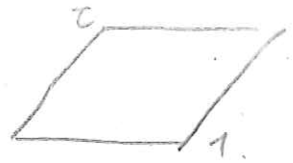
courbe elliptique

We begin w/ an ~~elliptic curve~~ $E_\tau = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$



$\mathrm{Im}\,\tau > 0$

$\mathcal{L} = \mathbb{C}\times\mathbb{C}/\mathbb{Z}^2 = (\mathcal{L}[0])$ fibré en droite

$(n,m)(z,u) = (z+n+m\tau,\ (-1)^m e^{2\pi i m z} u)$

$\chi(\mathcal{L}) =$ '1' generator of $H^2(\mathbb{T},\mathbb{Z}) = \mathbb{Z}$

section f: $\{ f(z+1) = f(z),\ f(z+\tau) = -e^{-2\pi i z} f(z), \}$ $\rightarrow$ modula relation

holomorphic function on $\mathbb{C}$ (can't be reduced to $\mathbb{T}^2$)

generated by $\theta_0(z,\tau) = \prod_{j=0}^{\infty}(1-e^{2\pi i((j+1)\tau - z)})(1-e^{2\pi i(j\tau+z)})$ $\rightarrow$ Jacobi theta function (comes from the first)

$\rightarrow$ could be extended to $\mathrm{Im}\,\tau \neq 0$ by $\theta_0(z,-\tau) := \theta_0(z+\tau,\tau)^{-1} = \theta_0(-z,\tau)^{-1}$ for $\mathrm{Im}\,\tau<0$.

holomorphic but has 0 at $n\tau$.

Now we have our elliptic gamma functions,

Recall Γ function (Euler): $\Gamma(z+1) = z\Gamma(z)$, $\leftarrow$ $\left(\Gamma(z) = \int_0^\infty x^{z-1}e^{-t}dt,\ \Gamma(n) = (n-1)!\right)$

$\zeta(z)\Gamma(z) = \int_0^\infty \frac{u^{z-1}}{e^u-1}du$ $\mathrm{Re}\,z>1$ $\left(\zeta(n) = \sum_{k=1}^{\infty}\frac{1}{k^n}\right)$

trigonometric (q-) Γ-function (Jackson 1900)

$\Gamma(z+1) = \frac{\sin\omega z}{\omega}\Gamma(z)$. $\omega\to 0$

change échelle
rescale

$\Gamma(z+\sigma) = \sin\pi z\, \Gamma(z)$

elliptic Γ-function (Rujsenaars–Baxter) (θ is the elliptic version of exp).

$\Gamma(z+\sigma) = \theta_0(z,\tau)\Gamma(z)$ $\quad \tau\to i\infty,\ \theta_0(z,\tau)\to(1-e^{2\pi i z}) = e^{\pi i z}\sin\pi z$

More precisely, there exists a unique solution of the diff. eq (Ruijsenaars, integrable quantum system)

$\mathrm{Im}\,\tau,\ \mathrm{Im}\,\sigma > 0$ $\quad \begin{cases} u(z+\sigma) = \theta_0(z,\tau)u(z) \\ u(z+1) = u(z) \end{cases}$

s.t. • $u(z)$ holom. in $\mathrm{Im}(z) > 0$

• $u(\frac{\tau+\sigma}{2}) = 1$

Then we call $u(z,\sigma) = \Gamma(z,\tau,\sigma)$),

$\Gamma(z,\tau,\sigma) = \prod_{j,k=0}^{\infty}\frac{1-e^{2\pi i((j+1)\tau+(k+1)\sigma - z)}}{1-e^{2\pi i(j\tau+k\sigma+z)}}$ $\rightarrow$ meromor.

& we can extend it to $\mathrm{Im}\,\tau<0$ on $\mathrm{Im}\,\sigma<0$ by

$\Gamma(z,\tau,\sigma) = \frac{1}{\Gamma(z-\tau,-\tau,\sigma)} = \frac{1}{\Gamma(z-\sigma,\tau,-\sigma)}$

For our purpose, we are interested in the gamma function w/ $\mathrm{Im}\,\tau<0$, $\mathrm{Im}\,\sigma>0$ it's $\Gamma(z,\tau,\sigma) = \prod_{j,k=0}^{\infty}\frac{1-e^{2\pi i(z-(j+1)\tau+k\sigma)}}{1-e^{2\pi i(z-j\tau+(k+1)\sigma)}}$.

