Reduction of Dirac Structures, Implicit Lagrangian Systems and Variational Principles

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Joint work with Jerrold E. Marsden

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Contents

Background

- An ***almost Dirac structure*** D_P on P *(Courant and Weinstein [1988])* is a subbundle

$$D_P \subset TP \oplus T^*P$$

such that $D_P = D_P^{\perp}$, where, for each $x \in P$,

$$D_P^{\perp}(x) = \{ (u_x, \beta_x) \in T_xP \times T_x^*P \, | \\ \langle\!\langle \, (v_x, \alpha_x), (u_x, \beta_x) \, \rangle\!\rangle = \alpha_x(u_x) + \beta_x(v_x) = 0, \\ \text{for all } \; (v_x, \alpha_x) \in D_P(x) \}.$$

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- A ***Dirac structure*** is one that satisfies

$$\langle \pounds_{X_1}\alpha_2, X_3\rangle + \langle \pounds_{X_2}\alpha_3, X_1\rangle + \langle \pounds_{X_3}\alpha_1, X_2\rangle = 0,$$

for all pairs of vector fields and one-forms (X_1, α_1), (X_2, α_2), (X_3, α_3) that take values in D_P.

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- ***Dirac*** started off with ***Hamilton's principle:***

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Especially, for the case of ***degenerate Lagrangians*** (*Dirac, Lectures on quantum mechanics [1964]).*

- But, Dirac went to work on the Hamiltonian side by introducing the associated Poisson brackets:

$$\{f, g\}_S = \{f, g\}_P - \{f, \varphi^\alpha\}_P \, c_{\alpha\beta} \{\varphi^\beta, g\}_P,$$

where a symplectic submanifold $S \subset P$ is defined as

$$S = \{x \in P \mid \varphi^\alpha(x) = 0\}.$$

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- A notion of ***implicit Hamiltonian systems*** was developed by *van der Schaft and Maschke [1995]:*

$$(X, \mathbf{d}H) \in D_P.$$

- Researches on the Lagrangian side has been left out and there has been a gap between Dirac structures and Lagrangian systems.
- Recently, a notion of ***implicit Lagrangian systems*** has been developed by *Yoshimura and Marsden (Journal of Geometry and Physics [published online, April, 2006]):*

$$(X, \mathfrak{D}L) \in D_{\Delta_Q}.$$

Induced Dirac Structures

- Let Q be a configuration manifold. Given a distribution $\Delta_Q \subset TQ$ and a distribution on T^*Q can be defined by

$$\Delta_{T^*Q} = (T\pi_Q)^{-1}(\Delta_Q) \subset TT^*Q,$$

where $\pi_Q : T^*Q \to Q$ is the canonical projection.

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where $\pi_Q : T^*Q \to Q$ is the canonical projection.

- Then, an ***induced Dirac structure*** can be defined by, for each $z \in T^*Q$,

$$\begin{aligned} D_{\Delta Q}(z) = \{ (v_z, \alpha_z) \in T_z(T^*Q) \times T_z^*(T^*Q) \mid v_z \in \Delta_{T^*Q}(z), \\ \alpha_z(w_z) = \Omega_{\Delta_Q}(v_z, w_z) \text{ for all } \ w_z \in \Delta_{T^*Q}(z) \}, \end{aligned}$$

where Ω is the canonical two–form on T^*Q and Ω_{Δ_Q} is the restriction of Ω to Δ_{T^*Q}.

Dirac Differential Operator

- Let $L : TQ \longrightarrow \mathbb{R}$ be a Lagrangian (possibly, degenerate) and $\mathbf{d}L : TQ \longrightarrow T^*TQ$ be locally given by

$$\mathbf{d}L = \left(q, v, \frac{\partial L}{\partial q}, \frac{\partial L}{\partial v} \right).$$

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$$\mathbf{d}L = \left(q, v, \frac{\partial L}{\partial q}, \frac{\partial L}{\partial v} \right).$$

- Define a Dirac differential of L by

$$\mathfrak{D}L = \gamma_Q \circ \mathbf{d}L : TQ \to T^*(T^*Q),$$

where $\gamma_Q : T^*TQ \to T^*T^*Q$ is a natural diffeomorphsim.

- In local, it reads

$$\mathfrak{D}L = \left(q, \frac{\partial L}{\partial v}, -\frac{\partial L}{\partial q}, v \right),$$

where $p = \partial L / \partial v$.

Implicit Lagrangian Systems

- An ***implicit Lagrangian system*** can be defined as a triple (L, Δ_Q, X) that satisfies, for each $(q,v) \in \Delta_Q$,

$$(X(q,p), \mathfrak{D}L(q,v)) \in D_{\Delta_Q}(q,p)$$

where $(q,p) = \mathbb{F}L(q,v)$.

