Sheaf quantization of Hamiltonian isotopies and applications

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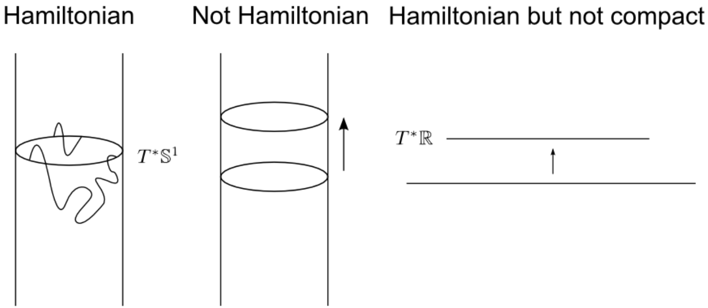
Arnold non displaceability conjecture has been solved for long. See Chaperon, Conley–Zehnder, Hofer, Laudenbach–Sikorav, etc. .
Consider a compact manifold M and a Hamiltonian isotopy $\Phi = \{\varphi_t\}_{t\in I}$, that is, the $\varphi_t\colon T^*M \to T^*M$ are symplectomorphisms and $\frac{\partial}{\partial_t}\Phi$ is the Hamiltomian vector field of a time dependant function f defined on T^*M. Then $\varphi_t(T^*_M M) \cap T^*_M M \neq \emptyset$ for all $t \in I$.



Recently Tamarkin gave a totally new proof using the microlocal theory of sheaves of Kashiwara-S. However, the microlocal theory of sheaves is associated with the homogeneous symplectic structure of the cotangent bundle T^*M and Tamarkin had to develop a non homogeneous microlocal theory of sheaves by adding a variable, which makes his proofs really intricated.
Here, we do the contrary, which is much simpler. We transform the geometrical problems on T^*M viewed as a symplectic manifold to problem on $\dot{T}^*(M \times \mathbb{R})$ (the cotangent bundle minus the zero section) viewed as a homogeneous symplectic manifold.
The main tool is a theorem of quantization of homogeneous Hamiltonian isotopy in the framework of sheaves.

Non displaceability (homogeneous symplectic version)

Consider a manifold M, a map $\psi\colon M \to \mathbb{R}$ and assume $d\psi(x) \neq 0$ for all $x \in M$. Set $\Lambda_\psi = \{(x; d\psi(x)); x \in M\} \subset T^*M$.
Let N be a compact non empty manifold (eventually with boundary or even corners). Among other results of non-displaceability, we shall prove:

Theorem Let $\{\varphi_t\}_{t\in I}$ be a homogeneous symplectic isotopy.
Then $\varphi_t(T^*_N M) \cap \Lambda_\psi \neq \emptyset$ for all $t \in I$.
Moreover, assume that for some $t_0 \in I$ the intersection $\varphi_{t_0}(T^*_N M) \cap \Lambda_\psi$ is transversal. Then

$$\#(\varphi_{t_0}(T^*_N M) \cap \Lambda_\psi) \geq \sum_j b_j(N).$$

References

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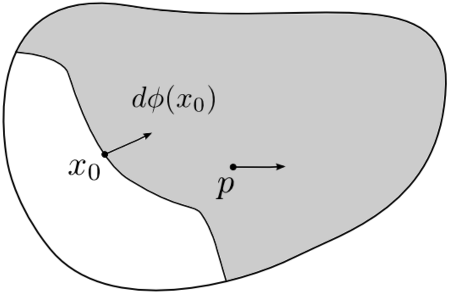
Microsupport

We work (for simplicity) over a field $\mathbf{k}$. We denote by $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ the bounded derived category of sheaves of $\mathbf{k}$-modules on M.
For a locally closed subset Z of M, we denote by $\mathbf{k}_Z$ the constant sheaf with stalk $\mathbf{k}$ on Z, extended by 0 on $M \setminus Z$.

