METRIC CONVEXITY IN THE SYMPLECTIC CATEGORY

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Some of this talk is work in progress

THE CONVEXITY THEOREMS

(M, ω) a connected paracompact symplectic manifold.

G connected Lie group acting properly and canonically on (M, ω).

$\mathfrak{g}$ the Lie algebra of G, $\mathfrak{g}^*$ its dual.

Assume that $\mathbf{J} : M \to \mathfrak{g}^*$ is an equivariant momentum map of the action: $\mathbf{dJ}^\xi = \omega(\xi_M, \cdot)$, where $\xi_M(m) = \left.\frac{d}{dt}\right|_{t=0} \exp(t\xi) \cdot m$.

Guillemin-Kirwan-Sternberg: Let M and G be compact, T a maximal torus of G, $\mathfrak{t}$ its Lie algebra, $\mathfrak{t}^*$ its dual, and $\mathfrak{t}^*_+$ the positive Weyl chamber relative to a fixed ordering of the roots. Then $\mathcal{P}_G := \mathbf{J}(M) \cap \mathfrak{t}^*_+$ is a compact convex polytope, called the G-**momentum polytope**. The fibers of $\mathbf{J}$ are connected.

For non-compact manifolds, the previous results no longer hold and a counterexample was given by Prato (1994).

Very important particular case: $G = T$.

Atiyah-Guillemin-Sternberg: $\mathbf{J}(M)$ is a convex compact polytope equal to Conv $\left(\mathbf{J}\left(M^T\right)\right)$, where $M^T := \{m \in M \mid t \cdot m = m, \forall t \in T\}$ is the fixed point set of the T-action and Conv(S) denotes the convex hull of the set S. The fibers of $\mathbf{J}$ are connected.

If $M = \mathcal{O}$, a coadjoint orbit in $\mathfrak{g}^*$, then there is a unique $\mu \in \mathfrak{t}^*_+$ such that $\mu \in \mathcal{O} \cap \mathfrak{t}^*$. Denote $\mathcal{O} = \mathcal{O}_\mu$ and let $i : \mathfrak{g} \hookrightarrow \mathfrak{t}$. Let $W := N(T)/T$ be the Weyl group of G. Particular case:

Kostant Linear Convexity: $i^*(\mathcal{O}_\mu) =$ Conv$(W \cdot \mu)$: the projection of the coadjoint orbit onto the Cartan algebra is the convex hull of the corresponding Weyl group orbit $\mathcal{O}_\mu \cap \mathfrak{t}^*$.

Let G be compact and T a maximal torus. What is the relationship between the images of the momentum maps for the two actions?

Let $\mathbf{J}_G : M \to \mathfrak{g}^*$. Then $\mathbf{J}_T = i^* \circ \mathbf{J}_G : M \to \mathfrak{t}^*$, where $i : \mathfrak{t} \hookrightarrow \mathfrak{g}$. Then

$$\mathbf{J}_T(M) = \text{Conv}\,(W \cdot \mathcal{P}_G)$$

Proof $\mathbf{J}_G$ equivariant implies $\mathbf{J}_G(M) = \bigcup_{\mu \in \mathcal{P}_G} \mathcal{O}_\mu$, so by Kostant

$$\mathbf{J}_T(M) = i^* \left(\bigcup_{\mu \in \mathcal{P}_G} \mathcal{O}_\mu \right) = \bigcup_{\mu \in \mathcal{P}_G} i^* \left(\mathcal{O}_\mu \right) = \bigcup_{\mu \in \mathcal{P}_G} \text{Conv}(W \cdot \mu).$$

For $\mu \in \mathcal{P}_G$ we have $W \cdot \mu \subset W \cdot \mathcal{P}_G \Longrightarrow \text{Conv}(W \cdot \mu) \subset \text{Conv}\,(W \cdot \mathcal{P}_G)$, so $\mathbf{J}_T(M) = \bigcup_{\mu \in \mathcal{P}_G} \text{Conv}(W \cdot \mu) \subset \text{Conv}(W \cdot \mathcal{P}_G)$.

Conversely, $W \cdot \mathcal{P}_G = \bigcup_{\mu \in \mathcal{P}_G} W \cdot \mu \subset \bigcup_{\mu \in \mathcal{P}_G} \text{Conv}(W \cdot \mu) = \mathbf{J}_T(M)$ and hence

$$\text{Conv}(W \cdot \mathcal{P}_G) \subset \text{Conv}\,(\mathbf{J}_T(M)) = \mathbf{J}_T(M)$$

since $\mathbf{J}_T(M)$ is convex. ■

There are other convexity theorems. Here is a famous one. G complex semisimple Lie group, $G^{\mathbb{R}}$ the same Lie group thought of as real, K maximal compact subgroup so its Lie algebra $\mathfrak{k}$ is a compact real form of the Lie algebra $\mathfrak{g}$ of G. Then put $\mathfrak{p} := i\mathfrak{k}$, $P := \exp \mathfrak{p}$ so $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, $G = KP$ are the **Cartan decompositions** at infinitesimal and global level. Let T be a maximal torus in K, so $\mathfrak{t}$ is a maximal toral subalgebra in $\mathfrak{k}$ and $\mathfrak{a} := i\mathfrak{t} \subset \mathfrak{p}$ is a maximal Abelian subspace of $\mathfrak{p}$. $\mathfrak{h} := \mathfrak{a} \oplus \mathfrak{t} \subset \mathfrak{g}$ is a Cartan subalgebra and let Δ be the roots it determines. Define $\mathfrak{n} := \oplus_{\alpha>0}\mathfrak{g}_\alpha$, $N := \exp \mathfrak{n}$, and $W := N(T)/T$, the **Weyl group** of K. Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ and $G^{\mathbb{R}} = KAN$ are the **Iwasawa decompositions** at infinitesimal and global level. K acts on P by conjugation and on $\mathfrak{p}$ by the adjoint action; $\exp : \mathfrak{p} \to P$ is a K-equivariant diffeomorphism.

$$b \in B := AN \longleftrightarrow b^*b \in P$$

is a diffeomorphism, where $g^* := \tau(g^{-1})$ and $\tau : G^{\mathbb{R}} \to G^{\mathbb{R}}$ is the **Cartan involution**. Its derivative $\tau : \mathfrak{g}^{\mathbb{R}} \to \mathfrak{g}^{\mathbb{R}}$ has only eigenvalues ± 1, the $+1$ eigenspace is $\mathfrak{k}$ and the -1 eigenspace is $\mathfrak{p}$.

Let $\rho_{\mathfrak{a}} : \mathfrak{g} \to \mathfrak{a}$ and $\rho_A : G \to A$ be the projections defined by the Iwasawa decompositions. For $a \in A$ let $\mathcal{O}_a$ be the K-orbit of a and identify A with $\mathfrak{a}$.

Kostant Nonlinear Convexity: $\rho_A(\mathcal{O}_a) = \mathrm{Conv}(W \cdot a)$.

What does this really say? Let $J = \log \circ \rho_A \circ \exp : \mathfrak{p} \to \mathfrak{a}$. Note $T_0 J = \rho_{\mathfrak{a}}$. Let $\eta \in \mathfrak{p}$ and $\mathcal{O}_\eta$ the K-orbit in $\mathfrak{p}$. Then

$$J(\mathcal{O}_\eta) = \rho_{\mathfrak{a}}(\mathcal{O}_\eta) = W \cdot \eta.$$

Example. $G = \mathrm{SL}(n,\mathbb{C})$, $K = \mathrm{SU}(n)$, $P = \{X \in \mathrm{SL}(n,\mathbb{C}) \mid X = X^*,$ positive definite$\}$, $A = \{X \in P \mid A$ diagonal$\}$, $N = \{X \in \mathrm{SL}(n,\mathbb{C}) \mid X$ upper triangular with ones on the diagonal$\}$,

$$\rho_{\mathfrak{a}}(X) = \text{diagonal of } X, \quad X \in \mathfrak{p}$$

but

$$J(X) = \frac{1}{2}\left(\log \Delta_1\left(e^{2X}\right), \frac{\log \Delta_2\left(e^{2X}\right)}{\log \Delta_1\left(e^{2X}\right)}, \ldots, \frac{\log \Delta_n\left(e^{2X}\right)}{\log \Delta_{n-1}\left(e^{2X}\right)}\right).$$

Here $\Delta_k(S)$ for a symmetric matrix $S = [s_{ij}]_{i,j=1,\ldots,n}$ denotes the determinant of the submatrix $[s_{ij}]_{i,j=1,\ldots,k}$. If $X = \operatorname{diag}(a_1, \ldots, a_n) \in \mathfrak{a}$, then $\mathcal{O}_X$ is the set of Hermitian matrices with eigenvalues $a_1, \ldots, a_n$.