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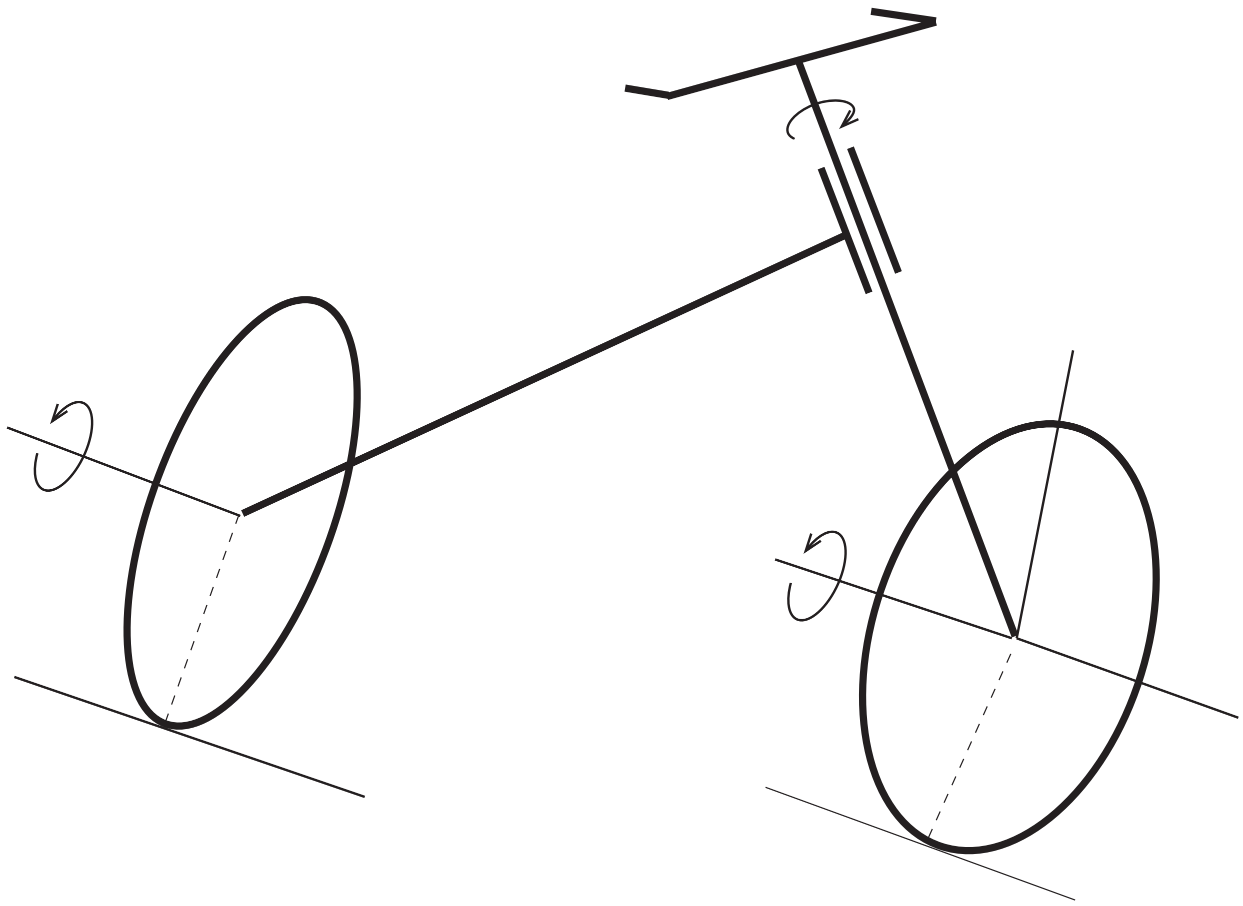
  where $(q,p) = \mathbb{F}L(q,v)$.

- The local expression of implicit Lagrangian systems can be given by

$$\dot{p} - \frac{\partial L}{\partial q} \in \Delta^{\circ}(q), \quad \dot{q} = v \in \Delta(q), \quad p = \frac{\partial L}{\partial v}.$$