Definition (Microsupport or singular support of a sheaf, K-S 81.)
Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ and let $p \in T^*M$. One says that $p \notin \mathrm{SS}(F)$ if there exists an open neighborhood U of p such that for any $x_0 \in M$ and any real C^1-function φ on M defined in a neighborhood of x_0 with $(x_0; d\varphi(x_0)) \in U$, one has $\mathrm{R}\Gamma_{\{x;\varphi(x)\geq\varphi(x_0)\}}(F)_{x_0} \simeq 0$.
In other words, $p \notin \mathrm{SS}(F)$ if the sheaf F has no cohomology supported by "half-spaces" whose conormals are contained in a neighborhood of p.

- The microsupport is $\mathbb{R}^+$-conic, that is, invariant by the action of $\mathbb{R}^+$ on T^*M.
- $\mathrm{SS}(F) \cap T^*_M M = \pi_M(\mathrm{SS}(F)) = \mathsf{Supp}(F)$.
- The microsupport is additive: if $F_1 \to F_2 \to F_3 \xrightarrow{+1}$ is a distinguished triangle in $\mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$, then $\mathrm{SS}(F_i) \subset \mathrm{SS}(F_j) \cup \mathrm{SS}(F_k)$ for all $i, j, k \in \{1, 2, 3\}$ with $j \neq k$.
- The microsupport is involutive (i.e., co-isotropic).

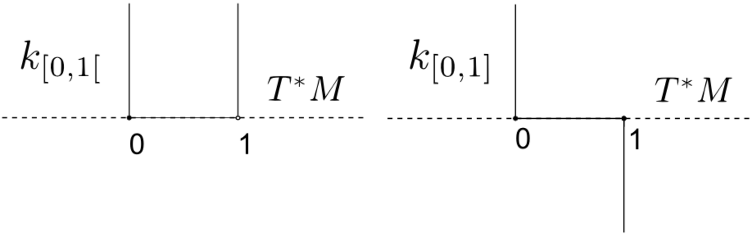


Examples

Example

(i) If F is a non-zero local system on M and M is connected, then $\mathrm{SS}(F) = T^*_M M$, the zero-section.
(ii) If N is a closed submanifold of M and $F = \mathbf{k}_N$, then $\mathrm{SS}(F) = T^*_N M$, the conormal bundle to N in M.
(iii) Let φ be a C^1-function such that $d\varphi(x) \neq 0$ whenever $\varphi(x) = 0$. Let $U = \{x \in M; \varphi(x) > 0\}$ and let $Z = \{x \in M; \varphi(x) \geq 0\}$. Then

$$\mathrm{SS}(\mathbf{k}_U) = U \times_M T^*_M M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \leq 0\},$$
$$\mathrm{SS}(\mathbf{k}_Z) = Z \times_M T^*_M M \cup \{(x; \lambda d\varphi(x)); \varphi(x) = 0, \lambda \geq 0\}.$$



Operations

Let $f\colon M \to N$ be a morphism of real manifolds. To f are associated the diagrams

$$
\begin{tikzcd}
TM \arrow[r, "f'"] \arrow[d, "\tau_M"] & M \times_N TN \arrow[r, "f_\tau"] \arrow[d] & TN \arrow[d, "\tau_N"] \\
M \arrow[r, equal] & M \arrow[r, "f"] & N.
\end{tikzcd}
\qquad
\begin{tikzcd}
T^*M \arrow[d, "\pi_M"] & M \times_N T^*N \arrow[l, "f_d"'] \arrow[r, "f_\pi"] \arrow[d] & T^*N \arrow[d, "\pi_N"] \\
M \arrow[r, equal] & M \arrow[r, "f"] & N.
\end{tikzcd}
$$

Let $\Lambda_M \subset T^*M$ be a closed $\mathbb{R}^+$-conic subset. Then f_π is proper on $f_d^{-1}\Lambda_M$ if and only if f is proper on $\Lambda_M \cap T^*_M M$.