Symplectic interpretation (using Poisson-Lie group theory) was given by Lu-Ratiu (1991) with a gap for real groups filled in by Sleewagen (2001).

Another problem: What if a canonical action of G on (M, ω) does *not* admit a momentum map? Is there absolutely no convexity result associated to this action? Convexity of what?

There are two possible objects associated to any canonical action: the **optimal momentum** map and the **cylinder valued momentum map**. Both exist for *any* symplectic action.

CYLINDER VALUED MOMENTUM MAPS

(M, ω) connected, paracompact. $\mathfrak{g}$ acts canonically on M.

$\pi : M \times \mathfrak{g}^* \to M$ trivial principal $(\mathfrak{g}^*, +)$-bundle relative to the action $\nu \cdot (m, \mu) := (m, \mu - \nu)$, with $m \in M$ and $\mu, \nu \in \mathfrak{g}^*$.

Define the flat connection one-form $\alpha \in \Omega^1(M \times \mathfrak{g}^*; \mathfrak{g}^*)$ by

$\langle \alpha(m, \mu)(v_m, \nu), \xi \rangle := (\mathbf{i}_{\xi_M}\omega)(m)(v_m) - \langle \nu, \xi \rangle$, $v_m \in T_m M, \xi \in \mathfrak{g}, \mu, \nu \in \mathfrak{g}^*$.

For $(z, \mu) \in M \times \mathfrak{g}^*$, let $(M \times \mathfrak{g}^*)(z, \mu) \subset M \times \mathfrak{g}^*$ be the **holonomy bundle** through (z, μ): all points in $M \times \mathfrak{g}^*$ that can be joined to (z, μ) by a horizontal curve.

$\mathcal{H}(z,\mu)$ denotes the **holonomy group** of α with reference point (z,μ): all $g \in G$ determined by all the loops in M based at m via horizontal lift. This is an Abelian zero dimensional Lie subgroup of $(\mathfrak{g}^*, +)$ by the flatness of α.

The **Bundle Reduction Theorem** guarantees that the principal bundle $((M \times \mathfrak{g}^*)(z,\mu), M, \pi|_{(M\times\mathfrak{g}^*)(z,\mu)}, \mathcal{H}(z,\mu))$ is a reduction of the principal bundle $(M \times \mathfrak{g}^*, M, \pi, \mathfrak{g}^*)$ and that the connection one-form α is reducible to a connection one-form on $(M \times \mathfrak{g}^*)(z,\mu)$. It is only here that paracompactness of M is used, since it is a technical hypothesis in the Bundle Reduction Theorem.

Notation: $(\widetilde{M}, M, \widetilde{p}, \mathcal{H}) := ((M \times \mathfrak{g}^*)(z,\mu), M, \pi|_{(M\times\mathfrak{g}^*)(z,\mu)}, \mathcal{H}(z,\mu))$. Let $\widetilde{\mathbf{K}} : \widetilde{M} \subset M \times \mathfrak{g}^* \to \mathfrak{g}^*$ be the projection into the $\mathfrak{g}^*$-factor.

$\overline{\mathcal{H}}$ closure of $\mathcal{H}$ in $\mathfrak{g}^*$. Since $\overline{\mathcal{H}}$ is a closed subgroup of $(\mathfrak{g}^*, +)$, the quotient $C := \mathfrak{g}^*/\overline{\mathcal{H}}$ is a cylinder, that is, it is isomorphic to the Abelian Lie group $\mathbb{R}^a \times \mathbb{T}^b$ for some $a, b \in \mathbb{N}$. Let $\pi_C : \mathfrak{g}^* \to \mathfrak{g}^*/\overline{\mathcal{H}} = C$ be the projection. Define $\mathbf{K} : M \to C$ to be the map that makes the following diagram commutative:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\widetilde{\mathbf{K}}} & \mathfrak{g}^* \\ \widetilde{p}\downarrow & & \downarrow\pi_C \\ M & \xrightarrow{\mathbf{K}} & \mathfrak{g}^*/\overline{\mathcal{H}}. \end{array}$$

In other words, $\mathbf{K}$ is defined by $\mathbf{K}(m) = \pi_C(\nu)$, where $\nu \in \mathfrak{g}^*$ is any element such that $(m, \nu) \in \widetilde{M}$. This is a good definition because if we have two points $(m, \nu), (m, \nu') \in \widetilde{M}$, this implies that $(m, \nu), (m, \nu') \in \widetilde{p}^{-1}(m)$ and, as $\mathcal{H}$ is the structure group of the principal fiber bundle $\widetilde{p} : \widetilde{M} \to M$, there exists an element $\rho \in \mathcal{H}$ such that $\nu' = \nu + \rho$. Consequently, $\pi_C(\nu) = \pi_C(\nu')$.

$\mathbf{K} : M \to \mathfrak{g}^*/\overline{\mathcal{H}} =: C$ is a **cylinder valued momentum map** associated to the canonical $\mathfrak{g}$-action on (M, ω). It is a strict generalization of the standard momentum map since *the G-action has a standard momentum map if and only if the holonomy group $\mathcal{H}$ is trivial. In this case, the cylinder valued momentum map* **is** *a standard momentum map.*

Notice that we refer to *"a"* and not to *"the"* cylinder valued momentum map since each choice of the holonomy bundle of the connection α defines such a map. How does it depend on choices?

Let $\widetilde{M_1}$ and $\widetilde{M_2}$ be two holonomy bundles of $(M \times \mathfrak{g}^*, M, \pi, \mathfrak{g}^*)$.

• $\exists \tau \in \mathfrak{g}^*$ such that $\widetilde{M_2} = R_\tau(\widetilde{M_1})$, where $R_\tau(m, \mu) := (m, \mu + \tau)$, for any $(m, \mu) \in M \times \mathfrak{g}^*$.

• Since $(\mathfrak{g}^*, +)$ is Abelian all the holonomy groups based at any point are the same and hence $\pi_C : \mathfrak{g}^* \to \mathfrak{g}^*/\overline{\mathcal{H}}$ does not depend on the choice of $\widetilde{M}$. This is why we call $\mathcal{H}$ the **Hamiltonian holonomy** of the G-action on (M, ω).

• π_C is a group homomorphism.

Let $\widetilde{p}_{\widetilde{M}_i} : \widetilde{M}_i \to M$, $\widetilde{\mathbf{K}}_{\widetilde{M}_i} : \widetilde{M}_i \to \mathfrak{g}^*$, and $\mathbf{K}_{\widetilde{M}_i} : M \to \mathfrak{g}^*$ be the maps in the diagram constructed using the holonomy bundles $\widetilde{M}_i$, $i \in \{1, 2\}$. Then $\mathbf{K}_{\widetilde{M}_2} = \mathbf{K}_{\widetilde{M}_1} + \pi_C(\tau)$.

Remark: So far, only the closedness of ω was used.

Remark: The Hamiltonian holonomy $\mathcal{H}$ is the image of the **period homomorphism** $P_\omega : \pi_1(M, z) \to \mathfrak{g}^*$ defined by

$$\langle P_\omega([\gamma]), \xi \rangle := \int_\gamma \mathbf{i}_{\xi_M} \omega, \text{ for any } \xi \in \mathfrak{g}.$$

Properties of $\mathbf{K}$:

(i) $\mathbf{K}$ *is a smooth map that satisfies Noether's Theorem:* $\forall h \in C^\infty(M)^{\mathfrak{g}} := \{f \in C^\infty(M) \mid \mathbf{d}h(\xi_M) = 0, \forall \xi \in \mathfrak{g}\}$, *the flow* F_t *of* X_h *satisfies the identity* $\mathbf{K} \circ F_t = \mathbf{K}|_{\mathrm{Dom}(F_t)}$.