Moreover it's generalised by Narukawa / Nishizawa to multiple gamma functions with

$G_0(z,\tau) = \theta_0(z,\tau)$ $\quad G_1(z,\tau,\sigma) = \Gamma(z,\tau,\sigma)$ and

$G_n(z+\tau_i,\tau_0,\dots,\tau_n) = G_{n-1}(z,\tau_0,\dots,\hat{\tau_i},\dots,\tau_n)\,G_n(z,\tau_0,\dots,\tau_n)$

(w/ modular relations

$\prod_{k=1}^{r} G_{r-2}\left(\frac{w}{x_k},\frac{x_1}{x_k},\dots,\widehat{\frac{x_k}{x_k}},\dots,\frac{x_r}{x_k}\right) = \exp\left(-\frac{2\pi i}{r!}B_{r,r}(w,x)\right)$

where $e^{wt}\prod_{k=1}^{r}\frac{t}{\dots} = \sum_{n=0}^{\infty} B_{r,n}(w,x_1,\dots,x_r)\frac{t^n}{n!}$)

This elliptic gamma function satisfies also modular relations. (2)

$$\Gamma(z+\tau,\tau,\sigma)=\theta_0(z,\sigma)\,\Gamma(z,\tau,\sigma) \qquad \Gamma(z+1,\tau,\sigma)=\Gamma(z,\tau,\sigma)$$

$$\Gamma(z+\sigma,\tau,\sigma)=\theta_0(z,\tau)\,\Gamma(z,\tau,\sigma) \quad (3\text{-period}).$$

& 3-term relation (Felder & Varchenko)

$$\Gamma(z,\tau,\sigma)=\Gamma(z,\tau,\tau+\sigma)\,\Gamma(z+\sigma,\tau+\sigma,\sigma)$$

$$\Gamma\left(\frac{w}{x_3},\frac{x_1}{x_3},\frac{x_2}{x_3}\right)\Gamma\left(\frac{w}{x_2},\frac{x_3}{x_2},\frac{x_1}{x_2}\right)\Gamma\left(\frac{w}{x_1},\frac{x_2}{x_1},\frac{x_3}{x_1}\right)=e^{-\pi i\frac{P_3(w,x)}{3}}$$

$\in P[w](x)$ degree 3 in w polynomial w/ $\frac{x_i}{x_j}$, $\frac{w}{x_j}$.

$$\Gamma(z,\tau,\sigma)=\prod_{j,k=0}^{\infty}\frac{\left(1-e^{2\pi i(-(j+1)\tau+(k\sigma+z))}\right)}{\left(1-e^{2\pi i(-j\tau+(k+1)\sigma-z)}\right)}$$

$\operatorname{Im}\tau<0 \quad \operatorname{Im}\sigma>0.$

Γ. (other important roles in physics like qKZB equation | solution of Knizhnik-Zamolodchikov-Bernard hypergeom. solution)

Now we ask ourselves, what is the geometric interpreation of Γ ??

Since gerbe is the next level of line bds, we conj the modula relation of Γ must be some sort of appearance of a gerbe. Now we leave the special functions a little bit and introduce gerbes.

Recall a holo line bd L over a mfd M, w/ an open covering $\{U_i\}$ is made up by a collection of transaction function $\varphi_{ij}\in\mathcal{O}^{\times}(U_i\cap U_j)$, s.t. $\varphi_{ij}\,\varphi_{jk}=\varphi_{ik}$ on $U_i\cap U_j\cap U_k$

Then $L=\coprod U_i\times\mathbb{C}/\sim$.

a holo gerbe $\mathcal{G}$ over a mfd M w/ open covering $\{U_i\}$ is made up by a collection of holo line bds L_{ij} on $U_i\cap U_j$, s.t. $L_{ij}\otimes L_{jk}\xleftarrow{\varphi_{ijk}} L_{ik}$ on $U_i\cap U_j\cap U_k$

s.t. $(\delta\varphi)_{ijkl}=1$, i.e. $\varphi_{ij,k}\,\varphi_{j,k,l}\,\varphi^{-1}_{i,k,l}\,\varphi^{-1}_{i,j,l}=1$.

This is a description by local charts, to get a global object as the case of line bds, we need to use the terminology (diff) stacks since when we put together $U_i\times\mathbb{C}$, it is a mfd, but when we put together L_{ij}, it is something on the next level. (It turns out) it is a (diff) stack. Here we introduce a stack (diff) via Lie gpds.