Examples: Nonholonomic Systems

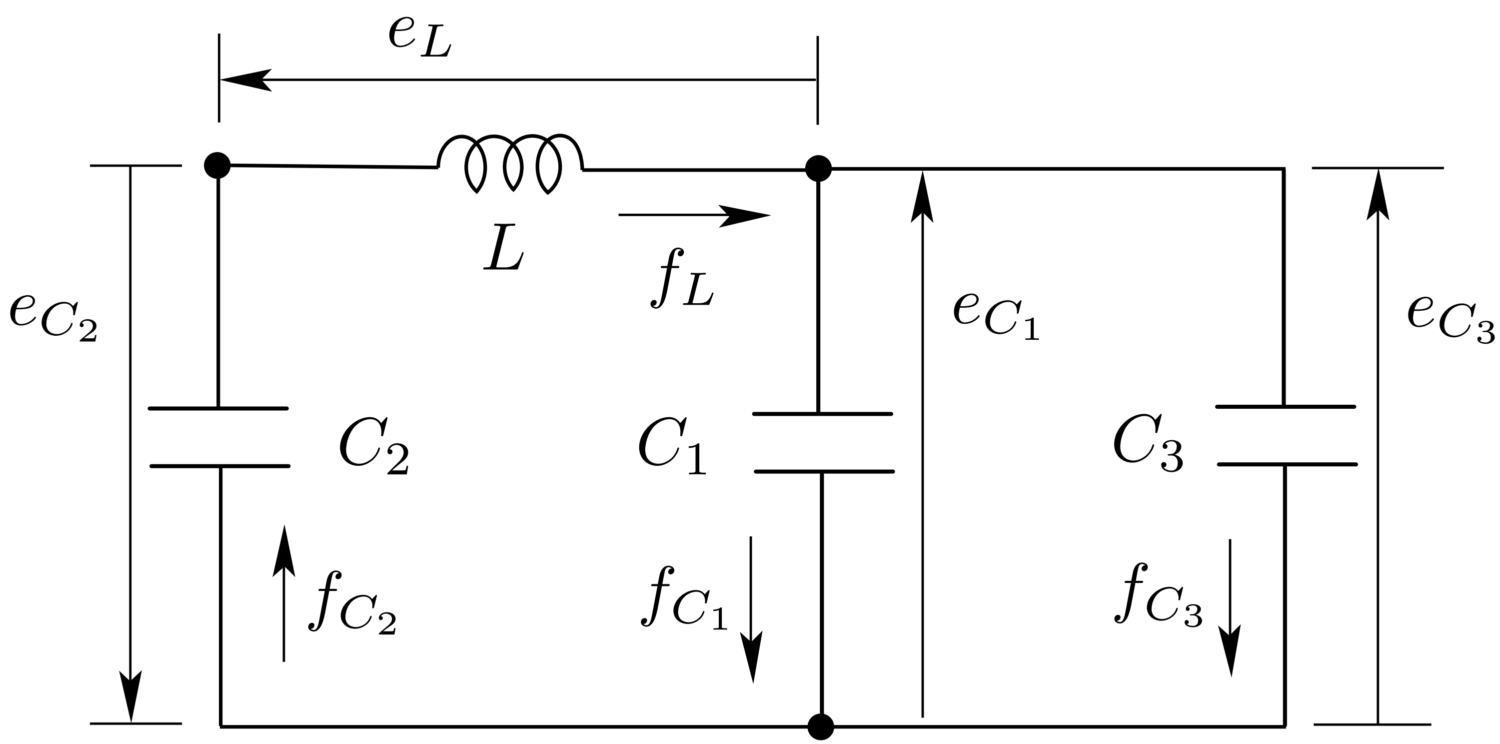
- Nonholonomic mechanical systems.



$$\text{Constraint distribution} : \Delta_Q \subset TQ$$

Examples: Electric Circuits

- Holonomic systems with degenerate Lagrangians.



$$\mathcal{L} = \frac{1}{2}L\left(f_L\right)^2 - \frac{1}{2}C_1\left(q_{C_1}\right)^2 - \frac{1}{2}C_2\left(q_{C_2}\right)^2 - \frac{1}{2}C_3\left(q_{C_3}\right)^2$$

Examples: The Case $\Delta_Q = TQ$

- The ***canonical Dirac Structure*** D on T^*Q may be defined as

$$D = \mathrm{graph}(\Omega^\flat) \subset TT^*Q \oplus T^*T^*Q,$$

or

$$D = \mathrm{graph}(B^\sharp) \subset TT^*Q \oplus T^*T^*Q,$$

where $\Omega^\flat : TT^*Q \to T^*T^*Q$ and $B^\sharp : T^*T^*Q \to TT^*Q$.

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- The ***standard implicit Lagrangian system***
$$(X, \mathfrak{D}L) \in D$$
reads
$$p = \frac{\partial L}{\partial v}, \quad v = \dot{q}, \quad \dot{p} = \frac{\partial L}{\partial q},$$
which are ***implicit Euler–Lagrange equations.***

Hamilton–Pontryagin Principle

- The ***Hamilton–Pontryagin principle*** *(originally developed by Livens [1919])* is given by

$$\delta \int_{t_1}^{t_2} \{L(q(t), v(t)) + p(t) \cdot (\dot{q}(t) - v(t))\} \, dt = 0.$$

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Keeping the endpoints of $q(t)$ fixed,

$$\int_{t_1}^{t_2} \left\{ (\dot{q} - v) \, \delta p + \left(-\dot{p} + \frac{\partial L}{\partial q} \right) \delta q + \left(-p + \frac{\partial L}{\partial v} \right) \delta v \right\} dt = 0$$

is satisfied for all $\delta q, \delta v$ and δp, one can directly obtain the ***implicit Euler–Lagrange equations:***

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is satisfied for all $\delta q, \delta v$ and δp, one can directly obtain the ***implicit Euler–Lagrange equations:***

$$p = \frac{\partial L}{\partial v}, \qquad \dot{q} = v, \qquad \dot{p} = \frac{\partial L}{\partial q}.$$

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- Can we *develop an* ***implicit analogue of Euler–Poincaré and Lie–Poisson reductions ?***

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- How can we *develop reduction of the* ***Hamilton–Pontryagin principle?***

- Can we *develop an* ***implicit analogue of Euler–Poincaré and Lie–Poisson reductions ?***

Our goals are to answer these questions!

Developments in Reduction

- There is a rich history on reduction; refer to *Marsden and Weinstein [2001]*, Comments on the history, theory, and applications of symplectic reduction.

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- Reduction of Dirac structures and Hamiltonian systems with symmetry was also shown by *Dorfman [1993]*.
- Reduction of implicit Hamiltonian systems was developed by *van der Schaft and Blankenstein [2000]*. Singular reduction of implicit Hamiltonian systems was developed by *Blankenstein and Ratiu [2002]*.