(ii) $T_m\mathbf{K}(v_m) = T_\mu \pi_C(\nu), \forall m \in M, \forall v_m \in T_mM$, *where* $\mu \in \mathfrak{g}^*$ *is any element s.t.* $\mathbf{K}(m) = \pi_C(\mu)$ *and* $\nu \in \mathfrak{g}^*$ *is uniquely determined by:*

$$\langle \nu, \xi \rangle = (\mathbf{i}_{\xi_M}\omega)(m)(v_m), \quad \forall \xi \in \mathfrak{g}.$$

(iii) Reduction Lemma: $\ker(T_m\mathbf{K}) = \left(\left(\mathrm{Lie}(\overline{\mathcal{H}})\right)^\circ \cdot m\right)^\omega$.

(iv) Bifurcation Lemma:

$$\mathrm{range}\,(T_m\mathbf{K}) = T_\mu\pi_C\left((\mathfrak{g}_m)^\circ\right),$$

where $\mu \in \mathfrak{g}^*$ *is any element such that* $\mathbf{K}(m) = \pi_C(\mu)$.

$\mathbf{K}$ is not equivariant, in general. There is a whole theory how to drop the coadjoint action to the cylinder $\mathfrak{g}^*/\overline{\mathcal{H}}$ and associate to it a cocycle to make it equivariant – has to do with central extensions.

There is a good reduction theory associated to $\mathbf{K}$. Free case:

The main result. *Let (M, ω) be a connected paracompact symplectic manifold and G a Lie group acting freely and properly on it by symplectic diffeomorphisms. Let $\mathbf{K} : M \to \mathfrak{g}^*/\overline{\mathcal{H}}$ be a cylinder valued momentum map for this action. Then $\mathfrak{g}^*/\overline{\mathcal{H}}$ carries a natural Poisson structure and there exists a smooth G-action on it with respect to which $\mathbf{K}$ is equivariant and Poisson. Moreover:*

(i) *The Marsden-Weinstein reduced space $M^{[\mu]} := \mathbf{K}^{-1}([\mu])/G_{[\mu]}$, $[\mu] \in \mathfrak{g}^*/\overline{\mathcal{H}}$, has a natural Poisson structure inherited from the symplectic structure (M, ω) that is, in general, degenerate. $M^{[\mu]}$ will be referred to as the* **Poisson reduced space.**

(ii) *The optimal reduced spaces can be naturally identified with the symplectic leaves of $M^{[\mu]}$.*

(iii) *The reduced spaces obtained by foliation reduction equal the orbit spaces* $M_{[\mu]} := \mathbf{K}^{-1}([\mu])/N_{[\mu]}$*, where* N *is a normal connected Lie subgroup of* G *whose Lie algebra is the annihilator* $\mathfrak{n} := \left(\mathrm{Lie}\left(\overline{\mathcal{H}}\right)\right)^\circ \subset \mathfrak{g}$ *of* $\mathrm{Lie}\left(\overline{\mathcal{H}}\right) \subset \mathfrak{g}^*$ *in* $\mathfrak{g}$*. The manifolds* $M_{[\mu]}$ *will be referred to as the* **symplectic reduced spaces***.*

(iv) *The quotient Lie group* $H_{[\mu]} := G_{[\mu]}/N_{[\mu]}$ *acts canonically freely and properly on* $M_{[\mu]}$ *and the quotient Poisson manifold* $M_{[\mu]}/H_{[\mu]}$ *is Poisson diffeomorphic to* $M^{[\mu]}$*.*

All these reduced spaces are, in general, distinct. But they are equal if there is a momentum map.

Singular version of this theorem: get cone spaces. Reason: there is a Marle-Guillemin-Sternberg normal form theorem for $\mathbf{K}$.

Question: Is there a convexity result of $\mathbf{K}$ since reduction works so well?

Let's return to the classical convexity theorem.

The convexity theorem is intimately related to reduction. The position of $\mu \in \mathbf{J}(M)$ determines by how much $T_z\mathbf{J} : T_zM \to \mathfrak{g}^*$ fails to be surjective, where $z \in \mathbf{J}^{-1}(\mu)$. The momentum polytope should be regarded as some kind of *a priori* bifurcation diagram that is already imposed on all G-symmetric Hamiltonian systems.

The proof of the convexity theorem was initially done by Morse theory (for the function $\|\mathbf{J}\|^2$.) It turns out that this method does not lead to the most general convexity theorem. There is a convexity theorem that has as corollary the convexity theorem of Poisson-Lie group actions whose proof is done with other methods and no proof of this result is known that can be carried out with Morse theory.

Conditions under which the T or G-momentum polytopes are convex were given by Condevaux, Dazord, and Molino (1988) and later by Hilgert, Neeb, and Plank (1994). These papers show that the proof of the convexity of the image of the momentum map rests on the following result:

Lokal-Global-Prinzip: Let $\Psi : X \to V$ be a locally fiber connected map from a connected locally connected Hausdorff topological space X to a finite dimensional vector space V, with local convexity data $(C_x)_{x\in X}$ such that all convex cones C_x are closed in V. Suppose that Ψ is a proper map. Then $\Psi(X)$ is a closed locally polyhedral convex subset of V, the fibers $\Psi^{-1}(v)$ are all connected, and $\Psi : X \to \Psi(X)$ is an open mapping.

The Lokal-Global-Prinzip not applicable if $\Psi^{-1}(v)$ are either not compact or Ψ is not closed; both conditions are necessary for Ψ to be a proper map. This is one of the difficulties in the (direct) proof of the general convexity theorem leading to the convexity theorem for compact Poisson-Lie group actions on compact symplectic manifolds.

Question: What are the essential features guaranteeing convexity of the image of a map?

Answer: Open onto its image and having local convexity data.

The master statement underlying this is the following:

Let $f : X \to V$ be a continuous map from a connected Hausdorff topological space X to a Banach space V that is open onto its image and has local convexity data. Then the image $f(X)$ is locally convex. If, in addition, $f(X)$ is closed in V then it is convex.

We shall combine this with a generalization of the Lokal-Global-Prinzip to get various convexity theorems.

Explanations

Convexity. Let V be a topological vector space.

A subset $X \subset V$ is **locally convex** if each point $x \in X$ has a neighborhood V_x such that $V_x \cap X$ is convex.

The connection between local convexity and convexity is given by:

Klee (1951): A closed connected and locally convex subset of a topological vector space is convex.

This theorem is very useful when dealing with momentum maps, because local convexity is always known - precise statement later.

The theorem of Klee is itself a generalization of a theorem of Tietze (1928) and Nakajima (1928) which is the case for V finite dimensional. This was used, precisely as we do, by Duistermaat and Pelayo (2006) to completely classify symplectic toral actions for which some (and hence all) principal orbits are coisotropic.

Compactness. In the definition of a **compact set** we *do not assume* that it is Hausdorff. Bourbaki would call this "quasi-compact".

A continuous map $f : X \to Y$, X, Y topological spaces and X Hausdorff, is **proper** if it is closed and $f^{-1}(y)$ compact in X, $\forall y \in Y$.

In the hypotheses above, if $K \subset Y$ is compact then $f^{-1}(K)$ is compact in X. The converse is true if Y is Hausdorff and is the quotient of a locally compact space.

Useful lemma in dealing with convexity properties:

Vaĭnšteĭn: If $f : X \to Y$ is a closed mapping from a metrizable space X onto a metrizable space Y, then for every $y \in Y$ the boundary $\mathrm{bd}(f^{-1}(y)) := \overline{f^{-1}(y)} \cap \overline{\left(X \setminus f^{-1}(y)\right)}$ is compact.

