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STRICT DEFORMATION QUANTIZATION: THE EXAMPLE OF TORIC MANIFOLDS

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Quantum mechanics suggests:

Given a compact Poisson manifold M, try to deform $A = C^{\infty}(M)$ into non-commutative algebras $A_{\hbar}$. One approach:

Strict deformation quantization:

Want a product $\times_{\hbar}$, an involution $*^{\hbar}$, and a C*-norm $\|\cdot\|$ on vector space A so that for $\hbar = 0$ have original algebra structure of A, and

1. $\hbar \mapsto \bar{A}_{\hbar}$ is a continuous field of C*-algebras, where $\bar{A}_{\hbar}$ is the completion of $A_{\hbar}$ for the norm.
2. $\|(f \times_{\hbar} g - g \times_{\hbar} f)/\hbar - i\{f, g\}\|_{\hbar} \to 0$ as $\hbar \to 0$.

(These express the "correspondence principle".) One can also apply the above definition to a non-commutative C*-algebra equipped with a "Poisson bracket".

We examine a simple class of examples:

Let A be a unital C*-algebra, e.g. $A = C(M)$. Let α be an action of $\mathbb{R}^d$ on A, e.g. coming from a smooth action on M. Let A^∞ be the dense *-subalgebra of smooth elements for α. Let ∂_j be the derivation of A^∞ in the j-th direction for α, e.g. the vector field on M in the j-th direction for α.
For any given $\theta \in M_d^{sk}(\mathbb{R})$ define a Poisson bracket on A^∞ by

$$\{a, b\} = \sum \theta_{jk}(\partial_j a)(\partial_k b).$$

We will restrict attention to actions of $\mathbb{T}^d$.

View $\mathbb{T}^d$ as $(\mathbb{R}^d/\mathbb{Z}^d)$, and define $e(r) = \exp(2\pi i r)$.

For $n \in \mathbb{Z}^d$ set

$$A_n = \{a \in A : \alpha_t(a) = e(n \cdot t)a \ \forall t\}.$$

Then $\oplus_n A_n$ is a dense $*$-subalgebra of A, and

$$A^\infty = \{\sum a_n : a_n \in A_n \text{ and } (\|a_n\|) \in \mathcal{S}(\mathbb{Z}^d)\}.$$

For θ as above, and $a, b \in A^\infty$ set

$$a \times_\theta b = \sum_{m,n} a_m \, b_n \, e(m \cdot \theta n).$$

Denote the resulting algebra by A_θ^∞.

For any covariant representation $(\pi, U, \mathcal{H})$ of $(A, \mathbb{T}^d, \alpha)$, (so $U_t \pi(a) U_t^* = \pi(\alpha_t(a))$ we have $\mathcal{H} = \oplus \mathcal{H}_n$ for U.
For $\xi \in \mathcal{H}$ with $\xi = (\xi_n)$, set

$$\pi^\theta(a)\xi = \sum_{m,n} \pi(a_m)\, \xi_n \;\; e(m \cdot \theta n).$$

Then π^θ is a $*$-representation of A_θ^∞. If π is injective, then we have the C*-norm $\|a\|_\theta = \|\pi_\theta(a)\|$. Can show this norm is independent of $(\pi, U, \mathcal{H})$. The completion of A_θ^∞ for this norm is the C*-algebra $\bar{A}_\theta$.

Then $\hbar \to \bar{A}_{\hbar\theta}$ is a strict deformation quantization of A in the direction of the Poisson bracket from θ. [R. 1993]

The algebras A_θ share some properties with A. For example, $K_*(A_\theta) \cong K_*(A)$. [R. 1993] But the positive cones coming from actual "vector bundles", i.e. projective modules, can be quite different.

For the case of a manifold M let $\Omega^*(M)$ be its deRham complex. The action of $\mathbb{T}^d$ on $C^\infty(M)$ lifts to an action on $\Omega^*(M)$ commuting with the exterior derivative d. So by the same techniques the exterior product on $\Omega^*(M)$ can be deformed, but the exterior derivative d left unchanged, to get a complex $\Omega^*(M_\theta)$ with $\Omega^0(M) = C^\infty(M_\theta)$.
(It is not (graded) commutative in general.)

Lie groups can be deformed into quantum groups by our construction.