Lie–Poisson Reduction

Lie–Poisson Reduction

- The group action of a Lie group G on T^*G is by cotangent lift of the left (or right) translation of G on itself and the quotient space is naturally diffeomorphic to the dual of the Lie algebra, namely,

$$(T^*G)/G \cong \mathfrak{g}^*$$

with the $\pm$ ***Lie–Poisson bracket***

$$\{f,h\}_{\pm} = \pm \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu} \right] \right\rangle ,$$

where $f,h \in \mathcal{F}(\mathfrak{g}^*)$ *(Marsden and Weinstein [1983]).*

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where $f, h \in \mathcal{F}(\mathfrak{g}^*)$ *(Marsden and Weinstein [1983]).*
- The ***Lie–Poisson equation*** can be obtained as
$$\dot{\mu} = \mp \operatorname{ad}^*_{\frac{\delta h}{\delta \mu}} \mu.$$

Euler–Poincaré Reduction

Euler–Poincaré Reduction

- The Lagrangian analogue of the Lie–Poisson reduction is given by a ***reduced constrained variational principle*** for $l : \mathfrak{g} \to \mathbb{R}$ *(Marsden and Ratiu [1999]):*

$$\delta \int_{t_1}^{t_2} l(\xi(t))\, dt = 0$$

with variations of the form

$$\delta\xi = \dot{\eta} + [v, \eta],$$

where $\eta(t)$ is a curve in $\mathfrak{g}$ such that $\eta(t_1) = \eta(t_2) = 0$.

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- The ***Euler–Poincaré equations*** are obtained as

$$\frac{d}{dt}\frac{\delta l}{\delta \xi} = \mathrm{ad}^*_{\xi}\frac{\delta l}{\delta \xi}.$$

Lie-Poisson Variational Principle

- See, Cendra, Marsden, Pekarsky and Ratiu [2003], Variational principles for Lie-Poisson and Hamilton-Poincaré equations, Moscow Mathematical Journal **3**, 833-867.

Lie-Poisson Variational Principle

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- Let $H : T^*G \to \mathbb{R}$ be a left–invariant Hamiltonian. Hamilton's phase space principle is given by

$$\delta \int_{t_1}^{t_2} \left(p(t) \cdot \dot{g}(t) - H(g(t), p(t)) \right) dt = 0,$$

where the function $B(g, \dot{g}, p) = p \cdot \dot{g} - H(g, p)$ is defined on $TG \oplus T^*G$.

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where the function $B(g, \dot{g}, p) = p \cdot \dot{g} - H(g, p)$ is defined on $TG \oplus T^*G$.
- The group G acts on B by simultaneously left translating on each factor by the left-action and the tangent and contangent lifts. Then, B is to be G–invariant.

Lie-Poisson Variational Principle

- Using $\bar{\lambda} : T^*G \to G \times \mathfrak{g}^*$ and $\lambda : TG \to G \times \mathfrak{g}$, one can identify

$$TG \oplus T^*G \cong G \times (\mathfrak{g} \oplus \mathfrak{g}^*).$$

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$$TG \oplus T^*G \cong G \times (\mathfrak{g} \oplus \mathfrak{g}^*).$$
- Define the trivialized Hamiltonian on $G \times \mathfrak{g}^*$ by
$$\bar{H} := H \circ \bar{\lambda}^{-1}$$
and define $h : \mathfrak{g}^* \to \mathbb{R}$ by the restriction of $\bar{H}$ to $\mathfrak{g}^*$.

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and define $h : \mathfrak{g}^* \longrightarrow \mathbb{R}$ by the restriction of $\bar{H}$ to $\mathfrak{g}^*$.
- The function B on $TG \oplus T^*G$ ***drops to quotients***, namely, the function $b : \mathfrak{g} \oplus \mathfrak{g}^* \longrightarrow \mathbb{R}$ given by
$$b(\xi, \mu) = \mu \cdot \xi - h(\mu),$$
where $\xi = T_g L_{g^{-1}} \cdot \dot{g} \in \mathfrak{g}$ and $\mu = T_e^* L_g \cdot p_g \in \mathfrak{g}^*$.

Lie-Poisson Variational Principle

- Then, the ***Lie–Poisson variational principle*** is given by

$$\delta \int_{t_1}^{t_2} \{\mu(t) \cdot \xi(t) - h(\mu(t))\} \; dt = 0$$

with variations of the form

$$\delta\xi = \dot{\eta} + [v, \eta],$$

where

$$\eta(t) = T_g L_{g^{-1}} \delta g(t)$$

is a curve in $\mathfrak{g}$ such that

$$\eta(t_1) = \eta(t_2) = 0.$$

Lie-Poisson Variational Principle

- Taking variations of $(\xi(t), \mu(t)) \in \mathfrak{g} \oplus \mathfrak{g}^*$, it reads

$$\int_{t_1}^{t_2} \left\{ \left(\xi - \frac{\delta h}{\delta \mu} \right) \delta\mu + (-\dot{\mu} + \mathrm{ad}_\xi^* \mu) \cdot \eta \right\} dt = 0.$$

Then, we can obtain

$$\dot{\mu} = \mathrm{ad}_\xi^* \, \mu, \quad \xi = \frac{\delta h}{\delta \mu}.$$

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- But, the variational principle suggests one should regard the *proper space for the Lie–Poisson equations not as simply $\mathfrak{g}^*$ but as* ***the larger space*** *$\mathfrak{g} \oplus \mathfrak{g}^*$.*

Lie-Poisson Variational Principle

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- But, the variational principle suggests one should regard the *proper space for the Lie–Poisson equations not as simply $\mathfrak{g}^*$ but as* ***the larger space*** *$\mathfrak{g} \oplus \mathfrak{g}^*$.*

This is the key to our questions!