Local Convexity Data. X connected, locally connected, Hausdorff

V topological vector space. A subset $C \subset V$ is a **cone** with **vertex** v_0 if $\forall \lambda \geq 0$, $\forall v \in C$, $v \neq v_0$, we have $(1-\lambda)v_0 + \lambda v \in C$. If C is also convex then it is called a **convex cone**.

V locally convex topological vector space.

$f : X \to V$ has **local convexity data** if $\forall x \in X$ and $\forall U_x$ open neighborhood of x, $\exists C_{x,f(x),U_x}$ convex cone with vertex $f(x)$ in V such that

A $f(U_x) \subset C_{x,f(x),U_x}$ is a neighborhood of the vertex $f(x)$ in the cone $C_{x,f(x),U_x}$

B $f|_{U_x} : U_x \to C_{x,f(x),U_x}$ is an open map and for any neighborhood U'_x of x, $U'_x \subset U_x$, the set $f(U'_x)$is a neighborhood of the vertex $f(x)$ in the convex cone $C_{x,f(x),U_x}$

Here $C_{x,f(x),U_x}$ is endowed with the subspace topology inherited from V.

Remarks.

• If the cones $C_{x,f(x),U_x}$ are closed in $f(X)$ the second condition in **B** is implied by the openness of $f|_{U_x} : U_x \to C_{x,f(x),U_x}$.

• $C_{x,f(x),U_x}$ does not depend on U_x: if $U'_x \subset U_x$ is another neighborhood of x, then $C_{x,f(x),U_x} = C_{x,f(x),U'_x}$. So we write $C_{x,f(x)}$.

• $C_{x,f(x)}$ depends only on the connected components of $f^{-1}(f(x))$: if y is in the same connected component of $f^{-1}(f(x))$ as x, then $C_{x,f(x)} = C_{y,f(y)}$.

Our strategy to prove local convexity for the image of a map that has local convexity data is to prove that it is open onto its image.

$f : X \to V$ continuous map with local convexity data. If f is open onto its image then $f(X)$ is a locally convex subset of V. Moreover, if $f(X)$ is closed in a convex subset of V then it is convex.

This is the basic theorem that will be applied every time when we prove convexity. It will be combined with the Lokal-Global-Prinzip.

- V is allowed to be infinite dimensional.

- Unlike the usual local convexity data condition, it is not assumed that the cones $C_{x,f(x)}$ are closed since this is not a reasonable assumption in infinite dimensions.

- To prove convexity: find necessary and sufficient conditions for a map that has local convexity data to be open onto its image.

- Montaldi and Tokieda (2003) proved that the openness of the momentum map (relative to its image endowed with the subspace topology) implies persistence of extremal relative equilibria under every perturbation of the value of the momentum map, provided the isotropy subgroup of this value is compact. So the openness property of the momentum map onto its image has interesting dynamical consequences.

• Need a generalization of the Lokal-Global-Prinzip that only requires the map to be closed and to have a normal topological space as domain, instead of using the properness condition. This theorem would enable us to

- extend a result of Prato (1994) by requiring the properness of a single component of the momentum map in order to conclude convexity and

- to drop the compactness hypothesis on the manifold in the convexity result leading to the Poisson-Lie convexity theorem.

• Use this theorem to deal with infinite dimensional convexity problems. This is not so easy and remains an open problem. Reason? Lack of a Marle-Guillemin-Sternberg normal form in the infinite dimensional setting which makes the local convexity data property very difficult to check.

Local Fiber Connectedness. $f : X \to Y$ continuous.

$A \subset X$ satisfies the **locally fiber connected condition (LFC)** if A does not intersect two different connected components of the fiber $f^{-1}(f(x))$, for all $x \in A$.

X connected, locally connected, Hausdorff, V locally convex topological vector space, and $f : X \to V$ continuous. f is **locally fiber connected** if $\forall x \in X$, any open neighborhood of x contains an open neighborhood U_x of x that satisfies (LFC).

Let X be a connected, locally connected, Hausdorff topological space, V a locally convex topological vector space, and $f : X \to V$ a continuous map that has local convexity data. Assume that f is a closed map onto its image $f(X)$ and that it has connected fibers. Then f is open onto its image $f(X)$ and $f(X)$ is locally convex. Moreover, if $f(X)$ is closed (e.g. f proper) then it is convex.

The key technical object in proving this is the study of the space X_f whose points are the connected components of the fibers of f.

THE LOKAL-GLOBAL-PRINZIP
FINITE DIMENSIONAL CASE

Let $f : X \to V$ be a closed map with values in a finite dimensional Euclidean vector space V and X a connected, locally connected, first countable, and normal topological space. Assume that f has local convexity data and is locally fiber connected. Then:

(i) All the fibers of f are connected.

(ii) f is open onto its image.

(iii) The image $f(X)$ is a closed convex set.

The theorem remains true if V is replaced by a convex subset $C \subset V$.

INFINITE DIMENSIONAL CASE

In the proof of the finite dimensional theorem, compactness of the balls in V was essential. So one cannot blindly pass to infinite dimensions. Need a second topology in which the balls are compact. So assume that $V = W^*$ for another Banach space W. By Alaoglu's Theorem the balls are weak* compact, which is what we need.

$(V, \|\cdot\|)$ is the Banach space dual W^*

(V, w^*) is V endowed with the weak* topology of W^*.

Since the weak* topology is weaker than the norm topology we have:

- $f : X \to (V, \|\cdot\|)$ continuous $\Longrightarrow f : X \to (V, w^*)$ continuous

- $f : X \to (V, w^*)$ closed $\Longrightarrow f : X \to (V, \|\cdot\|)$ closed

$(V, \|\cdot\|)$ Banach, $V = W^*$, for W a Banach space. $f : X \to (V, \|\cdot\|)$ continuous and $f : X \to (V, w^*)$ closed, where X is a connected, locally connected, and normal topological space. Assume that f has local convexity data and is locally fiber connected. Then:

(i) All the fibers of f are connected.

(ii) $f : X \to (V, w^*)$ is open onto its image.

(iii) The image $f(X) \subset (V, w^*)$ is a closed convex set.

It follows that $f : S \to (V, \|\cdot\|)$ is open onto its image and that $f(X)$ is closed in $(V, \|\cdot\|)$. The proof shows that on $f(X)$ the norm induced and weak* topologies coincide. This seems very strong, but it is automatic for the case when W is reflexive. Then the weak and weak* topologies coincide on V. Mazur's theorem states that the weak and norm closures of a convex set in a normed space coincide.

OPENNESS AND LOCAL CONVEXITY FOR TORAL MOMENTUM MAPS

Let V be finite dimensional vector space.

A subset $K \subset V$ is **polyhedral** if it is the intersection of a finite family of closed halfspaces of V.

Consequently, a polyhedral subset of V is closed and convex.

A subset $K \subset V$ is called **locally polyhedral** if $\forall x \in K$, $\exists P_x$ polytope in V such that $x \in \text{int}(P_x)$ and $K \cap P_x$ is a polytope.

The key result towards convexity is the Marle-Guillemin-Sternberg normal form. The detailed statement below is exactly as it is used in the context above to apply the general theorems on convexity.

(M,ω) symplectic, $\mathbf{J}_T : M \to \mathfrak{t}^*$ invariant momentum map of a T-action. Let $m \in M$ and $T_0 := (T_m)^0$ be the connected component of the stabilizer T_m. Let $T_1 \subset T$ be a subtorus such that $T = T_0 \times T_1$.

(i) $\exists$ (V, ω_V) symplectic v. s., a T-invariant open nbhd $U \subset M$ of $T \cdot m$, and a symplectic covering of a T-invariant open subset U' of $T_1 \times \mathfrak{t}_1^* \times V$ onto U under which the T-action on M is modeled by

$$\begin{array}{rcl} (T_0 \times T_1) \times ((T_1 \times \mathfrak{t}_1^*) \times V) & \to & ((T_1 \times \mathfrak{t}_1^*) \times V) \\ ((t_0, t_1), (t_1', \beta, v)) & \mapsto & (t_1 t_1', \beta, \pi(t_0) v), \end{array}$$

where $\pi : T_0 \to Sp(V)$ is a symplectic representation.