A groupoid $G_1 \overset{t}{\underset{s}{\rightrightarrows}} G_0$ is made up by

(a small cat where morp are all invertible)

- G_0 : obj space
- G_1 : space of arrows $\cdot \curvearrowleft \circ$ t: target map, s: source map.
- multiplication: comp of arrows : $m: G_1 \times_{G_0} G_1 \to G_1$, associativ $(gh)k = g(hk)$
- identity: $G_0 \xrightarrow{e} G_1$: identity arrow. s.t. $e(x) \cdot g = g = g \cdot e(y)$.
- Inverse : $x \underset{g^{-1}}{\overset{g}{\rightleftarrows}} y$ inverse arrow. s.t. $g \cdot g^{-1} = e(x)$, $g^{-1} g = e(y)$.

Lie gpd. G_1, G_0 mfd., smooth maps. s, t surj submersion. (gpd in diff cat)

Ex. ① Lie gp. $G_0 = pt$. $G_1 \rightrightarrows \bullet$ m..

e...

inverse.

② M. $\{U_i\}$ an open covering.

$x_i \in U_i$ $x_{ij} \in U_i \cap U_j$

rep. $x \in M$.

$$\begin{array}{c} U_i \cap U_j \\ \shortparallel \\ \sqcup U_{ij} \\ \downarrow\downarrow \\ \sqcup U_i \end{array} \qquad x_{ij} \to x_i,\ x_{ij} \to x_j$$

$x_{ij} \cdot x_{jk} = x_{ik}$.

$$\begin{array}{c} \sqcup U_{ij} \\ \downarrow\downarrow \\ \sqcup U_i \end{array} \ \text{Morita equivt} \ \begin{array}{c} \sqcup V_{ij} \\ \downarrow\downarrow \\ \sqcup V_i \end{array}$$

for the same of this talk one just has to remember it's some sort of equivc between Lie gpds.

$\Leftarrow$ $\{U_i\}$ $\{V_i\}$ diff cover of M.

this gpd $\begin{array}{c} \sqcup U_{ij} \\ \downarrow\downarrow \\ \sqcup U_i \end{array}$ "present" M.

<u>In general</u>, diff stack = Lie gpd / Morita equivc.

& $\mathfrak{X}$ a diff stack can be <u>"presented"</u> by $\begin{array}{c} X_1 \\ \downarrow\downarrow \\ X_0 \end{array}$ Lie gpds

though it also has internal descriptions via cat fibred in gpds we stay w/ this description in this lecture.

phylosophy: diff stack : Lie gpds

$\cup$ $\cup$

mfds : charts of mfds U_i

Now according to the desciption we have of gerbe

kernel $\quad \mathbb{C}^\times$-principal bd! another gpd.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \sqcup U_i \times \mathbb{C}^\times & \longrightarrow & \sqcup L_{ij}^\times & \longrightarrow & \sqcup U_{ij} & \longrightarrow & 1 \\ & & \downdownarrows & & \downdownarrows & & \downdownarrows & & \\ & & \sqcup U_i & \longrightarrow & \sqcup U_i & \xrightarrow{id} & \sqcup U_i & & \end{array}$$

morphism of gpd

the multiplication is formed by $\quad \underset{x_{ij}}{\overset{a}{\downarrow}} l_{ij} \quad \underset{x_{jk}}{\overset{\vec{a}}{\downarrow}} l_{jk} \xrightarrow{\varphi_{ijk}^{-1}} \underset{x_{ik}}{\downarrow} l_{ik}$

it's associative $\Leftrightarrow \delta\varphi = 1$. (we assume L_{ii} trivial (possible!)

It reminds us the centrale extension of gps, in fact it's a centrale extension of gpds, namely a short exact sequence of gpds w/ kernel in the center of the mid gpd.

So we define a $\mathbb{C}^\times$-gerbe $\mathcal{G} \dashrightarrow \begin{array}{c} G_1 \\ \downdownarrows \\ X_0 \end{array}$ s.t. $\begin{array}{ccccccccc} 1 & \to & X_0 \times \mathbb{C}^\times & \to & G_1 & \to & X_1 & \to & 1 \\ & & \downdownarrows & & \downdownarrows & & \downdownarrows & & \\ & & X_0 & \to & X_0 & \to & X_0 & & \end{array}$

$\quad\quad\quad\quad\quad\quad \downarrow$

$\quad\quad\quad\quad\quad\quad \mathcal{X} \dashrightarrow \begin{array}{c} X_1 \\ \downdownarrows \\ X_0 \end{array}$

extension centrale de gpd de Lie

Now our task is to find (a proper centrale extension from Γ.)

a gerbe over $\mathbb{C}/\mathbb{Z}^3$ (triptic curve) $\quad \mathbb{Z}^3$ acts by $\underbrace{(n+m\sigma+k\tau)+z}_{(n,m,k)\cdot z =}$, s.t. $\operatorname{Im}\tau \neq 0$, or $\operatorname{Im}\sigma \neq 0$.