Reduction of Variational Principle

- Let $L : TG \to \mathbb{R}$ be a left invariant Lagrangian and recall the Hamilton–Pontryagin principle is given by

$$\delta \int_{t_1}^{t_2} \left\{ L(g(t), v(t)) + p(t) \cdot (\dot{g}(t) - v(t)) \right\} dt = 0.$$

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$$\delta \int_{t_1}^{t_2} \{L(g(t), v(t)) + p(t) \cdot (\dot{g}(t) - v(t))\} \, dt = 0.$$
- The group G acts on the function
$$F(g, v, p) = L(g, v) + p \cdot (\dot{g} - v)$$
on $TG \oplus T^*G$ by simultaneously left translating such that, for an element $h \in G$,
$$h \cdot (g, v, p) = (hg, T_g L_h \cdot v, T^*_{hg} L_{h^{-1}} \cdot p).$$
Then, the function F is to be invariant under the action of G, since we assume that L is G–invariant.

Reduction of Variational Principle

- Using the diffeomorphism $\lambda : TG \to G \times \mathfrak{g}$, define the ***trivialized Lagrangian on*** $G \times \mathfrak{g}$ by
$$\bar{L} := L \circ \lambda^{-1}$$
and define ***the reduced Lagrangian*** $l : \mathfrak{g} \to \mathbb{R}$ by restriction of $\bar{L}$ to $\mathfrak{g}$.

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$$\bar{L} := L \circ \lambda^{-1}$$
and define ***the reduced Lagrangian*** $l : \mathfrak{g} \to \mathbb{R}$ by restriction of $\bar{L}$ to $\mathfrak{g}$.

- The function F on $TG \oplus T^*G$ given by
$$F(g, v, p) = L(g, v) + p \cdot (\dot{g} - v)$$
drops to the quotient, which is to be $f : \mathfrak{g} \oplus \mathfrak{g}^* \to \mathbb{R}$ given by
$$f(\eta, \mu) = l(\eta) + \mu \cdot (\xi - \eta),$$
where
$$\xi = T_g L_{g^{-1}} \cdot \dot{g}, \quad \eta = T_g L_{g^{-1}} \cdot v \in \mathfrak{g}, \text{ and } \mu = T_e^* L_g \cdot p \in \mathfrak{g}^*.$$

Reduction of Variational Principle

- ***Reduction of the Hamilton-Pontryagin principle*** is given by

$$\delta \int_{t_1}^{t_2} \{ l(\eta(t)) + \mu(t) \cdot (\xi(t) - \eta(t)) \} \; dt = 0$$

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with variations of the form

$$\delta \xi(t) = \dot{\zeta}(t) + [\xi(t), \zeta(t)],$$

where

$$\zeta = T_g L_{g^{-1}} \delta g$$

is a curve in $\mathfrak{g}$ such that

$$\zeta(t_1) = \zeta(t_2) = 0.$$

Reduction of Variational Principle

- Taking variations yields

$$\int_{t_1}^{t_2} \left\{ \left(\frac{\delta l}{\delta \eta} - \mu \right) \delta\eta + \delta\mu \cdot (\xi - \eta) \right.$$
$$\left. + (-\dot{\mu} + \mathrm{ad}^*_\xi \mu) \cdot \zeta \right\} dt = 0,$$

which is satisfied for any $\delta\eta \in \mathfrak{g}$, $\zeta \in \mathfrak{g}$ and $\delta\mu \in \mathfrak{g}^*$.

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which is satisfied for any $\delta\eta \in \mathfrak{g}$, $\zeta \in \mathfrak{g}$ and $\delta\mu \in \mathfrak{g}^*$.

- Then, ***reduction of implicit Euler-Lagrange equations*** can be given by

$$\mu = \frac{\delta l}{\delta \eta}, \quad \xi = \eta, \quad \dot{\mu} = \mathrm{ad}_{\xi}^{*} \mu,$$

Reduction of Variational Principle

- Taking variations yields

$$\int_{t_1}^{t_2} \left\{ \left(\frac{\delta l}{\delta \eta} - \mu \right) \delta\eta + \delta\mu \cdot (\xi - \eta) + (-\dot{\mu} + \mathrm{ad}^*_{\xi}\mu) \cdot \zeta \right\} dt = 0,$$

  which is satisfied for any $\delta\eta \in \mathfrak{g}$, $\zeta \in \mathfrak{g}$ and $\delta\mu \in \mathfrak{g}^*$.

- Then, ***reduction of implicit Euler-Lagrange equations*** can be given by

$$\mu = \frac{\delta l}{\delta \eta}, \quad \xi = \eta, \quad \dot{\mu} = \mathrm{ad}^*_{\xi}\mu,$$

  which we call ***implicit Euler–Poincaré equations*** on $\mathfrak{g} \oplus \mathfrak{g}^*$.