(ii) $\exists$ complex structure I on V such that $\langle v, w \rangle := \omega_V(Iv, w)$ defines a pos. def. scalar product on V. Then $V = \oplus_{\alpha \in \mathcal{P}_V} V_\alpha$, where $V_\alpha := \{v \in V \mid Y \cdot v = \alpha(Y) Iv, \text{for all } Y \in \mathfrak{t}_0\}$ and $\mathcal{P}_V := \{\alpha \in \mathfrak{t}_0^* \mid V_\alpha \neq \{0\}\}$. The corresponding T-momentum map $\Phi : T^*(T_1) \times V \to \mathfrak{t}_1^* \times \mathfrak{t}_0^* \simeq \mathfrak{t}^*$ is given by

$$\Phi\left((t_1, \beta), \sum_{\alpha \in \mathcal{P}_V} v_\alpha\right) = \Phi(1, 0.0) + \left(\beta, \frac{1}{2} \sum_{\alpha \in \mathcal{P}_V} \|v_\alpha\|^2 \alpha\right).$$

The Marle-Guillemin-Sternberg Normal Form provides the twisted product $(T_0 \times T_1) \times_{T_0} (\mathfrak{t}_1^* \times V)$ as a T-invariant local model for M. This is equivariantly diffeomorphic to $T_1 \times \mathfrak{t}_1^* \times V$ via the map

$$\begin{array}{ccc} (T_0 \times T_1) \times_{T_0} (\mathfrak{t}_1^* \times V) & \longrightarrow & T_1 \times \mathfrak{t}_1^* \times V \\ [(t_1, t_0), \eta, v] & \longmapsto & (t_1, \eta, t_0 \cdot v). \end{array}$$

(M, ω) symplectic manifold, $\mathbf{J}_T : M \to \mathfrak{t}^*$ invariant momentum map of a T-action. Then there exist a neighborhood U of m and a convex polyhedral cone $C_{\mathbf{J}(m)} \subset \mathfrak{t}^*$ with vertex $\mathbf{J}_T(m)$ such that:

(i) $\mathbf{J}_T(U) \subset C_{\mathbf{J}_T(m)}$ is an open neighborhood of $\mathbf{J}_T(m)$ in $C_{\mathbf{J}_T(m)}$;

(ii) $\mathbf{J}_T : U \to C_{\mathbf{J}_T(m)}$ is an open map;

(iii) If $\mathfrak{t}_0$ is the Lie algebra of the stabilizer T_m of m, then $C_{\mathbf{J}_T(m)} = \mathbf{J}_T(m) + \mathfrak{t}_0^\perp + \mathrm{cone}(\mathcal{P}_V)$;

(iv) $\mathbf{J}_T^{-1}(\mathbf{J}_T(m)) \cap U$ is connected for all $m \in U$.

So momentum maps of globally Hamiltonian toral actions always have local convexity data with closed cones and are locally fiber connected. Consequences for (M, ω) paracompact connected:

1. Generalization of Atiyah-Guillemin-Sternberg:

Assume $\mathbf{J}_T : M \to \mathfrak{t}^*$ is closed. Then $\mathbf{J}_T(M)$ is a closed convex locally polyhedral subset in $\mathfrak{t}^*$. The fibers of $\mathbf{J}_T$ are connected and $\mathbf{J}_T$ is open onto its image.

2. Generalization of Prato:

(i) If there exists $\xi \in \mathfrak{t}$ such that $\mathbf{J}_T^\xi := \langle \mathbf{J}_T, \xi \rangle \in C^\infty(M)$ is proper, then $\mathbf{J}_T(M)$ is a closed convex locally polyhedral subset in $\mathfrak{t}^*$. Moreover, the fibers of $\mathbf{J}_T$ are connected and $\mathbf{J}_T$ is open onto its image.

(ii) If there exists an integral element $\xi \in \mathfrak{t}$ such that $\mathbf{J}_T^\xi$ is a proper function having a minimum as its unique critical value then $\mathbf{J}_T(M)$ is the convex hull of a finite number of affine rays in $\mathfrak{t}^*$ stemming from the images of T-fixed points.

3. If G acts properly on M then $G \cdot m$ is **regular** if the dimension of nearby obits coincides with the dimension of $G \cdot m$.

Let M^{reg} denote the union of all regular orbits. For every connected component M^0 of M the subset $M^{reg} \cap M^0$ is connected, open, and dense in M^0. Similarly, if N is a T-invariant connected submanifold, then the set of regular points for the T-induced action on N equals $N \cap M^{reg}$. Hence $N \cap M^{reg}$ is open, dense, and connected in N.

A **region** in a topological space is a connected open set.

What topological conditions are needed that ensure that $\mathbf{J}_T$ is open onto its image?

Let $\mathbf{J}_T : M \to \mathfrak{t}^*$ be the momentum map of a torus action which has connected fibers. Then $\mathbf{J}_T$ is open onto its image if and only if $\mathbf{C}J_T(M^{reg})$ does not disconnect any region in $\mathbf{J}_T(M)$. Moreover, the image of the momentum map is locally convex and locally polyhedral.

Example, Prato (1994): $M := \mathbb{C}^2 \setminus (D^1 \times D^1)$, where D^1 is the closed unit disc in $\mathbb{C}$.

M is an open symplectic submanifold of $\mathbb{C}^2$ and T^2 acts on it by $(e^{i\theta_1}, e^{i\theta_2}) \cdot (z_1, z_2) := (e^{i\theta_1} z_1, e^{i\theta_2} z_2)$ with invariant momentum map $\mathbf{J}_{T^2}(z_1, z_2) = (|z_1|^2, |z_2|^2)/2$. Its fibers are connected.

Denote by $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x \geq 0\}$. Then

$$\mathbf{J}_{T^2}(M) = \mathbb{R}^2_+ \setminus \{(x, y) \mid x \leq 1/2 \text{ and } y \leq 1/2\}$$

which is *not* connected. But

$$\mathbf{C}(\mathbf{J}_{T^2}(M^{reg})) = \{(x, 0) \mid x > 1/2\} \cup \{(0, y) \mid y > 1/2\}$$

which does not disconnect any region in $\mathbf{J}_{T^2}(M)$. Consequently, according to the previous theorem, this momentum map is open onto its image and has a locally convex image, in agreement with the basic theorem. This can be seen directly looking at the image.

Example, Karshon and Lerman (1997): $M_1 := T^2 \times U$ where U is the subset of $\mathbb{R}^2$ obtained by removing the origin and the positive x-axis. M_1 is an open symplectic submanifold of $T^*(T^2)$. The restriction to M_1 of the lifted action of T^2 on its cotangent bundle has as momentum map the projection onto U. Hence, the image of this momentum map is $\mathbb{R}^2$ minus the origin and the positive x-axis.

Let M_2 be the symplectic manifold $\mathbb{C}^2$ minus the points whose first coordinate is nonzero. The momentum map for the T^2 action on M_2 is given by $(z, w) \mapsto (|z|^2, |w|^2)/2$ and the image is the set $\{(x, y) \in \mathbb{R}^2 \mid x > 0, y \geq 0\}$.

Gluing M_1 and M_2 along the pre-images of the positive quadrant we obtain another globally Hamiltonian T^2-space M with a momentum map $\mathbf{J}_{T^2}$ with connected fibers whose image is $\mathbb{R}^2$ minus the origin. $\mathbf{C}(\mathbf{J}_{T^2}(M^{reg}))$ is the positive x-axis which disconnects regions in $\mathbb{R}^2$ minus the origin. The previous theorem implies that this momentum map is not open onto its image.

4. What happens if the fibers of $\mathbf{J}_T$ are disconnected? There are reasonable topological conditions that insure openness onto its image, but for that we need a theorem in point set topology.

Definitions: A metric space is called a **generalized continuum** if it is locally compact and connected. In a topological space a **quasi-component** of a point is the intersection of all closed-and-open sets that contain that point. A topological space is called **totally disconnected** if the quasi-component of any point consists of the point itself. A continuous map $f : X \to Y$ is called **light** if all fibers $f^{-1}(y)$ are totally disconnected. A subset of a topological space is **non-dense** if it contains no open subsets.