In fact we are going to find a gerbe on the universal triptic curve

universal triptic curve $\quad \mathcal{X} = \{\mathbb{C}/\mathbb{Z}^3;\ \mathbb{Z}^3 \curvearrowright \mathbb{C}$ by $(n_1 n_2 n_3)\cdot z = \sum n_i x_i + w$, s.t. x_i are $\mathbb{R}$-linear independent $\}/\sim$ (action has rank 3, i.e. $\operatorname{Span}_{\mathbb{R}}\{x_i\} = \mathbb{C}$)

now more precisely defined as

moduli space of triptic curve $\quad \mathfrak{M} = [(\mathbb{CP}^2 - \mathbb{RP}^2)/SL(3,\mathbb{Z})]$ $\quad$ diff stack: $\begin{array}{c} (\mathbb{CP}^2 - \mathbb{RP}^2) \times SL(3,\mathbb{Z}) \\ \downdownarrows \\ \mathbb{CP}^2 - \mathbb{RP}^2 \end{array}$

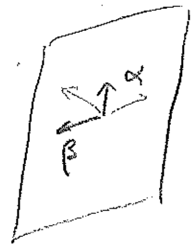
$$\mathcal{X} = [\mathbb{C} \times (\mathbb{C}^3 - \mathbb{C}\cdot\mathbb{R}^3) / \mathbb{C}^\times SL(3,\mathbb{Z}) \ltimes \mathbb{Z}^3] = [\mathcal{O}(1)|_{\mathbb{CP}^2 - \mathbb{RP}^2} / SL(3,\mathbb{Z}) \ltimes \mathbb{Z}^3]$$

$$(g, \mu) \cdot (w, x_1, x_2, x_3) = (w + \vec{\mu} g \vec{x},\ g\cdot\vec{x})$$

To find a suitable gpd pre. of $\mathcal{X}$, we begin w/ finding a suitable (covering) covering of $\mathbb{CP}^2-\mathbb{RP}^2$, $\mathcal{U}_{e_i}=\{x_i\neq 0\}$ not enough fine. (5)

$$\mathcal{U}_a^+ = \{ [x_1, x_2, x_3] \ : \ \operatorname{Im}(\alpha(\vec{x})/\beta(\vec{x})) > 0 \}$$

$a\in\Lambda=\mathbb{Z}^3$ where $\alpha,\beta\in\Lambda^\vee$ form a posi frame on $H(a)$



since $\operatorname{Im}\left(\frac{\alpha(\vec{x})}{\beta(\vec{x})}\right)>0$ stable under

$SL_2(\mathbb{Z})$-trans. namely $\operatorname{Im}\left(\frac{a\,\alpha(\vec{x})+b\,\beta(\vec{x})}{c\,\alpha(\vec{x})+d\,\beta(\vec{x})}\right)>0$

the above def. is well-defined.

$$\frac{a(ax_1+bx_2)}{d(cx_1+ad\,x_2)(}=\frac{a(ax_1+bx_2)}{acx_1+cbx_2+x_2}=\frac{a(ax_1+bx_2)}{c(ax_1+bx_2)+x_2}=\frac{a}{c+\frac{x_2}{ax_1+bx_2}}$$

$\begin{matrix} a & b \\ c & d \end{matrix}$

ok & $\mathcal{U}_a^+$ covers $\mathbb{CP}^2-\mathbb{RP}^2$. ($\mathcal{U}_{e_1}^+=\{[x]:\operatorname{Im}\frac{x_2}{x_3}>0\}$) in fact $\mathcal{U}_{e_i}^+$ covers $\mathbb{CP}^2-\mathbb{RP}^2$

$\Leftrightarrow \operatorname{Im}\frac{x_2}{ax_1+bx_2}<0$

$\frac{1}{a\frac{x_1}{x_2}+b} \Leftrightarrow \operatorname{Im}\frac{x_1}{x_2}>0$

$\rightsquigarrow$ lift to a covering V_a of $\mathcal{O}(1)|_{\mathbb{CP}^2-\mathbb{RP}^2}$

$\mathcal{X}$ is presented by $\hat{g}\in SL(3,\mathbb{Z})\ltimes\mathbb{Z}^3=:G$

$$\begin{array}{c} \coprod_{\substack{a,b\\ \hat{g}\in G}} V_{a,b,\hat{g}} : \{ \overset{\hat{g}}{V_a \frown V_b} \ (\hat{g},y),\ y\in V_a,\ \hat{g}^{-1}y\in V_b\} \\ \downarrow\downarrow \\ \coprod V_a \end{array}$$