Let $\Lambda_N \subset T^*N$ be a closed $\mathbb{R}^+$-conic subset. Then f_d is proper on $f_\pi^{-1}\Lambda_N$ if and only if $f_\pi^{-1}\Lambda_N \cap f_d^{-1} T^*_M M \subset M \times_N T^*_N N$. In this case, one says that f is non-characteristic for Λ_N.

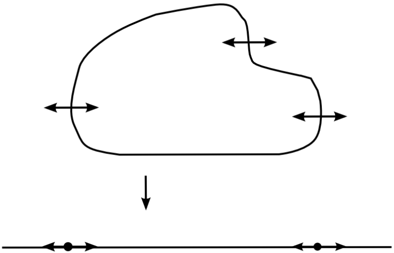
Operations

Let $f\colon M \to N$ be a morphism of manifolds, and recall the maps:
$T^*M \xleftarrow{f_d} M \times_N T^*N \xrightarrow{f_\pi} T^*N$.

Theorem Let $F \in \mathrm{D^b}(\mathbf{k}_M)$ and let $G \in \mathrm{D^b}(\mathbf{k}_N)$.

(i) (The stationary phase lemma.)
Assume that f is proper on $\mathrm{Supp}(F)$. Then $\mathrm{SS}(\mathrm{R}f_* F) \subset f_\pi f_d^{-1}\mathrm{SS}(F)$.

(ii) Assume that f is non-characteristic for $\mathrm{SS}(G)$. Then
$\mathrm{SS}(f^{-1}G) \subset f_d f_\pi^{-1}\mathrm{SS}(G)$.



The Morse lemma

Theorem (The Morse lemma for sheaves.)
Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$, let $\psi\colon M \to \mathbb{R}$ be a function of class C^1 and assume that ψ is proper on $\mathrm{Supp}(F)$. For $t \in \mathbb{R}$, set $M_t = \psi^{-1}(]-\infty, t[)$. Let $a < b$ in $\mathbb{R}$ and assume that $d\varphi(x) \notin \mathrm{SS}(F)$ for $a \leq \psi(x) < b$. Then the restriction morphism $\mathrm{R}\Gamma(M_b; F) \to \mathrm{R}\Gamma(M_a; F)$ is an isomorphism.

Proof Consider $G = \mathrm{R}\psi_* F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{R}})$. Then $\mathrm{SS}(G) \cap \{(t; dt); t \in [a, b[\} = \emptyset$. Then $\mathrm{R}\Gamma(]-\infty, b[; G) \to \mathrm{R}\Gamma(]-\infty, a[; G)$ is an isomorphism by the definition of the micro-support.

Morse inequalities

For E a bounded complex of $\mathbf{k}$-vector spaces with finite-dimensional cohomology, one sets

$$b_j(E) = \dim H^j(E), \quad b_l^*(E) = (-1)^l \sum_{j \le l} (-1)^j b_j(E).$$

Consider a map $\psi\colon M \to \mathbb{R}$ of class C^2 and define Λ_ψ as above.
Let $F \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ with compact support. Assume $\Lambda_\psi \cap \mathrm{SS}(F)$ is finite, say $\{p_1, \dots, p_N\}$, and, setting $x_i = \pi(p_i)$, $V_i := (\mathrm{R}\Gamma_{\{\psi(x) \ge \psi(x_i)\}}(M; F))_{x_i}$, also assume that the cohomologies of the V_i's are finite-dimensional $\mathbf{k}$-vector spaces. Set

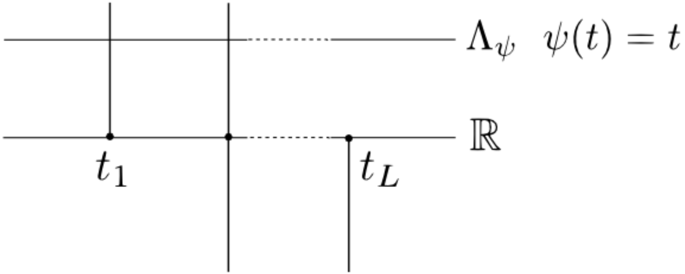
$$b_j(F) = \dim H^j(\mathrm{R}\Gamma(M; F)).$$

Then the Morse inequalities for sheaves are stated as:

$$b_l^*(F) \le \sum_{i=1}^{N} b_l^*(V_i). \tag{1}$$

Morse inequalities: idea of the proof.