Extension of Openness, Whyburn (1964): Let X and Y be locally connected generalized continua and let $f : X \to Y$ be an onto light mapping which is open on $X \setminus f^{-1}(F)$, where F is a closed non-dense set in Y which separates no region in Y and is such that $f^{-1}(F)$ is non-dense. Then f is open on X.

If the fibers of $\mathbf{J}_T$ are disconnected we still need a control on the connected components of the fibers of $\mathbf{J}_T$.

$\mathbf{J}_T : M \to \mathfrak{t}^*$ momentum map of a torus action on a connected symplectic manifold (M, ω). $\mathbf{J}_T$ satisfies the **connected component fiber condition (CCF)** if $\mathbf{J}_T(x) = \mathbf{J}_T(y)$ and $E_x \cap M^{reg} \neq \varnothing$, implies that $E_y \cap M^{reg} \neq \varnothing$, where E_x and E_y are the connected components of the fiber $\mathbf{J}_T^{-1}(\mathbf{J}_T(x))$ that contain x and y respectively.

$M_{\mathbf{J}_T}$ is the quotient topological space whose points are the connected components of the fibers of $\mathbf{J}_T$.

Suppose that $M_{\mathbf{J}_T}$ is a Hausdorff space. Then $\mathbf{J}_T$ is open onto its image if and only if $\mathbf{J}_T(M)$ is locally compact, $\mathbf{C}\mathbf{J}_T(M^{reg})$ does not disconnect any region in $\mathbf{J}_T(M)$, and $\mathbf{J}_T$ satisfies (CCF). Moreover, under these hypotheses, the image of the momentum map is locally convex and locally polyhedral.

OPENNESS AND LOCAL CONVEXITY FOR GENERAL MOMENTUM MAPS

$\mathbf{J}_G : M \to \mathfrak{g}^*$ equivariant momentum map of a canonical G-action. In general, even with G compact and $\mathbf{J}_G$ proper, it does not follow that $\mathbf{J}_G$ is open onto its image. So how does one recover the Guillemin-Kirwan-Sternberg convexity theorem?

Let $\pi_G : \mathfrak{g}^* \to \mathfrak{g}^*/G \equiv \mathfrak{t}^*_+$ be the projection map which is always proper if G is compact. Define $\mathbf{j}_G := \pi_G \circ \mathbf{J}_G : M \to \mathfrak{t}^*_+$. The idea is to use the general convexity theorems for $\mathbf{j}_G$. But for that we need local convexity data. This is ensured by:

Sjamaar (1998): M connected Hamiltonian G-manifold. Then $\forall x \in M$, $\exists!$ $C_x \subset \mathfrak{t}^*_+$ closed polyhedral convex cone with vertex at $\mathbf{j}_G(x)$ such that for every sufficiently small G-invariant neighborhood U of x the set $\mathbf{j}_G(U)$ is an open neighborhood of $\mathbf{j}_G(x)$ in C_x.

What about local fiber connectedness of $\mathbf{j}_G$? Using Lerman's symplectic cut technique,

Knop (2002): $\mathbf{j}_G$ is locally fiber connected.

Conclusion: $\mathbf{j}_G$ is locally fiber connected and has local convexity data.

$\widetilde{\mathbf{j}}_G := \pi \circ \mathbf{j}_G : M/G \to \mathfrak{t}^*_+$, where $\pi : M \to M/G$ is the projection.

The G-equivariant momentum map $\mathbf{J}_G : M \to \mathfrak{g}^*$ is G-**open** onto its image whenever $\widetilde{\mathbf{j}}_G$ is open onto its image.

Apply the finite dimensional Lokal-Global-Prinzip to the map $\mathbf{j}_G$ to obtain a generalization of the Guillemin-Kriwan-Sternberg convexity theorem.

Let M paracompact connected Hamiltonian G-manifold, G compact connected. If $\mathbf{J}_G$ is closed then $\mathbf{J}_G(M)\cap\mathfrak{t}^*_+$ is a closed convex locally polyhedral set. Moreover, $\mathbf{J}_G$ is G-open onto its image and all its fibers are connected.

So $\mathbf{J}_G$ is G-open, but is not open in general. Montaldi-Tokieda have given countexamples. Recall $\mathbf{j}_G := \pi_G \circ \mathbf{J}_G : M \to \mathfrak{t}^*_+$.

Suppose that $\mathbf{J}_G$ has connected fibers. Then $\mathbf{J}_G$ is G-open onto its image if and only if $\mathbf{C}((\pi_G \circ \mathbf{J}_G)(M^{reg}))$ does not disconnect any region in $\mathbf{J}_G(M) \cap \mathfrak{t}^*_+$. Moreover, in this context, $\mathbf{J}_G(M) \cap \mathfrak{t}^*_+$ is a locally convex and locally polyhedral set.

Suppose that $(M/G)_{\mathbf{j}_G}$ is Hausdorff. Then $\mathbf{J}_G$ is G-open onto its image if and only if $\mathbf{J}_G(M)$ is locally compact, $\mathbf{C}((\pi_G \circ \mathbf{J}_G)(M^{reg}))$ does not disconnect any region in $\mathbf{J}_G(M) \cap \mathfrak{t}^*_+$, and $\mathbf{j}_G$ satisfies (CCF). Moreover, in this context, the image $\mathbf{J}_G(M)\cap\mathfrak{t}^*_+$ is a locally convex and locally polyhedral set.

CONVEXITY FOR POISSON ACTIONS

Poisson Lie Groups. A Poisson manifold (H, π_H) is said to be a **Poisson Lie group** if H is a Lie group and the multiplication $(h_1, h_2) \mapsto h_1 h_2$ is a Poisson map from $H \times H$, equipped with the product Poisson structure, to H.

π_H vanishes at e. One can then define the *intrinsic derivative* $\epsilon : \mathfrak{h} \to \mathfrak{h} \wedge \mathfrak{h}$ by $\epsilon(\xi) = (\mathcal{L}_V \pi_H)(e)$, where V is any vector field on H with $V(e) = \xi \in \mathfrak{h}$. The dual map $\epsilon^* : \mathfrak{h}^* \wedge \mathfrak{h}^* \longrightarrow \mathfrak{h}^*$ satisfies the Jacobi identity, so $\mathfrak{h}^*$ is a Lie algebra as well. The corresponding connected and simply connected Lie group H^d is called the *dual group of* H. It has a unique Poisson structure π_{H^d} making it into a Poisson Lie group such that the intrinsic derivative of π_{H^d} is the Lie bracket on $\mathfrak{h}$. ϵ is a cocycle and it determines uniquely the Poisson Lie structures on H and H^d.

Lu-Weinstein Structure. K compact connected semisimple Lie group. Since any compact Lie group K is the commuting product $(Z_K)_0 K_{ss}$ of the connected component of the identity $(Z_K)_0$ of the center Z_K and of a closed semisimple subgroup K_{ss}, one can work only on the semisimple part, because on the toral part the analysis has been already done (the discrete part does not matter for the momentum map).

Since K admits a complexification, think of it as the compact real form of a connected complex semisimple Lie group G. Denote by $G^{\mathbb{R}}$ the real Lie group underlying G and let $G^{\mathbb{R}} = KAN$ be its Iwasawa decomposition. Denote by $\mathfrak{g}^{\mathbb{R}}, \mathfrak{k}, \mathfrak{a}$, and $\mathfrak{n}$ the real Lie algebras of $G^{\mathbb{R}}, K, A$, and N, respectively. Then $\mathfrak{g}^{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is the Iwasawa decomposition of $\mathfrak{g}^{\mathbb{R}}$. If $\mathfrak{t} = i\mathfrak{a}$ then $T = \exp \mathfrak{t}$ is a maximal torus of K. Define $B := AN$ whose Lie algebra is $\mathfrak{b} := \mathfrak{a} \oplus \mathfrak{n}$.