Let G be a compact Lie group, and let H be a subgroup of G with $H \cong \mathbb{T}^d$. Let α be the action of $H \times H$ on $A = C(G)$ given by

$$(\alpha_{(s,t)} f)(x) = f(s^{-1}xt)$$

For $\tilde{\theta}$ of size $2d \times 2d$ we can form $C(G_{\tilde{\theta}})$. But this doesn't respect the coproduct

$$(\Delta f)(x, y) = f(xy),$$

unless $\tilde{\theta}$ is of the form $\begin{pmatrix} \theta & 0 \\ 0 & -\theta \end{pmatrix}$, which we assume. Then the corresponding Poisson bracket makes G into a Poisson Lie group, and one gets a quantum group, $C(G_\theta)$, with unchanged coproduct, coidentity, and antipode. One also gets a quantum group structure on $C^\infty(G_\theta)$.
[R. 1993]

[Connes and Landi 2001]
[Connes and Dubois-Violette 2002]:

They searched for universal non-commutative spaces A having a projective module given by a projection $p \in M_4(A)$ with A generated by the entries of p and with reduced Chern classes $ch_j(p) = 0$ for $j = 0, 1$ as happens for the 4-sphere S^4. They were led to $C(S^4_\theta)$, using the action of a maximal torus in $SO(5)$.

More generally, they show:

Let M be a compact Riemannian "spin-manifold" so that it has a vector bundle $\mathcal{S}$ of spinors, and so has a Dirac operator D acting on $L^2(M, \mathcal{S})$. Suppose that α is a smooth action of $\mathbb{T}^d$ by isometries of M. Then α lifts to a projective representation on $L^2(M, \mathcal{S})$ which commutes with D.

Consequently, one can twist the action on $C^\infty(M, \mathcal{S})$ so that it becomes a $C^\infty(M_\theta)$ projective module, $C^\infty(M_\theta, \mathcal{S})$, with D still defined on it, and with completion still $L^2(M, \mathcal{S})$. We thus obtain a Dirac operator for $C(M_\theta)$.

Connes' view: The way to define a "non-commutative Riemannian manifold" is by means of a "spectral geometry" $(A, \mathcal{H}, D)$ satisfying suitable axioms.

These axioms are satisfied for $A = C(M_\theta)$, $\mathcal{H} = L^2(M, \mathcal{S})$, and D.
One recovers the "real" structure by means of the "charge conjugation operator" J on $L^2(M, \mathcal{S})$, twisted appropriately.

The Hodge operator $*$ on $\Omega^*(M)$ from the Riemannian metric can be twisted to give a Hodge operator $*_\theta$ on $\Omega^*(M_\theta)$.

[Varilly 2001] :

Let G be a compact Lie group, with closed subgroups $K \supset H$ with $H \cong \mathbb{T}^d$. Then for θ as before we can form $C((G/K)_\theta)$ and the quantum group $C(G_\theta)$, and for appropriate definitions, $C(G_\theta)$ "acts" on $C((G/K)_\theta)$, and $C((G/K)_\theta)$ is a "homogeneous subspace" of $C(G_\theta)$.

In this way one finds that $C((SO(5))_\theta)$ acts on $C(S^4_\theta)$, and $C(S^4_\theta)$ is a homogeneous space of $C((SO(5))_\theta)$.

Furthermore, the action of $C((SO(5))_\theta)$ is by "isometries" for the "Riemannian structure" on $C(S^4_\theta)$ given by the Dirac operator $(D, L^2(M_\theta, \mathcal{S}))$.

[Sitarz, 2001]:

Let G, K and H be as above, and let $\mathfrak{g} = \mathrm{Lie}(G)$, with universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, which is a Hopf algebra. Then the Hopf algebra $\mathcal{U}(\mathfrak{g})$ has an "action" on the algebra $C^\infty(G/K)$.

By a Drinfeld twist using $\mathfrak{h} = \mathrm{Lie}(H)$ one can deform $\mathcal{U}(\mathfrak{g})$ to a Hopf algebra $\mathcal{U}_\theta(\mathfrak{g})$. Only the coalgebra structure is deformed — the product is unchanged. Then $\mathcal{U}_\theta(\mathfrak{g})$ acts on $C^\infty((G/K)_\theta)$.

If G/K has a Dirac operator as above, so that $(G/K)_\theta$ does too, then this is compatible with the action of $\mathcal{U}_\theta(\mathfrak{g})$.

Thus $\mathcal{U}_\theta(so(5))$ acts on $C^\infty(S^4_\theta)$, in a way compatible with the Dirac operator for $C^\infty(S^4_\theta)$.

[Landi and van Suijlekom 2006]:

They study Yang-Mills for $C(S^4_\theta)$.

Classically there is an action of $SU(2)$ on S^7 with quotient S^4 (a Hopf fibration). This action commutes with an action of $Spin(5)$ on S^7. Then any f.d. representation of $SU(2)$ "induces" a vector bundle on S^4 which is $Spin(5)$-equivariant. For the representation of $SU(2)$ on $\mathbb{C}^2$ one gets the classical "instanton" bundle on S^4 having interesting Yang-Mills minima.

