Mathematical Foundations of Quantum Gravity, `gr-qc/0602120`; L. Crane, What is the Mathematical Structure of Quantum Spacetime, `arXiv:0706.4452`.

Categories of Modular Spectral Geometries 2

Let ω be just a state over a C*-algebra of observables $\mathcal{C}$.
Consider the family $\mathfrak{A}(\omega, \mathcal{C})$ of C*-subalgebras $\mathcal{A}$ of $\mathcal{C}$ such that ω is a KMS-state when restricted to $\mathcal{A}$.
Consider the family $\mathfrak{M}(\omega, \mathcal{C})$ of Hilbert C*-bimodules $\mathcal{M}$ over pairs of algebras in $\mathfrak{A}(\omega, \mathcal{C})$ whose linking C*-category is KMS for the state ω.
The family $\mathfrak{M}(\omega, \mathcal{C})$ becomes an involutive category under involution and tensor product of bimodules.
We have a tautological Fell bundle over the *-category $\mathfrak{M}(\omega, \mathcal{C})$ whose total space is the disjoint union of the bimodules in $\mathfrak{M}(\omega, \mathcal{C})$ and the projection functor assign to every $x \in \mathcal{M}$ the base point $\mathcal{M}$.
The modular geometry over this Fell bundle and over its "convolution algebra" are under investigation (as an example of bivariant spectral geometry).

Categories of Modular Spectral Geometries 3

In AQFT a quantum field theory is defined as a functor from a category of "geometries" (usually globally hyperbolic Lorentzian manifolds) to the category of *-homomorphisms of unital C*-algebras (localized observables).[90]

In our context it seems more appropriate to "reverse the functor": to every state over a unital C*-algebra we associate a functor from the category $\mathfrak{A}(\omega, \mathcal{C})$ with "suitable inclusions of algebras" to a category of modular spectral geometries.

[90] R. Brunetti, K. Fredenhagen, R. Verch, The Generally Covariant Locality Principle - A New Paradigm for Local Quantum Physics, Commun. Math. Phys., **237**, 31-68 (2003), `math-ph/0112041`.
R. Brunetti, M. Porrmann, G. Ruzzi, General Covariance in Algebraic Quantum Field Theory, `math-ph/0512059`.

Finding the Macroscopic Geometry 1

Tomita-Takesaki modular flow is trivial for commutative algebras: it is impossible to recover classical geometries in this way.

One possibility is to try to obtain macroscopic "emergent" classical geometries via some form of "coarse graning":

- "decoherence/einselection"[91],[92]
- "coherent states"[93],[94]

[91] H.D.Zeh, Roots and Fruits of Decoherence, quant-ph/0512078.
[92] W.H.Zurek, Decoherence, Einselection and the Quantum Origin of the Classical, quant-ph/0105127.
[93] B.Hall, The Segal-Bargmann "Coherent State" Transform for Compact Lie groups, J. Funct. Anal. 122 (1994) 103-151
[94] J.R.Klauder, On the Role of Coherent States in Quantum Foundations, arXiv:1008.4307v1.

Finding the Macroscopic Geometry 2

- "emergence/noiseless subsystems"[95],[96]
- or the "cooling" procedure developed by A.Connes-K.Consani-M.Marcolli[97],

Since the obstacles here are the essentially the same of those encountered in obtaining classical mechanics as a limit of quantum mechanics, other interesting possibilities are still open:

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[95] T.Konopka, F.Markopoulou, Constrained Mechanics and Noiseless Subsystems, `gr-qc/0601028`; F.Markopoulou, Towards Gravity form the Quantum, `hep-th/0604120`.

[96] O.Dreyer, Emergent Probabilities in Quantum Mechanics, `quant-ph/0603202`.; O.Dreyer, Emergent General Relativity, `gr-qc/0604075`.

[97] A.Connes, M.Marcolli, Noncommutative Geometry, Quantum Fields and Motives, 2008; A.Connes, K.Consani, M.Marcolli, Noncommutative Geometry and Motives: the Thermodynamics of Endomotives, `math.QA/0512138`.

Finding the Macroscopic Geometry 3

- classical geometries from algebraic "superselection theory"[98]
- "macro-observables subalgebras"[99]
- classical/quantum representations of observable algebras [100] and of "Poisson-Rinehart algebras".[101]

[98] K.Fredenhagen, Observables, Superselection Sectors and Gauge Groups, NATO Adv. Sci. Inst. Ser. B Phys., 295 (1992) 177-194.

[99] M.Requardt, An Alternative to Decoherence by Environment and the Appearance of a Classical World, `arXiv:1009.1220v1`.

[100] D.Mauro, A New Quantization Map, `arXiv:quant-ph/0305063v1`.

[101] G.Morchio, F.Strocchi, The Noncommutative Poisson Algebra of Classical and Quantum Mechanics, `arXiv:0805.2870v1`;
G.Morchio, F.Strocchi, Classical and Quantum Mechanics from the Universal Poisson-Rinehart Algebra of a Manifold, `arXiv:0901.0870v1`.