Let κ be the Killing form of $\mathfrak{g}$. Its imaginary part $\operatorname{Im}\kappa$ is a non-degenerate invariant symmetric bilinear form on $\mathfrak{g}^{\mathbb{R}}$. Since $\operatorname{Im}\kappa(\mathfrak{k},\mathfrak{k}) = \operatorname{Im}(\mathfrak{b},\mathfrak{b}) = 0$, the vector spaces $\mathfrak{k}$ and $\mathfrak{b}$ are dual to each other relative to $\langle\,,\rangle := \operatorname{Im}\kappa$. The Cartan decomposition $G^{\mathbb{R}} = PK$ defines the Cartan involution $\tau : G^{\mathbb{R}} \to G^{\mathbb{R}}$. Define $g^* := \tau(g^{-1})$ for any $g \in G$. The derivative of these maps at the identity will be denoted by the same symbols. The map $\tau : \mathfrak{g}^{\mathbb{R}} \to \mathfrak{g}^{\mathbb{R}}$ has eigenvalues ± 1. The $+1$ eigenspace is $\mathfrak{k}$ and the -1 is denoted by $\mathfrak{p}$. Note that $\langle\,,\rangle$ identifies $\mathfrak{k}^*$ with $\mathfrak{p}$. The exponential map is a diffeomorphism from $\mathfrak{p}$ to P. Let $\mathfrak{a}_+$ be the positive Weyl chamber in $\mathfrak{a} \cong \mathfrak{t}^*$ corresponding to the subgroup B.

Let $\rho_{\mathfrak{k}} : \mathfrak{g} \to \mathfrak{k}$, $\rho_{\mathfrak{b}} : \mathfrak{g} \to \mathfrak{b}$ be the projections associated to $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{b}$.

The Poisson-Lie structures on K and B are defined at the identity and then right extended to the whole group.

The bivector fields π_K and π_B given by

$$\pi_K(k)(T_k^* R_{k^{-1}}\eta_1, T_k^* R_{k^{-1}}\eta_2) = -\left\langle \rho_{\mathfrak{k}}(\mathrm{Ad}_{k^{-1}}\,\eta_1), \rho_{\mathfrak{b}}(\mathrm{Ad}_{k^{-1}}\,\eta_2)\right\rangle$$

for $\eta_1, \eta_2 \in \mathfrak{b} \cong \mathfrak{k}^*$, and

$$\pi_B(b)(T_b^* R_{b^{-1}}\xi_1, T_b^* R_{b^{-1}}\xi_2) = \left\langle \rho_{\mathfrak{k}}(\mathrm{Ad}_{b^{-1}}\,\xi_1), \rho_{\mathfrak{b}}(\mathrm{Ad}_{b^{-1}}\,\xi_2)\right\rangle$$

for $\xi_1, \xi_2 \in \mathfrak{k} \cong \mathfrak{b}^*$, make K and B into dual Poisson Lie groups.

The Poisson tensor π_K vanishes on T.

Classification. As is explained above, the Poisson tensor on K is determined by a cocycle $\epsilon : \mathfrak{k} \to \mathfrak{k} \wedge \mathfrak{k}$. Let ϵ_0 be the cocycle defining the Lu-Weinstein Poisson tensor. We consistently identify $\mathfrak{k}^*$ with $\mathfrak{p}$ via the pairing $\mathrm{Im}\,\kappa$. Let $\mathfrak{a}^\perp \cap \mathfrak{p}$ be the orthogonal complement, with respect to the Killing form κ (*not* $\mathrm{Im}\,\kappa$), of $\mathfrak{a}$ in $\mathfrak{p}$.

Levendorskiĭ-Soibelman (1991): Up to Poisson isomorphism, the Poisson-Lie structures on a simple compact Lie group K are given by

$$\epsilon = a\epsilon_0 + u, \quad a \in \mathbb{R},\ u \in \mathfrak{t} \wedge \mathfrak{t}.$$

Here we think of u as a constant map, sending $\mathfrak{k}$ to an element of $\mathfrak{k} \wedge \mathfrak{k}$ by extending it to be zero on $i(\mathfrak{a}^{\perp} \cap \mathfrak{p})$. These Poisson-Lie structures are all non-isomorphic for distinct a and u.

Remark. When K is a product of simple factors, one would have different a_j and u_j for each simple component.

Remark. When K acts symplectically, there is no preferred maximal torus. Nontrivial Poisson-Lie tensors, however, cannot even be defined until a maximal torus and a positive Weyl chamber (=positive roots of $\mathfrak{g}$) are chosen. A nonzero u in the theorem evidently requires a choice of maximal torus. A nonzero a does also: the proposition defining the Lu-Weinstein structure involves a distinguished KAN factorization.

To compute these Poisson-Lie structures, one uses Manin triples and one can determine them explicitly, including the dual groups.

For the convexity theorem, all that matters is whether a and u vsanish or not. Thus, there are just four cases of interest:

(1) $a = 0, u = 0$. This gives the Lie-Poisson structure on $K^d = \mathfrak{k}^*$.
(2) $a = 1, u = 0$. This is the Lu-Weinstein structure.
(3) $a = 1, u \neq 0$. This is a perturbation of (2).
(4) $a = 0, u \neq 0$. This is a perturbation of (1).

For each case there is diffeomorphism $\psi : K^d \to \mathfrak{p}$. For (1) this is just the usual identification of $K^d = \mathfrak{k}^* \cong \mathfrak{p}$. For (2) it is the map sym : $K^d = B \to \mathfrak{p}$ given by $b \mapsto \log(b^* b)$. There are similar explicit formulas for cases (3) and (4).

Poisson-Lie Momentum Maps. The Poisson-Lie group H acts on a Poisson manifold (P, π_P) in a Poisson fashion. Identify $\mathfrak{h} \cong \left(\mathfrak{k}^d\right)^* = T_e^* K^d$. So $\forall \xi \in \mathfrak{h}$ defines a left-invariant one-form ξ^ℓ on K^d.

Lu (1990): $J : P \to K^d$ is a **momentum mapping** for the action of K if

$$\xi_P = \pi_P(\cdot, J^*(\xi^\ell)).$$

Poisson-Lie Convexity Theorem.

Flaschka-Ratiu (1995): Let K be a compact connected semisimple Lie group, equipped with a Poisson-Lie structure. Let M be a compact connected symplectic manifold, and suppose that there is a Poisson action of K on M, with equivariant momentum map $J : M \to K^d$. Define $j := \psi \circ J : M \to \mathfrak{p}$. Then $j(M) \cap \mathfrak{a}_+$ is a compact convex polytope.

There are two proofs of this theorem.

• Direct original proof is based on a general convexity theorem all of whose hypotheses hold for actions of compact Lie groups, with any of the possible Poisson-Lie structures, on compact symplectic manifolds.

• A proof by Alekseev that reduces the theorem to the classical Guillemin-Kirwan-Sternberg theorem by modifying the symplectic structure in terms of the Poisson-Lie structure on K such that the resulting action becomes symplectic.

Open question: Is the same result true if one drops compactness of M and replaces it with properness of J?

The existing proofs use in an essential way compactness of M and no method is known how to replace this by properness of J. Using the general convexity theorems, it is now possible to modify the general convexity theorem whose corollary is the Poisson-Lie convexity theorem.

The compact connected Lie group K acts on a paracompact connected symplectic manifold (M, ω). A maximal torus T of K acts on (M, ω) with invariant momentum map $\mathbf{J}_T : M \to \mathfrak{t}^*$. Suppose there exists a *closed* map $\mathcal{P} : M \to \mathfrak{p}$ with the following properties:

(i) $\mathcal{P}$ is equivariant with respect to the adjoint action of K on $\mathfrak{p}$;

(ii) $\forall x \in M$, $T_x\mathcal{P}(T_xM) = \mathfrak{k}_x^{\mathrm{ann}} := \{\mu \in \mathfrak{p} \mid \langle \mu, \xi \rangle = 0, \forall \xi \in \mathfrak{k}\}$;

(iii) $\forall x \in M$, the kernel of $T_x\mathcal{P}$ equals

$$(\mathfrak{k} \cdot x)^{\omega} := \{v \in T_xM \mid \omega(x)(v, \xi_M(x)) = 0, \forall \xi \in \mathfrak{k}\};$$

(iv) the restriction of $\mathcal{P}$ to $\mathcal{P}^{-1}(\mathfrak{a}_+)$ is proportional to $\mathbf{J}_T$.