The Canonical Forms on $G \times \mathfrak{g}^*$

- Using $\bar{\lambda} : T^*G \to G \times \mathfrak{g}^*$, the canonical one-form θ is locally represented by, for each $(g, \mu) \in G \times \mathfrak{g}^*$,

$$\theta(g, \mu) \cdot (v, \rho) = \mu(T_g L_{g^{-1}} v),$$

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- The canonical two-form $\omega = -\mathbf{d}\theta$ on $G \times \mathfrak{g}^*$ can be locally denoted by, for each $(g, \mu) \in G \times \mathfrak{g}^*$,

$$\begin{aligned} &\omega(g, \mu)((v, \rho), (w, \sigma)) \\ &= -\rho(T_g L_{g^{-1}} w) + \sigma(T_g L_{g^{-1}} v) + \mu([T_g L_{g^{-1}} v, T_g L_{g^{-1}} w]), \end{aligned}$$

  where $(w, \sigma) \in T_{(g,\mu)}(G \times \mathfrak{g}^*) \cong T_g G \times \mathfrak{g}^*$ *(see, Abraham and Marsden [1978], pp.315).*

The Canonical Dirac Structure

- ***The canonical Dirac structure*** $\mathcal{D}$ on $G \times \mathfrak{g}^*$ is a subbundle whose fibers, for each $(g, \mu) \in G \times \mathfrak{g}^*$,

$$\begin{aligned}\mathcal{D}(g, \mu) &\subset T_{(g,\mu)}(G \times \mathfrak{g}^*) \times T^*_{(g,\mu)}(G \times \mathfrak{g}^*) \\ &\cong (T_g G \times \mathfrak{g}^*) \times (T^*_g G \times \mathfrak{g})\end{aligned}$$

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- Using the canonical two-form ω, the canonical Dirac structure $\mathcal{D}$ is locally given by, for each $(g, \mu) \in G \times \mathfrak{g}^*$,

$$\begin{aligned}\mathcal{D}(g, \mu) = \{((v, \rho), (\beta, \eta)) \in (T_gG \times \mathfrak{g}^*) \times (T^*_gG \times \mathfrak{g}) \mid \\ \beta(w) + \sigma(\eta) = \omega(g, \mu)((v, \rho), (w, \sigma)) \\ \text{for all } \ (w, \sigma) \in T_gG \times \mathfrak{g}^*\}.\end{aligned}$$

Canonical Dirac Reduction

- Using $\bar{\lambda} : T^*G \to G \times \mathfrak{g}^*$ and $\lambda : TG \to G \times \mathfrak{g}$, we employ the identification

$$\begin{aligned}
\mathcal{D} \subset T(T^*G) \oplus T^*(T^*G) \\
&\cong T(G \times \mathfrak{g}^*) \oplus T^*(G \times \mathfrak{g}^*) \\
&\cong (G \times \mathfrak{g}^*) \times [(\mathfrak{g} \times \mathfrak{g}^*) \oplus (\mathfrak{g}^* \times \mathfrak{g})] \\
&\cong (G \times \mathfrak{g}^*) \times (V \oplus V^*),
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  where $V = \mathfrak{g} \times \mathfrak{g}^*$ and $V^* = \mathfrak{g}^* \times \mathfrak{g}$.

- By simply taking quotients by G, it reads

$$\begin{aligned} \mathcal{D}/G &\subset \frac{T(T^*G) \oplus T^*(T^*G)}{G} \\ &\cong \mathfrak{g}^* \times (V \oplus V^*). \end{aligned}$$

Canonical Dirac Reduction

- Then, we can define ***a reduced Dirac structure*** $\mathcal{D}_{\mu}^{/G}$ on $V_{\mu} = \mathfrak{g} \oplus \mathfrak{g}^{*}$, depending on $\mu \in \mathfrak{g}^{*}$, by

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where $\xi = T_g L_{g^{-1}} v$, $\zeta = T_g L_{g^{-1}} w \in \mathfrak{g}$, $\nu = T_e^* L_g \beta \in \mathfrak{g}^*$, and the ***reduced symplectic structure on*** V_μ is

$$\omega_\mu^{/G}((\xi,\rho),(\zeta,\sigma)) = -\rho(\zeta) + \sigma(\xi) + \mu([\xi,\zeta]).$$

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- Notice that the reduced Dirac structure $\mathcal{D}_\mu^{/G}$ includes *the Lie-Poisson structure or the coadjoint orbit symplectic structure (through $\mu \in \mathfrak{g}^*$).*

Dirac Differential Operator

- The differential of the Lagrangian $\bar{L}$ on $G \times \mathfrak{g}$ is
$$\mathbf{d}\bar{L} : G \times \mathfrak{g} \to (G \times \mathfrak{g}) \times (\mathfrak{g}^* \times \mathfrak{g}^*),$$
which is represented, in coordinates, by
$$\mathbf{d}\bar{L} = \left(g, T_g L_{g^{-1}} v, T_e^* L_g \frac{\partial L}{\partial g}, T_e^* L_g \frac{\partial L}{\partial v} \right).$$

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- Then, the Dirac differential
$$\mathfrak{D}\bar{L} = \bar{\gamma}_Q \circ \mathbf{d}\bar{L} : G \times \mathfrak{g} \to (G \times \mathfrak{g}^*) \times (\mathfrak{g}^* \times \mathfrak{g})$$
is locally denoted by
$$\mathfrak{D}\bar{L} = \left(g, T_e^* L_g \frac{\partial \bar{L}}{\partial v}, -T_e^* L_g \frac{\partial \bar{L}}{\partial g}, T_g L_{g^{-1}} v \right).$$