Links with the Aastrup Grimstrup Nest Spectral Triples 1

As a first tentative step in the direction of the construction of spectral triples for loop quantum gravity by J.Aastrup, J.Grimstrup, M.Paschke, R.Nest, we give an alternative family of spectral triples in loop quantum gravity that arise from a direct application of C.Antonescu-E.Christensen's construction of spectral triples for AF C*-algebras.

These new spectral triples depend on the choice of a countable filtration of graphs and on the choice of a sequence of positive real numbers and, although very similar to J.Aastrup-J.Grimstrup-R.Nest triples, they still differ from them because of the following notable points:

Links with the Aastrup Grimstrup Nest Spectral Triples 2

- the Hilbert spaces on which these spectral triples naturally live are identified with subspaces of the kinematical Hilbert space of usual loop quantum gravity, without the need of resorting to tensorization with a continuous part and a matricial part,
- their construction does not depend on the arbitrary choices of Riemannian structures on Lie groups and of Dirac operators on the Clifford bundle of such classical manifolds: their Dirac operators, coming directly from the C.Antonescu-E.Christensen's recipe for spectral triples for AF C*-algebras, seem to be completely quantal,
- they appear to be spectral triples and not semi-finite spectral triples.

Links with the Aastrup Grimstrup Nest Spectral Triples 3

- Consider a family of finite oriented graphs Γ that is directed by inclusion
- Let G be a compact Lie group (usually $SU(2)$ in loop quantum gravity)
- The kinematical Hilbert space of loop quantum gravity is an inductive limit of the net of Hilbert spaces $\Gamma \mapsto \mathcal{H}_\Gamma := L^2(G^{\Gamma_1}, \mu_\Gamma)$
- For any given countable chain of inclusions of finite oriented graphs $\Gamma^0 \subset \Gamma^1 \subset \cdots \subset \Gamma^n \subset \cdots$, the net of C*-algebras $\mathcal{A}_{\Gamma^1} \subset \mathcal{A}_{\Gamma^2} \subset \cdots \subset \mathcal{A}_{\Gamma^n} \subset \cdots$, generated by the holonomies associated to the respective graphs, form a filtration of finite dimensional C*-algebras with inductive limit $\mathcal{A}_{(\Gamma^n)} \subset \mathcal{A}$ a C*-subalgebra of the algebra $\mathcal{A}$ generated by holonomies acting on the inductive limit Hilbert space $\mathcal{H}_{(\Gamma^n)} \subset \mathcal{H}$.

Links with the Aastrup Grimstrup Nest Spectral Triples 4

- The same applies to the partial observable algebras of holonomies and fluxes $\mathcal{F}_\Gamma$ that are irreducibly reprensented on the respective kinematical Hilbert spaces

An immediate application of C.Antonescu-E.Christensen construction of spectral triples for AF C*-algebras provides:

Theorem

For every given countable inclusion $\Gamma^0 \subset \Gamma^1 \subset \cdots \subset \Gamma^n \subset \cdots$ of finite oriented graphs there is a family of spectral triples $(\mathcal{F}_{(\Gamma^n)}, \mathcal{H}_{(\Gamma^n)}, D_{(\Gamma^n),(\alpha_n)})$ depending on the choice of the filtration of graphs (Γ^n) and on a sequence of positive real numbers (α_n) as in Antonescu Christensen construction.

Modular Theory in Covariant Quantum Theories 1

Exactly as in quantum statistical mechanics (where KMS-states and modular theory subsume the theory of Gibbs equilibrium states extending it to systems with infinite degrees of freedom), in a general covariant quantum theory (with truncated degrees of freedom) the dynamical constraint can be specified by the choice of a KMS-state.

- The C*-algebra $\mathcal{F}$ of partial observables (generated by fluxes and holonomies) in loop quantum gravity is irreducibly represented on the kinematical Hilbert space GNS-Hilbert space $\mathcal{K}_\phi$ of the covariant Fock vacuum ϕ.

Modular Theory in Covariant Quantum Theories 2

- Consider the case of a theory with finite (truncated) degrees of freedom. Let H be the self adjoint dynamical constraint. The state $\omega_\beta(x) := \tau(\rho_\beta x)$, where τ is the canonical trace on $\mathcal{B}(\mathcal{K}_\phi)$ and $\rho_\beta := e^{-\beta H}/\tau(e^{-\beta H})$ is a density operator, is the unique Gibbs KMS state at inverse temperature β with respect to the one-parameter group $t \mapsto \mathrm{Ad}_{e^{itH}}$.
- On the Liouville Koopman von Neumann Hilbert space $\mathcal{K}_\phi \otimes \mathcal{K}^*_\phi$ the algebra $\mathfrak{F}$ is already represented in standard form and the "covariant vacuum" vector $\xi_\omega := \rho_\omega^{-1/2}$ is cyclic separating and generates a modular one parameter group with generator $H \otimes I - I \otimes (\Lambda_\phi H \Lambda_\phi^{-1})$.

