On deformations of bihamiltonian structures of hydrodynamic type

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1. Introduction

Consider a class of Poisson brackets defined on the space of local functionals of the formal loop space of $M \subset \mathbb{R}^n$. They have the form

$$\{w^i(x), w^j(y)\}$$
$$= g^{ij}(w(x))\delta'(x-y) + \Gamma_k^{ij}(w(x))w_x^k\,\delta(x-y)$$

$$\det(g^{ij}(w)) \neq 0$$

Poisson bracket (Hamiltonian structure) of hydrodynamic type (B. Dubrovin, S.P. Novikov, 1983)

Theorem. (Dubrovin & Novikov, 1983)
The above formula defines a Poisson bracket if and only if

$$(g_{ij}) = (g^{ij})^{-1}$$

defines a flat metric on M and

$$\Gamma^{ij}_k = -g^{il}\,\Gamma^j_{lk}.$$

Hamiltonian systems of hydrodynamic type

$$\frac{\partial w^i}{\partial t} = V^i_j(w) w^j_x = \{w^i(x), H\}, \quad i = 1, \ldots, n$$

with a Hamiltonian of the form

$$H = \int h(w(x)) dx.$$

The system is called diagonalizable if there exist coordinates $w^1, \ldots, w^n$ such that

$$\frac{\partial w^i}{\partial t} = \lambda^i(w) w^i_x$$

$w^1, \ldots, w^n$ are the Riemann invariants.

Theorem. (Novikov, Tsarëv) *A Diagonalizable Hamiltonian system of hydrodynamic type is integrable*

Integrability $\Longrightarrow$ existence of symmetry

$$\frac{\partial w^i}{\partial s} = \mu^i(w) w^i_x, \qquad \frac{\partial \mu^i}{\partial w^k} = \frac{\mu^k - \mu^i}{\lambda^k - \lambda^i} \frac{\partial \lambda^i}{\partial w^k}$$

Tsarëv's algorithm of integration

$$x + t\lambda^i(w) - \mu^i(w) = 0.$$

Generalized hodograph transform

Bihamiltonian structure of hydrodynamic type

A pair of compatible Poisson brackets of hydrodynamic type $(\{\,,\,\}_1, \{\,,\,\}_2)$

$$\{w^i(x), w^j(y)\}_a$$
$$= g_a^{ij}(w(x))\delta'(x-y) + \Gamma^{ij}_{a,k}(w(x))w^k_x\,\delta(x-y)$$
$$a = 1, 2$$

It is called semisimple if

$$\det(g_1^{ij} - \lambda g_2^{ij})$$

has n pairwise distinct real roots.

For a semisimple bihamiltonian structure, there exist canonical coordinates $u^1, \ldots, u^n$ satisfying

$$g_1^{ij} = f^i(u)\delta_{ij}, \quad g_2^{ij} = u^i f^i(u)\delta_{ij}$$

A system of hydrodynamic type is called Bihamiltonian if there exist a bihamiltonian structure of hydrodynamic type such that

$$w_t^i = V_j^i(w)w_x^j = \{w^i(x), H_1\}_1 = \{w^i(x), H_2\}_2.$$

A system of hydrodynamic type with semisimple bihamiltonian is diagonalizable.

The class of semisimple bihamiltonian structures of hydrodynamic type is characterized by the Lamé equation

$$\partial_k \gamma_{ij} = \gamma_{ik}\gamma_{kj}, \quad i,\, j,\, k \text{ distinct}$$
$$\partial_i \gamma_{ij} + \partial_j \gamma_{ji} + \sum_{k \neq i,\, j} \gamma_{ki}\gamma_{kj} = 0, \quad i \neq j$$

plus the equations

$$u^i \partial_i \gamma_{ij} + u^j \partial_j \gamma_{ji} + \sum_{k \neq i,\, j} u^k \gamma_{ki}\gamma_{kj} + \frac{1}{2}(\gamma_{ij} + \gamma_{ji}) = 0, \quad i \neq j$$