Similarly, for a θ' determined by θ, there is an action of (ordinary) $SU(2)$ on $C(S^7_{\theta'})$ such that the fixed-point algebra is exactly $C(S^4_\theta)$. For each representation of $SU(2)$ get a projective module over $C(S^4_\theta)$. For the 2-d representation get the "quantized instanton bundle", say $\mathcal{E}$.

Then $\mathcal{U}_\theta(so(5))$ acts on $C^\infty(S^4_\theta)$ and $C^\infty(S^7_{\theta'})$, and $\mathcal{E}$ is equivariant for this infinitesimal action.

One can look for connections on $\mathcal{E}$, that is, maps

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_A \Omega^1(S^4_\theta)$$

(for $A = C^\infty(S^4_\theta)$) satisfying the Leibniz rule

$$\nabla(\xi\omega) = (\nabla\xi)\omega \; + \; \xi \otimes d\omega.$$

Then ∇ extends to

$$\nabla : \mathcal{E} \otimes_A \Omega^p(S^4_\theta) \to \mathcal{E} \otimes_A \Omega^{p+1}(S^4_\theta).$$

The curvature of ∇ is $F = \nabla^2$, and it is determined by

$$\nabla^2 : \mathcal{E} \to \mathcal{E} \otimes_A \Omega^2(S^4_\theta).$$

On $\mathcal{E}$ have a natural Hermitian metric

$$\langle \xi,\ \eta \rangle_A \ \in \ A = C(S^4_\theta),$$

which extends to one on $\Omega^*(S^4_\theta)$. We say that a connection ∇ is compatible with this Hermitian metric if it satisfies the Leibniz rule

$$d(\langle \xi,\ \eta \rangle_A) \ = \ \langle \nabla\xi,\ \eta \rangle_A + (-1)^{|\xi|} \langle \xi,\ \nabla\eta \rangle_A.$$

We let $CC(\mathcal{E})$ denote the set of compatible connections on $\mathcal{E}$ for its Hermitian metric.

From the usual round Riemannian metric on S^4, for which $SO(5)$ acts by isometries, we get the deformed Hodge $*_\theta$ on $\Omega^*(S^4_\theta)$.

For any $\nabla \in CC(\mathcal{E})$ with curvature F set

$$YM(\nabla) = \int *_\theta \operatorname{tr}(F \ *_\theta F)$$

where tr is the A-valued trace on $\mathrm{End}_A(\mathcal{E})$ coming from the Hermitian metric on $\mathcal{E}$.

Let $\mathcal{G} = UEnd_A(\mathcal{E})$, the group of unitary elements of $End_A(\mathcal{E})$. It is the "gauge group" for $\mathcal{E}$, and acts on $CC(\mathcal{E})$ by conjugation. The functional YM on $CC(\mathcal{E})$ is invariant under the action of $\mathcal{G}$.

The set of minima of YM is carried into itself by the action of $\mathcal{G}$, and the set of $\mathcal{G}$–orbits in the set of minima is the "moduli space" of mimima.

The conformal Lie algebra $so(5,1)$ acts on $C^\infty(S^4)$, while $SO(5)$, and so $\mathbb{T}^2$, act by Ad on $so(5,1)$. Thus we can form $\mathcal{U}_\theta(so(5,1))$, and it acts on $C^\infty(S^4_\theta)$ and on $\Omega^*(S^4_\theta)$, leaving the Hodge structure invariant. It also acts on $C^\infty(S^7_{\theta'})$, and so infinitesimally on $CC(\mathcal{E})$.

There is a canonical connection, ∇^0, coming from the construction of $\mathcal{E}$, which is a minima for YM.

Landi and van S. use the action of $\mathcal{U}_\theta(so(5,1))$ to show that the tangent space at ∇^0 in the moduli space of minima for YM is exactly 5-dimensional.

Connes: Any spectral geometry $(A, \mathcal{H}, D)$ can be viewed as a "non-commutative metric space". The Lipschitz seminorm, L, on A is defined by

$$L(a) = \|[D, a]\|.$$

Let $S(A)$ denote the state-space of A, consisting of the positive linear functionals on A of norm 1 (the "non-commutative probability measures"). Define a metric, ρ_L on $S(A)$ by

$$\rho_L(\mu, \nu) = \sup\{|\mu(a) - \nu(a)| : L(a) \leq 1\}.$$

Me: Want (*) The topology on $S(A)$ from ρ_L should agree with the weak-* topology.

Then can define a quantum Gromov-Hausdorff distance.

[Hanfeng Li, 2005] For L on $C^\infty(M_\theta)$ coming from D the property (*) holds. Furthermore,

$$\mathrm{dist}_{qGH}(M_{\theta_1}, M_{\theta_2})$$

is continuous in θ_1 and θ_2.