Set $G = \mathrm{R}\psi_* F$. Then $G \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{\mathbb{R}})$ has compact support and we are reduced to prove the result when $M = \mathbb{R}$, $\psi(t) = t$.
Set $\{t_1, \dots, t_L\} = \psi(\{x_1, \dots, x_N\})$, the critical values of ψ w.r.t. $\mathrm{SS}(F)$. Then $\mathrm{SS}(G) \cap \{(t, dt)\}$ is contained in the set $\{(t_1; dt), \dots, (t_L; dt)\}$.



Set $I_j =]-\infty, t_j[$ and $Z_j =]-\infty, t_j]$. The proof uses the isomorphisms $\mathrm{R}\Gamma(I_{j+1}; G) \xrightarrow{\sim} \mathrm{R}\Gamma(Z_j; G)$ and the distinguished triangles

$$(\mathrm{R}\Gamma_{t \geq t_j}(G))_{t_j} \to \mathrm{R}\Gamma(Z_j; G) \to \mathrm{R}\Gamma(I_j; G) \xrightarrow{+1} .$$

Kernels

Let M_i $(i = 1, 2, 3)$ be manifolds. We set
$M_{ij} := M_i \times M_j$, $(1 \le i, j \le 3)$, $M_{123} = M_1 \times M_2 \times M_3$.
$q_i\colon M_{ij} \to M_i$ or $q_i\colon M_{123} \to M_i$, $q_{ij}\colon M_{123} \to M_{ij}$. $p_i\colon T^*M_{ij} \to T^*M_i$ or
$p_i\colon T^*M_{123} \to T^*M_i$, $p_{ij}\colon T^*M_{123} \to T^*M_{ij}$,
p_{12^a}: the composition of p_{12} and the antipodal map on T^*M_2.
We consider the operation of convolution of kernels:

$$\begin{aligned}
\circ\colon \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{12}}) \times \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{23}}) &\to \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M_{13}}) \\
(K_1, K_2) &\mapsto K_1 \circ K_2 := \mathrm{R}q_{13!}(q_{12}^{-1}K_1 \otimes q_{23}^{-1}K_2).
\end{aligned}$$

Assume that

$$\begin{cases}
\text{(i) } q_{13} \text{ is proper on } q_{12}^{-1}\operatorname{Supp}(K_1) \cap q_{23}^{-1}\operatorname{Supp}(K_2), \\
\text{(ii) } \quad {p_{12^a}}^{-1}\mathrm{SS}(K_1) \cap {p_{23}}^{-1}\mathrm{SS}(K_2) \cap (T^*_{M_1}M_1 \times T^*M_2 \times T^*_{M_3}M_3) \\
\qquad \subset T^*_{M_1\times M_2\times M_3}(M_1 \times M_2 \times M_3).
\end{cases}$$

Then

$$\mathrm{SS}(K_1 \circ K_2) \subset p_{13}(p_{12^a}^{-1}\mathrm{SS}(K_1) \cap p_{23}^{-1}\mathrm{SS}(K_2)).$$

Hamiltonian isotopies

Consider $\Phi = \{\varphi_t\}_{t\in I}\colon \dot{T}^*M \times I \to \dot{T}^*M$ such that φ_t is a homogeneous symplectic isomorphism for each $t \in I$ and $\varphi_0 = \mathrm{id}_{\dot{T}^*M}$.
Then Φ is a Hamiltonian isotopy and there exists a conic Lagrangian submanifold Λ of $\dot{T}^*M \times \dot{T}^*M \times T^*I$ whose projection is the graph of Φ in $\dot{T}^*M \times \dot{T}^*M \times I$. Moreover, the set $\Lambda \cup T^*_{M\times M\times I}(M \times M \times I)$ is closed in $T^*(M \times M \times I)$ and for any $t \in I$ the inclusion $i_t\colon M \times M \to M \times M \times I$ is non-characteristic for Λ and the graph of φ_t is $\Lambda_t = \Lambda \circ T^*_t I$.