Reduction of Dirac Differential

- The naive quotient $\mathbf{d}^{/G}\bar{L} : \mathfrak{g} \longrightarrow \mathfrak{g} \times (\mathfrak{g}^* \times \mathfrak{g}^*)$ is locally given by, for $\eta \, (= T_g L_{g^{-1}} v) \in \mathfrak{g}$,

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$$\mathbf{d}^{/G}\bar{L} = \left(\eta, 0, \frac{\delta l}{\delta \eta} \right).$$
- The quotient $\mathfrak{D}^{/G}\bar{L} : \mathfrak{g} \longrightarrow \mathfrak{g}^* \times (\mathfrak{g}^* \times \mathfrak{g})$ is denoted by, for each $\eta \, (= T_g L_{g^{-1}} v) \in \mathfrak{g}$,
$$\mathfrak{D}^{/G}\bar{L} = \left(\frac{\delta l}{\delta \eta}, 0, \eta \right),$$
where $\mathbb{F}l : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ is given by, for each $\eta \in \mathfrak{g}$,
$$\mu = \frac{\delta l}{\delta \eta} \in \mathfrak{g}^*.$$

Reduction of Dirac Differential

- Define the reduction of $\mathbf{d}\bar{L} : G\times\mathfrak{g} \longrightarrow (G\times\mathfrak{g})\times(\mathfrak{g}^*\times\mathfrak{g}^*)$ by the operator $\mathbf{d}^{/G}l : \mathfrak{g} \longrightarrow \mathfrak{g}^* \oplus \mathfrak{g}^*$ such that it takes the value at each $\eta\,(= T_gL_{g^{-1}}v) \in \mathfrak{g}$
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- Define the ***reduction*** of $\mathfrak{D}\bar{L} : G \times \mathfrak{g} \longrightarrow (G \times \mathfrak{g}^*) \times (\mathfrak{g}^* \times \mathfrak{g})$ by the operator $\mathfrak{D}^{/G}l : \mathfrak{g} \longrightarrow \mathfrak{g}^* \oplus \mathfrak{g}$ such that for each $\eta \in \mathfrak{g}$, it takes the values at the new base point $\mu = \delta l/\delta\eta \in \mathfrak{g}^*$ via the partial Legendre transformation

$$\mathfrak{D}^{/G}l_\eta = (0, \eta) \in V_\mu^* = \mathfrak{g}^* \oplus \mathfrak{g}.$$

The Partial Vector Field

- Recall $\bar{\lambda} : T^*G \longrightarrow G \times \mathfrak{g}^*$ is equivariant relative to the cotangent lift of left translations on G and the G-action Λ on $G \times \mathfrak{g}^*$ given by, for $g, h \in G$ and $\mu \in \mathfrak{g}^*$,

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- Let X be a left invariant vector field on T^*G and the induced vector field $\bar{X} = \bar{\lambda}_* X$ on $G \times \mathfrak{g}^*$ is denoted by, for $g \in G, \mu \in \mathfrak{g}^*$,

$$\bar{X}(g, \mu) = \Big(Y_\mu(g), \mu, \dot{\mu}\Big) \in T_gG \times T_\mu\mathfrak{g}^*,$$

where $Y_\mu \in \mathfrak{X}(G)$ which depends on $\mu \in \mathfrak{g}^*$ is given by

$$Y_\mu(g) = \dot{g}.$$

The Partial Vector Field

- By equivariance of $\bar{\lambda}$, $\bar{X}$ is left invariant as
$$\Lambda_g^* \bar{X} = \bar{X},$$
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- The ***partial vector field*** $\bar{X}^{/G}$ is defined by the quotient of $\bar{X}$ on $G \times \mathfrak{g}^*$ such that it takes the value, at the base point $\mu \in \mathfrak{g}^*$,
$$\bar{X}_\mu^{/G} = \left(\xi, \dot{\mu}\right) \in V_\mu := \mathfrak{g} \oplus \mathfrak{g}^*,$$
where
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Euler-Poincaré-Dirac Reduction

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- ***The reduction of*** $(\bar{L},\, \Delta_G = TG,\, \bar{X})$ is given by a triple $(l,\ \mathfrak{g}, \bar{X}^{/G})$ that satisfies, at each base point $\eta \in \mathfrak{g}$,

$$(\bar{X}_{\mu}^{/G}, \mathfrak{D}^{/G} l_{\eta}) \in \mathcal{D}_{\mu}^{/G}$$

together with the partial Legendre transform $\mu = \mathbb{F}l(\eta)$.

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together with the partial Legendre transform $\mu = \mathbb{F}l(\eta)$.
- Locally, it reads
$$\sigma(\xi - \eta) + \left\langle -\dot{\mu} + \mathrm{ad}^*_{\xi}\mu, \zeta \right\rangle = 0,$$
which satisfies for all $(\zeta, \sigma) \in V_{\mu} = \mathfrak{g} \oplus \mathfrak{g}^*$.