Here γ_{ij} are the *rotation coefficients*

$$\gamma_{ij}(u) := H_i^{-1} \partial_i H_j, \quad i \neq j.$$

with the *Lamé coefficients*

$$H_i(u) := f_i^{-1/2}(u), \quad i = 1, \ldots, n$$

The “Lax pair” (Ferapontov)

$$\partial_i \psi_j = \gamma_{ji} \psi_i, \quad i \neq j$$
$$\partial_i \psi_i + \sum_{k \neq i} \gamma_{ki} \frac{u^k - \lambda}{u^i - \lambda} \psi_k + \frac{1}{2\,(u^i - \lambda)} \psi_i = 0.$$

Reconstruction of the bihamiltonian structure from solutions γ_{ij}:

To solve $\partial_k \phi_i = \gamma_{ki} \phi_k, \quad i \neq k$

To get a solution near a point u_0 such that

$$\phi_i(u_0) \neq 0, \quad i = 1, \ldots, n$$

then

$$g_1^{ij} = \phi_i^{-2} \delta_{ij}, \quad g_2^{ij} = u^i \phi_i^{-2} \delta_{ij}$$

Depends on n^2 functions of one variable.

Examples of bihamiltonian structures of hydrodynamic type come from:

- dispersionless limits of some bihamiltonian structures that are familiar in soliton theory
- 2d topological field theory
- Frobenius manifolds

2d topological field theory

Genus expansion of the free energy

$$\mathcal{F} = \mathcal{F}_0 + \varepsilon^2 \mathcal{F}_1 + \varepsilon^4 \mathcal{F}_2 + \dots$$

Here $\mathcal{F}_g = \mathcal{F}_g(t^{i,p})$, $i = 1, \dots, n$, $p \geq 0$. The genus zero two point correlation functions

$$v_i = \frac{\partial^2 \mathcal{F}_0}{\partial t^{1,0} \partial t^{i,0}}$$

give a solution of a hierarchy of bihamiltonian systems of hydrodynamic type

$$\frac{\partial v_i}{\partial t^{j,p}} = \{v_i(x), H_{j,p}\}_1 = \{v_i(x), H_{j,p-1}\}_2$$

$$F(v) = \mathcal{F}_0|_{t^{1,0}=v^i,\ t^{i,p\geq 1}=0}$$

Conjecturally, the full genera two point correlation functions

$$w_i = \frac{\partial^2 \mathcal{F}}{\partial t^{1,0} \partial t^{i,0}}$$

satisfy certain hierarchy of bihamiltonian system of the KdV type. The related bihamiltonian structures should be certain deformations of bihamiltonian structures of hydrodynamic type.

Known examples:

2d topological gravity and the KdV hierarchy (Kontsevich-Witten)

CP^1 topological sigma model and the extended Toda hierarchy (Getzler, Okounkov-Pandharipande, Dubrovin-Z.)

2. Deformations of Hamiltonian structures of hydrodynamic type

For a given Poisson bracket of hydrodynamic type with the flat metric (g^{ij}), we consider

$$\{w^i(x), w^j(y)\} = g^{ij}(w)\,\delta'(x-y) + \Gamma^{ij}_k\, w^k_x\, \delta(x-y)$$

$$+ \sum_{k\ge 1} \varepsilon^k \sum_{j=0}^{k+1} A^{ij}_{k,l}(w; w_x, \ldots, w^{(l)})\, \delta^{(k+1-l)}(x-y)$$

Here $A^{ij}_{k,l} \in \mathcal{A}$ with $\mathcal{A} = C^\infty(w)[w^{i,p}]_{p\ge 1}$ and

$$\deg w^{i,p} = p, \ \deg A^{ij}_{k,l} = l, \ w^{i,p} = \partial_x^p w^i$$

Miura type transformations

$$w^i \mapsto Q^i(w) + \sum_{k\geq 1} \varepsilon^k P_k^i(w, w_x, \dots, w^{(k)}).$$

Here $P_k^i \in \mathcal{A}$, $\deg P_k^i = k$, and $\det \left(\frac{\partial Q^i}{\partial w^j}\right) \neq 0$.