$\coprod V_{ab}$ "generated" by two sorts of charts $\cdot\ V_a\cap V_b$ $(\hat{g}=1)$
($\because$ G discreet.) $\cdot\ V_a(\hat{g})$ $(a=b)$

To describe a centrale extension, we need to describe

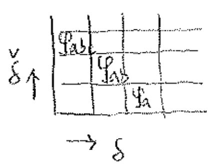
$L_{a,b}\xrightarrow{\text{over}} V_a\cap V_b$ $\quad L_a(\hat{g})\xrightarrow{\text{over}} V_a(\hat{g})$ & for the multiplication

we need $\quad L_{a,b}\otimes L_{b,c}\xleftarrow{\varphi_{a,b,c}} L_{a,c}$

$L_a(\hat{g})\otimes L_{a,b}\xleftarrow{\varphi_{a,b}(\hat{g})} (\hat{g})^{-1*}L_{\hat{g}^{-1}a,\hat{g}^{-1}b}\otimes L_b(\hat{g})$

$L_a(\hat{g})\otimes(\hat{g}^{-1})^{*}L_{\hat{g}^{-1}a}(\hat{h})\xleftarrow{\varphi_a(\hat{g},\hat{h})} L_a(\hat{g}\hat{h})$

gpd cohom ?: →



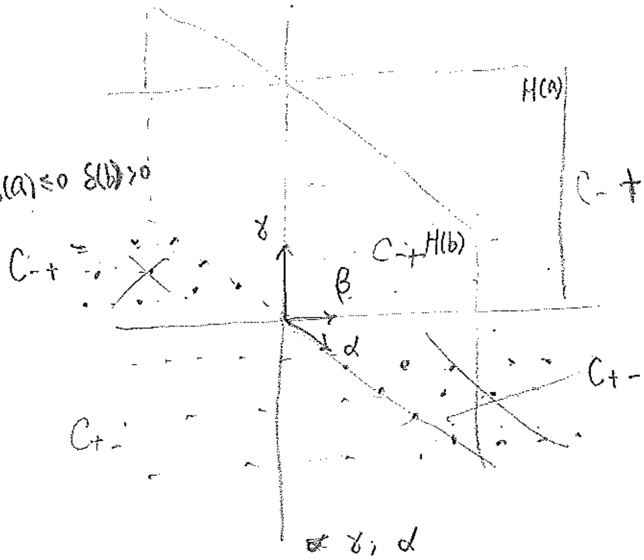
& $G_1=\coprod_{a,b,\hat{g}} L_{a,b}\otimes L_b^{-1}(\hat{g})$ will be a $\mathbb{C}^\times$-centrale extension.

& $\varphi_{a,b,c}^{-1}\cdot\varphi_{b,c}(\hat{g})\cdot\varphi_c(\hat{g},\hat{h})$, on $V_a\cap V_{\hat{g}^{-1}b}\cap V_{(\hat{g}\hat{h})^{-1}c}$ rep $H^2(\mathcal{X},\mathcal{O}^\times)$

is the char. of $\mathcal{G}\longrightarrow H^3(\mathcal{X},\mathbb{Z})$. D-D of $\mathcal{G}$

many charts, we doubt have enough gamma functions.
(serves as section of $L_{a,b}$'s)

$$\Gamma_{a,b}(w,x) = \frac{\prod_{\delta\in C_{+-}/\mathbb{Z}\gamma}\left(1-e^{2\pi i(\delta(x)-w)/\gamma(x)}\right)}{\prod_{\delta\in C_{-+}/\mathbb{Z}\gamma}\left(1-e^{2\pi i(\delta(x)+w)/\gamma(x)}\right)}$$



C_{+-} $\delta(a)\geq 0$ $\delta(b)\leq 0$ $\quad(>0,\ \leq 0)\quad(\leq 0$

$$\Gamma_{e_1,e_2}^{(w,x)} = \Gamma\left(\gamma, \frac{x_1}{x_3}, \frac{x_2}{x_3}\right).$$

$a=e_1 \quad \gamma=\varepsilon_3$
$b=e_2 \quad \alpha=\varepsilon_1$
$ \quad \beta=\varepsilon_2$

$$\prod_{j,k\geq 0}^{\infty} \frac{\left(1-e^{2\pi i((j+1)\tau - k\sigma-\gamma)}\right)}{\left(1-e^{2\pi i(-j\tau+(k+1)\sigma-\gamma)}\right)}$$