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which satisfies for all $(\zeta, \sigma) \in V_\mu = \mathfrak{g} \oplus \mathfrak{g}^*$.
- Thus, we obtain
$$\xi = \eta, \quad \dot{\mu} = \mathrm{ad}_\xi^* \mu, \quad \mu = \frac{\delta l}{\delta \eta},$$
which are ***implicit Euler-Poincaré equations.***

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- From $X = (g, p, \dot{g}, \dot{p})$ and $\mathfrak{D}L = (g, \partial L/\partial v, -\partial L/\partial g, v)$, it locally reads

$$\left\langle -\frac{\partial L}{\partial g}, u \right\rangle + \langle v, \alpha \rangle = \langle \alpha, \dot{g} \rangle - \langle \dot{p}, u \rangle$$

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$$p = \frac{\partial L}{\partial v}, \quad \dot{g} = v \in \Delta(g), \quad \dot{p} - \frac{\partial L}{\partial g} \in \Delta^{\circ}(g).$$

Dirac Reduction

- Consider a constraint distribution $\Delta_G \subset TG$ given by

$$\Delta_G = \{(g, v) \in TG \mid g \in U,\ v \in \Delta(g)\}.$$

Then, define the distribution $\Delta_{G\times\mathfrak{g}^*} \subset T(G \times \mathfrak{g}^*)$ by

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$$\Delta_{G\times\mathfrak{g}^*} = \{(g,\mu,v,\rho) \mid g \in U,\ v \in \Delta(g)\}.$$
- The ***induced Dirac structure on*** $G\times\mathfrak{g}^*$ is defined by, for each $(g,\mu) \in G\times\mathfrak{g}^*$,
$$\begin{aligned}\mathcal{D}_{\Delta_G}(g,\mu) = \{ & ((v,\rho),(\beta,\eta)) \in (T_gG\times\mathfrak{g}^*)\times(T_g^*G\times\mathfrak{g}^*) \mid \\ & (v,\rho) \in \Delta_{G\times\mathfrak{g}^*}(g,\mu), \text{ and} \\ & \beta(w)+\sigma(\eta) = \omega_{\Delta_G}(g,\mu)((v,\rho),(w,\sigma)) \\ & \text{for all } (w,\sigma) \in \Delta_{G\times\mathfrak{g}^*}(g,\mu)\}.\end{aligned}$$

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$$\begin{aligned}\mathcal{D}^{/G}_{\Delta_G}(\mu) = \Big\{ ((\xi,\rho),(\nu,\eta)) \in V_\mu \times V^*_\mu \mid (\xi,\rho) \in \mathfrak{g}^{\Delta} \oplus \mathfrak{g}^*, \\ \text{and } \nu(\zeta) + \sigma(\eta) = \omega^{/G}_{\Delta_G}(\mu)((\xi,\rho),(\zeta,\sigma)) \\ \text{for all } (\zeta,\sigma) \in \mathfrak{g}^{\Delta} \oplus \mathfrak{g}^* \Big\},\end{aligned}$$

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where $\omega^{/G}_{\Delta_G}(\mu)$ is the restriction of the ***reduced symplectic structure*** $\omega^{/G}(\mu)$ ***to*** $\mathfrak{g}^{\Delta} \oplus \mathfrak{g}^* \subset V_\mu$.

Euler-Poincaré-Suslov Reduction

- The ***reduction of*** $(\bar{L}, \Delta_G, \bar{X})$ is given by a triple $(l,\ \mathfrak{g}^{\Delta}, \bar{X}^{/G})$ that satisfies, at each $\eta \in \mathfrak{g}^{\Delta}$,

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with the partial Legendre transform $\mu = \mathbb{F}l(\eta) \in \mathfrak{g}^{*}$.

- In local coordinates,

$$\sigma(\xi - \eta) + \left\langle -\dot{\mu} + \mathrm{ad}_{\xi}^{*}\mu, \zeta \right\rangle = 0$$

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$$\mu = \frac{\delta l}{\delta \eta}, \quad \xi = \eta \in \mathfrak{g}^{\Delta}, \quad \dot{\mu} - \mathrm{ad}_{\xi}^{*}\mu \in (\mathfrak{g}^{\Delta})^{\circ},$$

where $(\mathfrak{g}^{\Delta})^{\circ} \subset \mathfrak{g}^*$ is an annihilator of $\mathfrak{g}^{\Delta} \subset \mathfrak{g}$.

Euler-Poincaré-Suslov Reduction

- The set of equations

$$\mu = \frac{\delta l}{\delta \eta}, \quad \xi = \eta \in \mathfrak{g}^{\Delta}, \quad \dot{\mu} - \mathrm{ad}^*_{\xi}\mu \in (\mathfrak{g}^{\Delta})^{\circ},$$

is called as ***Euler-Poincaré-Suslov Equations.***

(See, Bloch, Nonholonomic Mechanics and Control [2003].)

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- Example (Euler-Poincaré-Suslov Problem on $SO(3)$). The Euler-Poincaré-Suslov equations are to be given in coordinates as

$$p_a = I_{ab}\,\omega^b, \quad A_a\,\omega^a = 0, \quad \dot{p}_b - C_{ab}^{c} I^{ad}\,p_c\,\omega^d = \lambda A_b,$$

  where $\omega = \omega^i e_i \in \mathfrak{g}$ and $A = A_a e^a \in \mathfrak{g}^*$.

Concluding Remarks

- Taking simply quotients of the canonical Dirac structure on T^*G by G ends up with a reduced Dirac structure on $\mathfrak{g} \oplus \mathfrak{g}^*$ that depends points $\mu \in \mathfrak{g}^*$.

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- Generalization to the case in which G acting to a configuration space Q works ok, which results in the implicit analogue of the Lagrange-Poincaré and Hamilton-Poincaré equations (*see, Cendra, Marsden, Pekarsky, and Ratiu[2003]*), but needs to be worked out.