Problem:
Classify the equivalence classes of deformations of a given Poisson bracket of hydrodynamic type under the Miura type transformations.

Poisson cohomologies: infinite dimensional case

Local translation invariant k-vector field

$$\alpha = \sum \frac{1}{k!} \partial_{x_1}^{s_1} \dots \partial_{x_k}^{s_k} A^{i_1 \dots i_k} \frac{\partial}{\partial w^{i_1, s_1}(x_1)} \wedge \dots \wedge \frac{\partial}{\partial w^{i_k, s_k}(x_k)}$$

The components of α have the form

$$\begin{aligned} A^{i_1 \dots i_k} &= \sum_{p_2, \dots, p_k \geq 0} B^{i_1 \dots i_k}_{p_2 \dots p_k}(w(x_1); w_x(x_1), \dots) \\ &\quad \times \delta^{(p_2)}(x_1 - x_2) \dots \delta^{(p_k)}(x_1 - x_k) \end{aligned}$$

Here

$$B^{i_1 \dots i_k}_{p_2 \dots p_k} \in \mathcal{A}$$

and the distribution $A^{i_1 \dots i_k}$ are antisymmetric with respect to the simultaneous permutations

$$i_p, x_p \leftrightarrow i_q, x_q.$$

The operator of total derivative ∂_x is defined by

$$\partial_x f(w; w_x, w_{xx}, \dots) = \sum_{s\geq 0} w^{i,s+1} \frac{\partial f}{\partial w^{i,s}}.$$

The delta-function and its derivatives are defined formally by

$$\int f(w(y); w_y(y), w_{yy}(y), \dots)\, \delta^{(k)}(x-y)\, dy$$
$$= \partial_x^k f(w(x); w_x(x), w_{xx}(x), \dots).$$

Note the useful identity

$$f(w(y); w_y(y), w_{yy}(y), \dots)\, \delta^{(k)}(x-y)$$
$$= \sum_{m=0}^{k} \left(\begin{array}{c} k \\ m \end{array} \right) \partial_x^m f(w(x); w_x(x), w_{xx}(x), \dots)$$
$$\times \delta^{(k-m)}(x-y).$$

Denote

$$\Lambda^k: \text{ Space of local k-vectors}$$

$$\Lambda^0: \; I = \int f(w, w_x, \dots)\, dx, \quad f \in \mathcal{A}$$

The Schouten-Nijenhuis bracket is defined on the space of local multi-vectors

$$[\ ,\ ]: \; \Lambda^k \times \Lambda^l \to \Lambda^{k+l-1}, \quad k, l \geq 0.$$

It satisfies

$$[\alpha, \beta] = (-1)^{kl}[\beta, \alpha]$$

$$(-1)^{km}[[\alpha, \beta], \gamma] + (-1)^{kl}[[\beta, \gamma], \alpha]$$
$$+(-1)^{lm}[[\gamma, \alpha], \beta] = 0$$

for $\alpha \in \Lambda^k$, $\beta \in \Lambda^l$, $\gamma \in \Lambda^m$.

In particular, for two vector fields

$$\xi = \sum_{s\geq 0} \partial_x^s \xi^i \frac{\partial}{\partial w^{i,s}(x)}, \;\; \eta = \sum_{s\geq 0} \partial_x^s \eta^i \frac{\partial}{\partial w^{i,s}(x)}$$

the Schouten-Nijenhuis bracket is given by the usual commutator

$$[\xi, \eta] = \sum_{s\geq 0} \partial_x^s \zeta^i \frac{\partial}{\partial w^{i,s}(x)}$$

with

$$\zeta = \sum_{s\geq 0} \left(\partial_x^s \xi^k \frac{\partial \eta^i}{\partial w^{k,s}} - \partial_x^s \eta^k \frac{\partial \xi^i}{\partial w^{k,s}} \right)$$