$\frac{x_1}{x_3} \quad \frac{x_2}{x_3}$

$\Gamma_{a,b}$ is again a meromorphic function on $\underline{V_a\cap V_b}$. ($\operatorname{Im}(\tau)>0$) $(V_{e_1}\cap V_{e_2}$ $\operatorname{Im}\frac{x_2}{x_3}>0$ $\operatorname{Im}\frac{x_3}{x_1}>0$

$\rightsquigarrow$ $L_{a,b}$

$$\begin{array}{c} L_{a,b} \\ \downarrow \\ V_a\cap V_b \end{array} \quad \Gamma_{a,b}\in H^0(V_a\cap V_b, L_{a,b})$$

$L_{a,b}\otimes L_{b,c} \xleftarrow{\varphi_{a,b,c}} L_{a,c}$ given by $\underbrace{\Gamma_{a,b}\cdot\Gamma_{b,c}\cdot\Gamma_{a,c}^{-1}}_{\varphi_{a,b,c}=}$ $= \exp\left(-\frac{\pi i}{3}P_{a,b,c}(w,x)\right)$ ($\Gamma_{c,a}$)

$P_{a,b,c}(w,x)\in\mathbb{Q}(x)[w]$ w/ at most 3 in w.
$\rightsquigarrow$ product of theta functions.

Similarly. $\begin{array}{c} L_a(\hat{g}) \\ \downarrow \\ V_a(\hat{g}) \end{array}$ $\Delta_a(\hat{g};w,x) = \Delta_a((g,\mu);w,x)$

$$= \Delta_{g^{-1}a}(\mu; w, g^{-1}x).$$

$$= \prod_{j=0}^{\mu(g^{-1}a)-1}\theta_0\left(\frac{w+j\alpha_1(g^{-1}(x))}{\alpha_3(g^{-1}(x))}, \frac{\alpha_2(g^{-1}x)}{\alpha_3(g^{-1}x)}\right)$$

&

- $$\frac{\Gamma_{g^{-1}a,g^{-1}b}(w+\mu g^{-1}x),\ g^{-1}x)}{\Gamma_{a,b}(w,x)} = e^{\pi i P_{a,b}(\hat{g};w,x)}\,\frac{\Delta_a((g,\mu);w,x)}{\Delta_b((g,\mu);w,x)}$$ (g,μ)

- $$\Delta_a(\hat{g}\hat{h};w,x) = e^{2\pi i P_a(\hat{g},\hat{h};w,x)}\,\Delta_a(\hat{g};w,x)\,\Delta_{g^{-1}a}(\hat{h};w+\mu g^{-1}x, g^{-1}x)$$

$$\Rightarrow (\phi_{a,b,c},\ \phi_{a,b},\ \phi_a) = \left(e^{-\frac{2\pi i}{3!} P_{a,b,c}(w,x)},\ e^{-\frac{2\pi i}{2!} P_{a,b}(\hat{g};w,x)},\ e^{-\pi i P_a(\hat{g},\hat{h};w,x)} \right).$$

& Γ is a section of this gerbe in the sense:

$\Gamma_{a b}\Delta^{-1}_b(\hat{g};-) \to$ a meromorphic gpd. morphism

$$\begin{tikzcd}
\coprod V_a \times \mathbb{C}^\times \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & G_1 \arrow[d, shift left] \arrow[d, shift right] \arrow[r, shift left] & \coprod_{\hat{g}} V_{a,b,\hat{g}} \arrow[l, shift left] \arrow[d, shift left] \arrow[d, shift right] \\
\coprod V_a \arrow[r, "id"] & \coprod V_a \arrow[r, "id"] & \coprod V_a
\end{tikzcd}$$

It's now very easy to see $\mathcal{G}|_{\mathbb{C}/\mathbb{Z}^3}$

$\hookrightarrow$ one sepecific tripticcurve

$$\begin{tikzcd}
1 \arrow[r] & \mathbb{T}^2 \times \mathbb{C}^\times \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \coprod_n (L^{(n)})^\times \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \mathbb{T}^2 \times \mathbb{Z} \arrow[r] \arrow[l, bend right, "\Pi\,\theta_0^{-1}"'] \arrow[d, shift left] \arrow[d, shift right] & 1 \\
& \mathbb{T}^2 \arrow[r] & \mathbb{T}^2 \arrow[r] & \mathbb{T}^2 &
\end{tikzcd}$$

L = the line bd "twisted" by θ_0 that we introduced at first.