Main theorem

For $K \in \mathrm{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times I})$ and $t_0 \in I$, we set

$$K_{t_0} := K \circ \mathbf{k}_{t_0} \simeq K|_{t=t_0}.$$

Theorem
We consider $\Phi\colon \dot{T}^*M \times I \to \dot{T}^*M$ as above. Then there exists $K \in \mathrm{D}^{\mathrm{lb}}(\mathbf{k}_{M\times M\times I})$ satisfying

(a) $\mathrm{SS}(K) \subset \Lambda \cup T^*_{M\times M\times I}(M \times M \times I)$,

(b) $K_0 \simeq \mathbf{k}_\Delta$.

Moreover:

(i) both projections $\mathrm{Supp}(K) \rightrightarrows M \times I$ are proper,

(ii) $K_t \circ K_t^{-1} \simeq K_t^{-1} \circ K_t \simeq \mathbf{k}_\Delta$ for all $t \in I$,

(iii) such a K satisfying the conditions (a) and (b) above is unique up to a unique isomorphism,

Sketch of proof

Unicity. Assume K_1 and K_2 satisfy (a) and (b) and set $L = K_2^{-1} \circ K_1$. Then $\mathrm{SS}(L) \subset T^*M \times T^*M \times T_I^*I$. Therefore $L \simeq q^{-1}\mathrm{R}q_* L$ where $q\colon M \times M \times I \to M \times M$ is the projection. Since $L|_{t=0} \simeq \mathbf{k}_\Delta$, the result follows.

Existence. It decomposes into several steps, using the unicity.
(i) First we reduce to the case where the isotopy is the identity outside of a "conically compact" subset of $\dot{T}^*M$.
(ii) Next, again by using the unicity, we reduce to proving the result in a neighborhood of $t = 0$.
(iii) Then we construct a contact transform θ which interchanges the conormal to the diagonal with the conormal to a tubular neighborhood of the diagonal and show that one can find an invertible kernel L associated to θ.
(iv) By using θ and L we are reduced to quantize in a neighborhood of $t = 0$ a Lagrangian manifold $\Lambda \subset \dot{T}^*(M \times M) \times T^*I$ such that Λ_0 is the conormal to a tubular neighborhood of the diagonal, which is straightforward.

Non displaceability

Consider
a homogeneous Hamiltonian isotopy $\Phi = \{\varphi_t\}_{t\in I} \colon \dot{T}^*M \times I \to \dot{T}^*M$,
$\Lambda \subset \dot{T}^*(M \times M \times I)$ the conic Lagrangian submanifold associated to Φ,
$K \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M\times M\times I})$ the quantization of Φ.
Let $F_0 \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$ with compact support.
Set:

$$F = K \circ F_0 \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_{M\times I}),$$

$$F_{t_0} = F|_{\{t=t_0\}} \simeq K_{t_0} \circ F_0 \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M) \quad \text{for } t_0 \in I.$$

Lemma

(i) We have isomorphisms $\mathrm{R}\Gamma(M; F_t) \simeq \mathrm{R}\Gamma(M; F_0)$ *for all* $t \in I$.
(ii) $\mathrm{SS}(F_t) \subset \varphi_t(\mathrm{SS}(F_0) \cap \dot{T}^*M) \cup T^*_M M$.