Poisson bivector $\alpha \in \Lambda^2$: $\quad [\alpha, \alpha] = 0$

The natural gradation on the ring $\mathcal{A}$

$$\deg w^{i,m} = m, \ m \geq 1, \quad \deg f(u) = 0$$

can be extended to a gradation on the space of local multi-vectors by defining

$$\deg \frac{\partial}{\partial w^{i,s}} = -s, \ \deg \delta^{(s)} = s + 1$$

Denote

$$\Lambda^k_m = \{\xi \in \Lambda^k \big| \deg \xi = m\}$$

$$\hat{\Lambda}^k = \{\alpha \in \Lambda^k \otimes \mathbb{C}[[\varepsilon], \varepsilon^{-1}] | \deg \alpha = k\}$$

$$\hat{\Lambda} = \oplus_{k \geq 0} \hat{\Lambda}^k$$

Here $\deg \varepsilon = -1$.

A Poisson bracket of hydrodynamic type corresponds to a Poisson bivector

$$\varpi \in \Lambda_2^2$$

Its deformations are Poisson bivectors of the form

$$\varpi + \sum_{k \geq 1} \varepsilon^k P_k \in \hat{\Lambda}^2$$

Consider the Poisson cohomologies of $(\hat{\Lambda}, \varpi)$

$$H^k = \mathrm{Ker}\, \partial|_{\hat{\Lambda}^k} \,/\, \mathrm{Im}\, \partial|_{\hat{\Lambda}^{k-1}}\,, \qquad k \geq 0$$

$$H^k = \oplus_{m \geq -1}\, \varepsilon^m H^k_m$$

Theorem. *Let M be a ball of $\mathbb{R}^n$,*

$$H^k_m = 0, \quad m \geq 1.$$

(E. Getzler; for $k = 1, 2$ also by L. Degiovanni, F. Magri, V. Sciacca)

Corollary. *For any Poisson bivector*

$$\varpi + \sum_{k\geq 1} \varepsilon^k P_k$$

there exists a vector field

$$X = \sum_{k\geq 1} \varepsilon^k X_k \in \hat{\Lambda}^1$$

such that

$$\varpi + \sum_{k\geq 1} \varepsilon^k P_k = e^{-\mathrm{ad}_X} \varpi.$$

I.e., any deformation of a Hamiltonian structure of hydrodynamic type ϖ can be obtained by a Miura type transformation

$$w^i \mapsto e^X w^i = w^i + \sum_{k\geq 1} P_k^i(w, w_x, \dots, w^{(k)})$$

Example. *The Magri bracket*

$$\{w(x), w(y)\} = w(x)\delta' + \frac{1}{2} w_x \delta - \varepsilon^2 \delta'''$$

for the KdV equation

$$w_t = w w_x - \frac{2}{3}\varepsilon^2 w_{xxx}.$$

The Miura transformation

$$w = v - \varepsilon \frac{v_x}{\sqrt{v}}$$

reduces it to

$$\{v(x), v(y)\} = v(x)\delta' + \frac{1}{2} v_x \delta$$

3. Deformations of bihamiltonian structures of hydrodynamic type

A bihamiltonian structure of hydrodynamic type is given by a pair of bivectors $\varpi_1, \varpi_2 \in \Lambda^2_2$ with

$$[a\varpi_2 + b\varpi_1, a\varpi_2 + b\varpi_1] = 0$$

for arbitrary parameters a, b.