(more precisely $\mathbb{T}^2 = E_\sigma$, $L^{(n)} = L([0])^* \otimes L([-\tau])^* \otimes \cdots \otimes L([-(n-1)\tau])^*$, $n \geq 0$

$L([n\tau]) \otimes \cdots \otimes L([-\tau])$, $n < 0$.

$L^{(n)} \otimes L^{(m)} = L^{(n+m)}$ & $\prod_{j=0}^{n-1} \theta_0^{-1}(z + j\tau, \sigma)$ a section of $L^{(n)}$.

$\Pi_{j=-N}^{n-1} / \Pi_{j=-N}^{0}$ as the interpreation of $\Pi_{j=0}^{n-1}$ for $n-1<0$).

<u>Theorem</u> $H^3(X, \mathbb{Z})$ fits inside a bit more than $\mathbb{Q}$-version

$$0 \to \mathbb{Z} \to H^3(X, \mathbb{Z})/\text{torsion} \to H^3(\mathbb{Z}^3, \mathbb{Z}) \cong \mathbb{Z} \to 0$$

$Cl(\mathcal{G}) \in H^3(X, \mathbb{Z})$ $\overset{\text{one of the}}{\text{generators}}$ $\to$ the generator of $H^3(\mathbb{Z}^3, \mathbb{Z})$

$H^3(X, \mathbb{Q}) = \mathbb{Q} \times \mathbb{Q}$

$\downarrow$

$H^3(\mathbb{Z}^3, \mathbb{Q}) \cong \mathbb{Q}$ $Cl(\mathcal{G}) \leftrightarrow$ gen of this copy of $\mathbb{Q}$.

<u>Thm</u>

$Cl(\mathcal{G}|_E) \underset{\text{gen}}{\in} H^3(E, \mathbb{Z}) = \mathbb{Z}$. $\square$

$$\theta_0(z,\tau) = \prod_{j=0}^{\infty} \left(1 - e^{2\pi i((j+1)\tau - z)}\right)\left(1 - e^{2\pi i(j\tau + z)}\right) \qquad \mathrm{Im}\,\tau > 0$$

It could be extended to $\mathrm{Im}\,\tau \neq 0$ by

$$\theta_0(z,-\tau) := \theta_0(z+\tau,\tau)^{-1} \ \left(= \theta_0(-z,\tau)^{-1}\right) \qquad \text{for } \mathrm{Im}\,\tau < 0$$

modula relations

semi-periods • $\theta_0(z+1,\tau) = \theta_0(z,\tau)$, $\quad \theta_0(z+\tau,\tau) = -e^{-2\pi i z}\,\theta_0(z,\tau)$

τ-term • $\theta_0(z,\tau+1) = \theta_0(z,\tau)$, $\quad \theta_0\left(\frac{z}{\tau}, -\frac{1}{\tau}\right) = -i e^{\pi i\left(\frac{\tau}{6} - z + \frac{z^2}{\tau} + \frac{1}{6\tau} + \frac{z}{\tau}\right)}\theta_0(z,\tau)$

Elliptic gamma function Γ:

$\Gamma(z,\tau,\sigma)$ is defined by :

There exists a unique solution of the diff. eq. w/. $\mathrm{Im}\,\tau > 0$, $\mathrm{Im}\,\sigma > 0$

$$\begin{cases} u(z+\sigma) = \theta_0(z,\tau)\,u(z) \\ u(z+1) = u(z) \end{cases}$$

such that • $u(z)$ holomorphic in $\mathrm{Im}\,z > 0$

• $u\left(\frac{z+\sigma}{2}\right) = 1$

Then we call $u(z,\sigma) = \Gamma(z,\tau,\sigma)$.

• $\Gamma(z,\tau,\sigma) = \prod_{j,k=0}^{\infty} \left(1 - e^{2\pi i((j+1)\tau + (k+1)\sigma - z)}\right)\left(1 - e^{2\pi i(j\tau + k\sigma + z)}\right)^{-1}$

We can extend it to $\mathrm{Im}\,\tau < 0$ or $\mathrm{Im}\,\sigma < 0$ by

$\Gamma(z,\tau,\sigma) = \Gamma(z-\tau, -\tau, \sigma)^{-1} = \Gamma(z-\sigma, \tau, -\sigma)^{-1}$.