Non displaceability

We consider a C^2-map $\psi\colon M \to \mathbb{R}$ and we assume that the differential $d\psi(x)$ never vanishes. Hence

$$\Lambda_\psi := \{(x; d\psi(x));\ x \in M\} \subset \dot{T}^*M.$$

Theorem We consider $\Phi = \{\varphi_t\}_{t\in I}$, $\psi\colon M \to \mathbb{R}$ and $F_0 \in \mathrm{D}^{\mathrm{b}}(\mathbf{k}_M)$. We assume $\mathrm{R}\Gamma(M; F_0) \neq 0$. Then for any $t \in I$, $\varphi_t(\mathrm{SS}(F_0) \cap \dot{T}^*M) \cap \Lambda_\psi \neq \emptyset$.
Proof Let us assume that the conclusion of the theorem is false for some $t \in I$. We deduce $\Lambda_\psi \cap \mathrm{SS}(F_t) = \emptyset$. Since $\mathrm{Supp}(F_t)$ is compact we may find $a, b \in \mathbb{R}$ such that $\psi(\mathrm{Supp}(F_t)) \subset]a, b[$. Then the Morse lemma for sheaves gives

$$\mathrm{R}\Gamma(]-\infty, b[; \mathrm{R}\psi_* F_t) \xrightarrow{\sim} \mathrm{R}\Gamma(]-\infty, a[; \mathrm{R}\psi_* F_t) \simeq 0.$$

This contradicts the isomorphism $\mathrm{R}\Gamma(M; F_t) \simeq \mathrm{R}\Gamma(M; F_0) \neq 0$.

Non displaceability: Morse inequalities

Theorem Let $\Phi = \{\varphi_t\}_{t \in I}$, F_0 and $\psi \colon M \to \mathbb{R}$ be as above. Set

$$S_0 = \mathrm{SS}(F_0) \cap \dot{T}^* M.$$

Let $t_0 \in I$. Assume that $\Lambda_\psi \cap \varphi_{t_0}(S_0)$ is contained in $\Lambda_\psi \cap \varphi_{t_0}(S_{0,\mathrm{reg}})$ and the intersection is finite and transversal. Then

$$\#\left(\varphi_{t_0}(S_0) \cap \Lambda_\psi\right) \geq \sum_j b_j(F_0).$$

Positive isotopies

Let $\Phi = \{\varphi_t\}_{t\in I} \colon \dot{T}^*M \times I \to \dot{T}^*M$ be a homogeneous Hamiltonian isotopy, let $\Lambda \subset \dot{T}^*M \times \dot{T}^*M \times T^*I$ be the associated Lagrangian manifold and let

$$f = \langle \alpha_M, \partial\Phi/\partial t\rangle.$$

Definition (Y. Eliashberg, S. Kim and L. Polterovich.) The isotopy Φ is said to be non-negative if $\langle \alpha_M, H_f\rangle \geq 0$ or equivalently, if $\Lambda \subset \{\tau \leq 0\}$..

Let $K \in \mathrm{D^b}(\mathbf{k}_{M\times M\times I})$ be the quantization of Φ and let $F_0 \in \mathrm{D^b}(\mathbf{k}_M)$ with compact support. Set:

$$F = K \circ F_0 \in \mathrm{D^b}(\mathbf{k}_{M\times I}),$$

$$F_{t_0} = F|_{\{t=t_0\}} \simeq K_{t_0} \circ F_0 \in \mathrm{D^b}(\mathbf{k}_M) \quad \text{for } t_0 \in I.$$

Lemma

(i) For $a \leq b$, we have natural morphisms $F_a \to F_b$ which induce the isomorphisms $\mathrm{R}\Gamma(M; F_a) \simeq \mathrm{R}\Gamma(M; F_b)$.
(ii) $\mathrm{SS}(F_t) \subset \varphi_t(\mathrm{SS}(F_0 \cap \dot{T}^*M) \cup T^*_M M$.

Positive isotopies

The next theorem generalizes previous results of Chernov and S. Nemirovski and V. Colin, E. Ferrand and P. Pushkar.

Theorem

Let M be a connected and non-compact manifold and let $\Phi\colon \dot{T}^*M \times I \to \dot{T}^*M$ be a non-negative homogeneous Hamiltonian isotopy. Assume that $[0,1] \subset I$ and that there exists two compact connected submanifolds X and Y of M such that $\varphi_1(\dot{T}^*_X M) = \dot{T}^*_Y M$. Then $X = Y$.