Two differentials

$$\varepsilon\partial_a : \ \hat{\Lambda}^k \to \hat{\Lambda}^{k+1}, \quad \varepsilon\partial_a \xi = \varepsilon[\varpi_a, \xi], \quad a = 1, 2.$$

$$\partial_1^2 = \partial_2^2 = \partial_1\partial_2 + \partial_2\partial_1 = 0.$$

Define

$$\Omega^k = \mathrm{Ker}(\partial_1|_{\hat{\Lambda}^k})$$

Then $\varepsilon\partial_2$ defines a differential

$$\varepsilon\partial_2 : \ \Omega^k \to \Omega^{k+1}$$

The cohomologies of the complex $(\Omega, \varepsilon\partial_2)$ give the bihamiltonian cohomologies

$$H^0_m = \mathrm{Ker}(\partial_1|_{\Lambda^0_m}) \cap \mathrm{Ker}(\partial_2|_{\Lambda^0_m})$$

$$H^1_m = \mathrm{Ker}(\partial_1\partial_2|_{\Lambda^0_m})$$

$$H^k_m = \mathrm{Ker}(\partial_1\partial_2|_{\Lambda^{k-1}_m})/\mathrm{Im}(\partial_1|_{\Lambda^{k-2}_{m-2}}) \oplus \mathrm{Im}(\partial_2|_{\Lambda^{k-2}_{m-2}})$$

For $I \in \mathrm{Ker}(\partial_1\partial_2|_{\Lambda^0_m})$, the triviality of the first Poisson cohomology of ϖ_1 ensures the existence of $J \in \Lambda^0_m$ such that $\partial_2 I = \partial_1 J$.

So H^1_m corresponds to the space of bihamiltonian vector fields

For $X \in \mathrm{Ker}(\partial_1\partial_2|_{\Lambda^1_m})$ we have an infinitesimal deformation

$$(\varpi_1, \varpi_2) \mapsto (\varpi_1, \varpi_2 + \epsilon\partial_1 X)$$

So H^2_m corresponds to the space of infinitesimal deformations of the bihamiltonian structure modulo the trivial deformations caused by Miura type transformations.

Theorem. *For a semisimple bihamiltonian structure of hydrodynamic type, we have* $H^2_m = 0$ *for* $m = 1, 3, 4, \ldots$ *and*

$$H^2_2 = \{\sum_{i=1}^{n} \Big(\partial_1 \int (u^i c_i(u^i) u^i_x \log u^i_x) dx \\ -\partial_2 \int (c_i(u^i) u^i_x \log u^i_x) dx\Big)\}$$

Here $u^1, \ldots, u^n$ *are canonical coordinates of the semisimple bihamiltonian structure, under these coordinates, the related compatible flat metrics take the form*

$$g_1^{ij} = f^i(u)\delta_{ij}, \quad g_2^{ij} = u^i f^i(u)\delta_{ij}.$$

(B. Dubrovin, S.Q. Liu, Z.)

Any equivalence class of infinitesimal deformation of a semisimple bihamiltonian structure of hydrodynamic type (ϖ_1, ϖ_2) has a unique representative of the form

$$(\varpi_1, \varpi_2) \mapsto (\varpi_1, \varpi_2 + \varepsilon \partial_1 X), \qquad X \in H^2_2$$

Corollary. *For a given semisimple bihamiltonian structure of hydrodynamic type, any equivalence class of deformations is uniquely determined by a set of* n *functions of one variable*

$$c_1(u^1), \ldots, c_n(u^n)$$

Here $u^1, \ldots, u^n$ *are the canonical coordinates of the semisimple bihamiltonian structure of hydrodynamic type.*

The functions $c_1(u^1), \ldots, c_n(u^n)$ are called the central invariants of the deformed bihamiltonian structure.

Computation of the central invariants:

$$\{u^i(x), u^j(y)\}_a = g_a^{ij}(u)\delta' + \Gamma^{ij}_{k;a} u^k_x \, \delta$$
$$+\varepsilon \sum_{k=0}^{2} A^{ij}_{k,a} \, \delta^{(2-k)}(x-y) + \varepsilon^2 \sum_{k=0}^{3} B^{ij}_{k,a} \, \delta^{(3-k)}(x-y)$$
$$+\mathcal{O}(\varepsilon^3), \qquad a = 1, 2$$