Modular Theory in Covariant Quantum Theories 2

- As a corollary this implies that if the algebra of partial observables $\mathcal{F}$ if an AF C*-algebra, the Dirac operator obtained by C.Antonescu-E.Christensen construction $(\mathcal{F}, \mathcal{H}_\phi, D_{(\Gamma^n),(\alpha_n)})$, via the process of tensorial symmetrization, can be determined (modulo a global multiplication constant) by the specification of a KMS state ω, by imposing the equation $D_{(\Gamma^n),(\alpha_n)} \otimes I - I \otimes (\Lambda_\phi D_{(\Gamma^n),(\alpha_n)} \Lambda_\phi^{-1}) = K_\omega$ on the space $\mathcal{H}_\phi \otimes \mathcal{H}_\phi^*$. Again, at this "toy model" level, the specification of a KMS-states essentially determines a non-commutative geometry on the algebra of observables.

Connection with Other Approaches to Quantum Geometry

- Possible reproduction of quantum geometries already defined in the context of S.Doplicher-J.Roberts-K.Fredenhagen models[102] deserves to be investigated.
- Important connections of these ideas to "quantum information theory" and "quantum computation" are also under consideration[103].

[102] S.Doplicher, K.Fredenhagen, J.Roberts, The Structure of Spacetime at the Planck Scale and Quantum Fields, Commun. Math. Phys., **172**, 187 (1995), `hep-th/0303037`.

[103] P.B., Hypercovariant Theories and Spectral Space-time, unpublished (2001).

Foundations of Quantum Theory 1

Quantum Theory is notoriously plagued by unresolved conceptual problems:

- the problem of measurement
- the problem of time for covariant quantum theories
- conflicts with classical determinism
- no satisfactory operational/axiomatic foundation exists so far

Despite the recurrent claims of a need for a theory that supersedes, modifies or extends quantum theory (hidden variables, several alternative interpretations, collapse of wave function, deterministic derivations of quantum theory), in our view, the problems stem form the fact that quantum theory still now is essentially an incomplete theory, incapable of "standing on its own feet":

Foundations of Quantum Theory 2

- *In all the current formulations of quantum theory, the basic degrees of freedom of a theory are specifically introduced "by hand" and make always reference to a classical underlying geometry:*
    * Dirac canonical quantization via imposition of CCR for pairs of conjugated classical variables,
    * Weyl quantization from classical phase-spaces
    * second quantization from symplectic spaces
    * deformation quantization from Poisson manifolds or algebras
- *"unhealthy" usage of classical notions of space-time manifolds are usually recognized as responsible for the problems of divergences in Quantum Field Theory, but the attempted cures simply try to substitute "by hand" classical geometries with non-commutative conterparts instead of:*

Foundations of Quantum Theory 3

- **looking for an intrinsic characterization of degrees of freedom (space-time) "within Quantum Theory" itself.**

Notable exceptions in this direction:

* attempts to spectrally reconstruct space-time via operational data (states-observables) in AQFT: U.Bannier, S.J.Summers-R.White,[104]
* efforts in information theoretical foundation of quantum theory: J.A.Wheeler ["it from bit"], C.Rovelli [quant-ph/9609002v2], J.Bub-R.Clifton-H.Halvorson [quant-ph/0211089v2], A.Grimbaum [quant-ph/0306079v2, /0410071v1, /0509104v2], G.M.D'Ariano [arXiv:0807.4383v5],
* R.Haag's "quantum events" [Local Quantum Physics, Sec.VII],
* C.Rovelli's relational/relativistic quantum theory [QG, Sec.5.6.4].

[104] U.Bannier, Internat. J. Theor. Phys. 33 (1994) 1797-1809; S.J.Summers-R.White, Commun. Math. Phys. 237 (2003) 203-220.

Foundations of Quantum Theory 4.

Our main ideological stand on the issue is that:[105]

* *space-time should be spectrally reconstructed a posteriori from a basic operational theory of observables and states;*
* *A.Connes' non-commutative geometry provides the natural environment where to attempt an implementation of the spectral reconstruction of space-time;*
* *Tomita-Takesaki modular theory should be the main tool to achieve the previous goals, associating to operational data, spectral non-commutative geometries.*

[105] For related ideas on the reconstruction of space-time via purely quantum theoretical constructs see C.Rovelli-F.Vidotto [arXiv:0905.2983] and C.Rovelli [Quantum Gravity, Sec.5.6.4].

Foundations of Quantum Theory 5.

Contrary to most of the proposals for fundamental theories in physics,[106] our approach (if ever successful) will only provide an absolutely general operational formalism to model information acquisition and communication/interaction between quantum observers (described via certain categories of algebras of operators) and to extract from that some geometrical data in the form of a non-commutative geometry of the system.

Notable development in this same ideological direction can be found in the work of R.P.Kostecki.[107]

[106] that are usually of an ontological character, postulating basic microscopic degrees of freedom and their dynamics with the goal to explain known macroscopic behaviour

[107] Ryszard Paweł Kostecki, Quantum Theory as Inductive Inference, `arXiv:1009.2423v2`.

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  arXiv:0801.2826v1.[108]

[108] Version 2 coming soon ;-)

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