For our purpose, we like Γ in $\mathrm{Im}\,\tau < 0$, $\mathrm{Im}\,\sigma > 0$, it's

• $\Gamma(z,\tau,\sigma) = \prod_{j,k=0}^{\infty} \left(1 - e^{2\pi i(z - (j+1)\tau + k\sigma)}\right)\left(1 - e^{2\pi i(-z - j\tau + (k+1)\sigma)}\right)^{-1}$

modula relations (Felder & Varchenko)

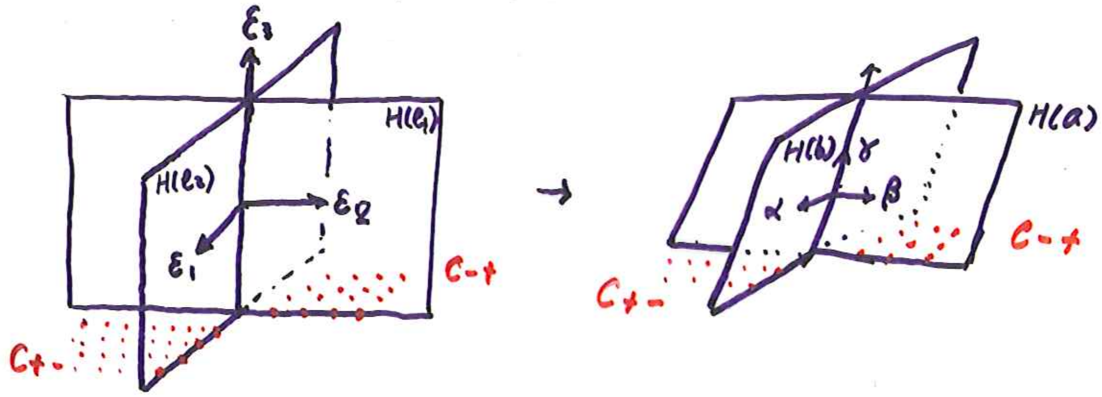
semi periods • $\Gamma(z+1,\tau,\sigma) = \Gamma(z,\tau,\sigma)$,

$\Gamma(z+\tau,\tau,\sigma) = \underline{\theta_0(z,\sigma)}\,\Gamma(z,\tau,\sigma)$, $\quad \Gamma(z+\sigma,\tau,\sigma) = \underline{\theta_0(z,\tau)}\,\Gamma(z,\tau,\sigma)$.

3-term • $\Gamma(z,\tau,\sigma)=\Gamma(z,\tau,\tau+\sigma)\,\Gamma(z+\sigma,\tau+\sigma,\sigma)$

$$\Gamma\left(\frac{w}{x_3},\frac{x_1}{x_3},\frac{x_2}{x_3}\right)\Gamma\left(\frac{w}{x_2},\frac{x_3}{x_2},\frac{x_1}{x_2}\right)\Gamma\left(\frac{w}{x_1},\frac{x_2}{x_1},\frac{x_3}{x_1}\right)=e^{-\pi i\frac{P_3(w,x)}{3}}$$

with $P_3(w,x)\in\mathbb{C}[w](x)$ homogeneous & degree in $w\leq 3$.



$\Gamma_{e_1,e_2}=\Gamma$ $\qquad\qquad$ $\Gamma_{a,b}$

- $\Gamma_{a,b}\,\Gamma_{b,c}\,\Gamma_{c,a}=\underline{\exp\left(-\frac{\pi i}{3}P_{a,b,c}(w,x)\right)}$
- $$\frac{\Gamma_{g^{-1}a,g^{-1}b}(w+\mu(A^{-1}x),A^{-1}x)}{\Gamma_{a,b}(w,x)}=\underline{\exp(\pi i\,P_{a,b}(\overset{(A,\mu)}{\overset{\shortparallel}{g}};w,x))}\,\frac{\Delta_a(g;w,x)}{\Delta_b(g;w,x)}$$
- $\Delta_a(\underset{\underset{(A,\mu)}{\shortparallel}}{g}h;w,x)=\underline{\exp(2\pi i\,P_a(g,\underset{\underset{(B,\nu)}{\shortparallel}}{h};w,x))}\,\Delta_a(\underset{\underset{(A,\mu)}{\shortparallel}}{g};w,x)\,\Delta_{g^{-1}a}(h;w+\mu(A^{-1}x),A^{-1}x)$

where $\Delta_a(g;w,x)\,(=\Delta_{A^{-1}a}((1,\mu);w,x))$

$$=\prod_{j=0}^{\mu(A^{-1}x)-1}\theta_0\left(\frac{w+j\alpha_1(A^{-1}x)}{\alpha_3(A^{-1}x)},\frac{\alpha_2(A^{-1}x)}{\alpha_3(A^{-1}x)}\right).$$