the central invariants are given by

$$c_i(u) = \frac{1}{3(f^i(u))^2} \left(Q_2^{ii} - u^i Q_1^{ii} + \sum_{k \neq i} \frac{(P_2^{ki} - u^i P_1^{ki})^2}{f^k(u)(u^k - u^i)} \right)$$
$$i = 1, \dots, n.$$

$$g_1^{ij} = \delta_{ij} f^i(u), \quad g_2^{ij} = \delta_{ij} u^i f^i(u)$$
$$P_a^{ij}(w) = A^{ij}_{0;a}(u), \quad Q_a^{ij}(w) = B^{ij}_{0;a}(u)$$
$$i, j = 1, \dots, n, a = 1, 2.$$

If we choose another representative

$$\{\,,\,\}\tilde{}_1 = c\{\,,\,\}_2 + d\{\,,\,\}_1, \quad \{\,,\,\}\tilde{}_2 = a\{\,,\,\}_2 + b\{\,,\,\}_1$$
$$ad - bc \neq 0$$

of the above bihamiltonian structure, then the functions $c_i(u)$ are changed to

$$\tilde{c}_i(\tilde{u}^i) = \frac{cu^i + d}{ad - bc}\, c_i(u^i), \quad i = 1, \dots, n.$$

where

$$\tilde{u}^i = \frac{au^i + b}{cu^i + d}, \quad i = 1, \dots, n$$

are the canonical coordinates of the bihamiltonian structure with respect to the new representative.

4. Examples and problems

Example 1.

$$\{w(x), w(y)\}_1 = \delta'(x-y)$$

$$\{w(x), w(y)\}_2 = w(x)\delta'(x-y) + \frac{1}{2} w_x \, \delta(x-y)$$

The second one is the Lie-Poisson bracket on the dual of the Lie algebra of smooth vector fileds on S^1.

Bihamiltonian structure for the Riemann hierarchy

$$\frac{\partial w}{\partial t_p} = \frac{1}{p!} w^p \, w_x, \quad p \geq 0$$

One deformation of the above bihamiltonian structure

$$\{w(x), w(y)\}_1 = \delta'(x-y)$$

$$\{w(x), w(y)\}_2 = w(x)\delta'(x-y) + \frac{1}{2} w_x \, \delta(x-y) + \frac{\varepsilon^2}{8} \delta'''(x-y)$$

It is the Gardner, Zakharov & Faddeev; Magri bihamiltonian structure for the KdV hierarchy, in particular, the KdV equation.

The central invariant $c_1 = \frac{1}{24}$.

A second deformation of the above bihamiltonian structure of hydrodynamic type

$$\begin{aligned}\{w(x), w(y)\}_1 &= \delta'(x-y) - \frac{\varepsilon^2}{8}\delta'''(x-y)\\ \{w(x), w(y)\}_2 &= w(x)\delta'(x-y) + \frac{1}{2}w_x\,\delta(x-y)\end{aligned}$$

It gives the bihamiltonian structure of the Camassa-Holm equation

$$\begin{aligned}&w_t = m m_x - \frac{\varepsilon^2}{12} m_x m_{xx} - \frac{\varepsilon^2}{24} m m_{xxx}\\ &w = m - \frac{\varepsilon^2}{8} m_{xx}.\end{aligned}$$

The central invariant $c_1 = \frac{u}{24},\ u = w$.

More examples of bihamiltonian structures

We consider the semisimple bihamiltonian structures of hydrodynamic type, that are defined on the formal loop space of the orbit spaces of Coxeter groups of type $A_n, B_, C_n$. These bihamiltonian structures are induced by the flat pencil of metrics discovered by Saito - Yano - Sekiguchi in 1980. These flat pencil of metrics also induce polynomial Frobenius manifold structures on these orbit spaces. We show that the bihamiltonian structure obtained from the Drinfeld-Sokolov construction are certain deformations of these semisimple bihamiltonian structures of hydrodynamic type with constant central invariants.

i) The A_n case, the first Poisson bracket of hydrodynamic type is given by

$$\sum_{i,\,j=1}^{n} \{w^i(x), w^j(y)\}_1\, p^{i-1} q^{j-1}$$
$$= \frac{1}{n+1}\left[\frac{\lambda'(p)-\lambda'(q)}{p-q}\,\delta'(x-y) + \left(\frac{\lambda_x(p)-\lambda_x(q)}{(p-q)^2} - \frac{\lambda'_x(q)}{p-q}\right)\delta(x-y)\right]$$

where

$$\lambda(p,x) := p^{n+1} + \sum_{i=1}^{n} w^i(x)\, p^{i-1}$$
$$\lambda(p) = \lambda(p,x), \quad \lambda(q) = \lambda(q,x)$$

The second one have the expression

$$\sum_{i,\,j=1}^{n} \{w^i(x), w^j(y)\}_2\, p^{i-1} q^{j-1}$$

$$= \frac{1}{n+1}\left[\left(\frac{\lambda'(p)\lambda(q) - \lambda'(q)\lambda(p)}{p-q} + \frac{1}{n+1}\lambda'(p)\lambda'(q)\right)\,\delta'\right.$$

$$+\left(\frac{\lambda_x(p)\lambda(q) - \lambda_x(q)\lambda(p)}{(p-q)^2} + \frac{\lambda_x(q)\lambda'(p) - \lambda'_x(q)\lambda(p)}{p-q}\right.$$

$$\left.\left. + \frac{1}{n+1}\lambda'(p)\lambda'_x(q)\right)\,\delta(x-y)\right].$$

A representative of the class of deformation of the above bihamiltonian structure with central invariants

$$c_1 = \cdots = c_n = \frac{n+1}{24}$$

can be described in terms of the differential operator

$$L = (\varepsilon\partial_x)^{n+1} + w^n(x)(\varepsilon\partial_x)^{n-1} + \cdots + w^1(x).$$

For a local functional F define the pseudo-differential operator

$$\frac{\delta F}{\delta L} = \sum_{i=1}^{n} (\varepsilon\,\partial_x)^{-i} \frac{\delta F}{\delta w^i(x)}.$$

Then we have the bihamiltonian structure

$$\{F,G\}_i = \frac{1}{n+1}\int \mathrm{res}\left(\frac{\delta F}{\delta L}\mathcal{H}_i\frac{\delta G}{\delta L}\right)dx, \qquad i=1,2.$$

The Hamiltonian mappings are defined by

$$\mathcal{H}_1 :\ A \mapsto [L,A]_+$$

$$\begin{aligned}\mathcal{H}_2 :\ A \mapsto\ & (LA)_+L - L(AL)_+ \\ & + \frac{1}{n+1}\left[L, \int^x \mathrm{res}[L,A]\,dx\right].\end{aligned}$$

(Adler, Gelfand-Dickey)

ii) For the semisimple bihamiltonian structures that correspond to the Coxeter groups of type B_n, C_n, D_n, the Drinfeld-Sokolov construction from the affine Kac-Moody algebras $B_n^{(1)}, C_n^{(1)}, D_n^{(1)}$ gives the deformations that belong to the class with central invariants

For B_n type: $c_1 = \frac{1}{6},\ c_2 = \cdots = c_n = \frac{1}{12}$

For C_n type: $c_1 = \frac{1}{24},\ c_2 = \cdots = c_n = \frac{1}{12}$

For D_n type: $c_1 = \cdots = c_n = \frac{1}{12}$

iii) For any semisimple Frobenius manifold, there is associated to it a semisimple bihamiltonian structure of hydrodynamic type. A construction of the so called topological deformation of this bihamiltonian structure of hydrodynamic type was proposed by Boris Dubrovin and Z., using properties of the Viasoro symmetries of the associated hierarchy of integrable systems. Such deformations have central invariants

$$c_1 = c_2 = \cdots = c_n = \frac{1}{24}.$$

To calculate H^2_m, we need to use the following

Theorem. (B. Dubrovin, S.Q. Liu, Z.) *Any deformation of a semisimple bihamiltonian structure of hydrodynamic type is quasitrivial, i.e., it can be obtained from its leading terms by a* quasi-Miura *transformation*

$$w^i \mapsto w^i + \sum_{k\geq 1} \varepsilon^k A^i_k(w, w_x, \ldots, w^{(m_k)})$$

$$A^i_k \in C^\infty(B)\left[w_x, \ldots, w^{(m_k)}\right]\left[\left(u^1_x u^2_x \ldots u^n_x\right)^{-1}\right]$$

$$m_k \leq \left[\frac{3\,k}{2}\right], \quad \deg A^i_k = k$$

Corollary. *Any two bihamiltonian systems w.r.t. a same semisimple bihamiltonian structure mutually commute.*

Examples of quasitriviality

The bihamiltonian structure for KdV

$$\{w(x), w(y)\}_1 = \delta'$$

$$\{w(x), w(y)\}_2 = w(x)\delta' + \frac{1}{2}w_x\delta + \frac{\varepsilon^2}{8}\delta'''$$

is reduced to the leading terms

$$\{v(x), v(y)\}_1 = \delta'$$

$$\{v(x), v(y)\}_2 = v(x)\delta' + \frac{1}{2}v_x\delta$$

by a quasi-Miura transformation

$$\begin{aligned} w = v &+ \frac{\varepsilon^2}{24}\partial_x^2 \log v_x \\ &+\varepsilon^4\partial_x^2\left(\frac{v^{(4)}}{1152v_x^2} - \frac{7v_{xx}v_{xxx}}{1920v_x^3} + \frac{v_{xx}^3}{360v_x^4}\right) + \dots \end{aligned}$$

This transformation coincides with the genus expansion in 2d topological gravity, $v = \frac{\partial^2\mathcal{F}_0}{\partial x\partial x}$, $w = \frac{\partial^2\mathcal{F}}{\partial x\partial x}$.

The bihamiltonian structure for the camassa-Holm equation

$$\{w(x), w(y)\}_1 = \delta' - \frac{\varepsilon^2}{8}\delta'''$$

$$\{w(x), w(y)\}_2 = w(x)\delta' + \frac{1}{2}\,w'(x)\delta$$

is reduced to the leading terms

$$\{v(x), v(y)\}_1 = \delta'$$

$$\{v(x), v(y)\}_2 = v(x)\delta' + \frac{1}{2}v_x\delta$$

by a quasi-Miura transformation

$$
\begin{aligned}
w = v + \epsilon^2\,\partial_x\!\left(\frac{v\,v_{xx}}{24\,v_x} - \frac{v_x}{48}\right) \\
+\epsilon^4\partial_x\!\left(\frac{7\,v_{xx}^2}{2880\,v_x} + \frac{v\,v_{xx}^3}{180\,v_x^3} - \frac{v^2\,v_{xx}^4}{90\,v_x^5} - \frac{v_{xxx}}{512}\right. \\
-\frac{59\,v\,v_{xx}\,v_{xxx}}{5760\,v_x^2} + \frac{37\,v^2\,v_{xx}^2\,v_{xxx}}{1920\,v_x^4} - \frac{7\,v^2\,v_{xxx}^2}{1920\,v_x^3} + \frac{5\,v\,v^{(4)}}{1152\,v_x} \\
\left.-\frac{31\,v^2\,v_{xx}\,v^{(4)}}{5760\,v_x^3} + \frac{v^2\,v^{(5)}}{1152\,v_x^2}\right) + \cdots.
\end{aligned}
$$

Question: Given a semisimple bihamiltonian structure of hydrodynamic type, and a set of functions $c_1(u^1), \ldots, c_n(u^n)$, whether there exists a deformation with $c_1, \ldots, c_n$ as the central invariants?

(sufficient condition: $H^3_m = 0,\ m \geq 3$)