Then $\mathcal{P}(M) \cap \mathfrak{a}_+$ is a closed convex set. If M is compact, the set $\mathcal{P}(M) \cap \mathfrak{a}_+$ is a compact convex polytope.

METRIC CONVEXITY

Question: In view of the fact that so far everything was point set topological, how much more can one generalize the convexity theorems? Can one replace the target space with more general spaces than locally convex topological vector spaces? Are such generalizations interesting?

Anwer: The target can be taken to be a "path metric space", "length space", "inner space" – terminology differs from author to author – with some good properties. There are interesting examples already in symplectic geometry. For symplectic actions that do not admit momentum maps, there is always the cylinder valued momentum map. Is there a convexity theorem for it? Strong indications are that yes, since there is an analogue of the Marle-Guillemin-Sternberg normal form (Ortega-Ratiu) which then gives local convexity data in the metric sense. But one needs some very general convexity theorem to prove such a result. This is possible and will be shown below.

In the definition of the metric space $+\infty$ is an allowed value.

X connected normal first countable topological space and (Y, d) complete locally compact geodesic metric space. Assume that $f : X \to Y$ is a continuous closed map, locally open onto its image, locally fiber connected, and has local convexity data. Then:

(i) $f(X) \subset (Y, d)$ is a weakly convex subset of Y.

(ii) If, in addition, (Y, d) is uniquely geodesic (that is, any two points can be joined by a unique geodesic) and $d(y_1, y_2) < \infty$ for all $y_1, y_2 \in Y$, then $f(X)$ is a convex subset of (Y, d), f has connected fibers, and it is open onto its image.

All terms need to be explained: geodesic metric space, map locally open onto its image, local convexity data (because we defined it so far only for maps with values in locally convex vector spaces), convex and weakly convex set in a geodesic metric space.

A continuous map $f : X \to Y$ is said to be **locally open onto its image** if for any $x \in X$ there exists an open neighborhood U_x of x such that the restriction $f|_{U_x} : U_x \to f(U_x)$ is an open map, where $f(U_x)$ has the topology induced by Y.

Let $f : X \to Y$ be a continuous map between two connected Hausdorff topological spaces. Let X_f be the topological quotient space whose points are the connected components of the fibers of f. $\pi_f : X \to X_f$ is the projection and $\widetilde{f} : X_f \to Y$ is uniquely characterized by $\widetilde{f} \circ \pi_f = f$. $\widetilde{f}$ is continuous and if the fibers of f are connected then it is also injective.

Benoist (1998): Suppose $f : X \to Y$ is a continuous map between two topological spaces. If f is locally fiber connected and locally open onto its image then π_f is an open map.

X normal first countable topological space and Y Hausdorff. $f : X \to Y$ continuous, locally open onto its image, and locally fiber connected. If f is a closed map, then

(i) the projection $\pi_f : X \to X_f$ is also a closed map,

(ii) the quotient X_f is a Hausdorff topological space.

Length Spaces. (X, d) metric space.

A **curve** or a **path** in X is a continuous map $c : I \to X$ with I a connected interval of $\mathbb{R}$.

The **length** $l_d(c)$ of a curve $c : [a, b] \to X$ induced by the metric d is

$$l_d(c) := \sup_{\Delta_n} \sum_{i=0}^{n-1} d(c(t_i), c(t_{i+1})),$$

where the supremum is taken over all possible partitions $\Delta_n : a = t_0 \leq t_1 \leq \cdots \leq t_n = b$ of the interval $[a, b] \subset \mathbb{R}$. So $l_d(c) \geq 0$ or $+\infty$. The curve c is said to be **rectifiable** if its length is finite.

Properties of the length function

1. $l_d(c) \geq d(c(a), d(c(b)))$, for any path $c : [a, b] \to X$.

2. If $\phi : [a', b'] \to [a, b]$ is an onto weakly monotonic map, then $l_d(c) = l_d(c \circ \phi)$.

3. Additivity: if c is the concatenation of two paths c_1 and c_2 then $l_d(c) = l_d(c_1) + l_d(c_2)$.

4. If c is rectifiable of length l, then the function $\lambda : [a, b] \to [0, l]$ defined by $\lambda(t) = l_d(c_{[a,t]})$ is a continuous weakly monotonic function.

5. Reparametrization by arc length: if c and λ are as in as in the previous point, then there is a unique path $\widetilde{c} : [0, l] \to X$ such that

$$\widetilde{c} \circ \lambda = c \quad \text{and} \quad l_d(\widetilde{c}_{[0,t]}) = t.$$

6. Lower semicontinuity: let (c_n) be a sequence of paths $[a, b] \to X$ converging uniformly to a path c. If c rectifiable, then for every $\varepsilon > 0$, exists an integer N_ε such that

$$l_d(c) \leq l_d(c_n) + \varepsilon$$

whenever $n > N_\varepsilon$.

The distance d is said to be a **path metric**, **length metric**, or an **inner metric** if the distance between every pair of points $x, y \in X$ is equal to the infimum of the length of rectifiable curves joining them. If there are no such curves then, by defintion, $d(x, y) = \infty$. If d is a length metric then (X, d) is called a **path metric space**, a **length space**, or an **inner space**.

Given (X, d) there always is a length metric $\overline{d}$ induced by d:

$$\overline{d}(x, y) := \inf_{R_{x,y}} l_d(\gamma),$$

where $R_{x,y} :=$ {all rectifiable curves connecting x and y}. If there are no such curves then define $\overline{d}(x, y) = +\infty$.

Properties of $\overline{d}$: (X, d) metric space. (Bridson and Haefliger)

1. $\overline{d}$ is a metric.

2. $\overline{d}(x, y) \geq d(x, y)$ for all $x, y \in X$.

3. If $c : [a, b] \to X$ is continuous with respect to topology induced by $\overline{d}$, then it is continuous with respect to the topology induced by d. The converse is false, in general. The topology induced by the metric d is coarser than the topology induced by the metric $\overline{d}$.

4. If a map $c : [a, b] \to X$ is a rectifiable curve in (X, d), then it is a continuous and rectifiable curve in $(X, \overline{d})$.

5. The length of a curve $c : [a, b] \to X$ in $(X, \overline{d})$ is the same as its length in (X, d).

6. $\overline{\overline{d}} = \overline{d}$.

7. (X, d) is a length space if and only if $\overline{d} = d$.

Classical examples are Riemannian manifolds (M, g). If $c : [a, b] \to M$ is a piecewise differentiable path, the Riemannian length is

$$l_g(c) := \int_a^b \sqrt{g_{ij}(t)\dot{c}^i(t)\dot{c}^j(t)}.$$

Let (M, g) be a connected Riemannian manifold. Given $x, y \in M$, let $d(x, y)$ be the infimum of the Riemannian lengths of piecewise continuously differentiable paths $c : [0, 1] \to M$ such that $c(0) = x$ and $c(1) = y$. d is called the **length metric**. Then:

1. d is a metric on X.

2. The topology on X defined by this distance is the same as the given manifold topology on X.

3. (X, d) is a length space.

Geodesic Metric Spaces.

A curve $c : [a,b] \to (X,d)$ is called a **shortest path** if its length is minimal among the curves with the same endpoints. Shortest paths in length spaces are also called **distance minimizers**.

(X,d) length space. A curve $c : I \subset \mathbb{R} \to X$ is called **geodesic** if $\forall t \in I$, $\exists J$ subinterval containing a neighborhood of t in I such that $c|_J$ is a shortest path. In other words, a geodesic is a curve which is locally a distance minimizer. A length space (X,d) is called a **geodesic metric space** if for any two points $x, y \in X$ there exist a shortest path between x and y.

Clearly in a length space a shortest path is a geodesic. The extension of the Hopf-Rinow theorem from Riemannian geometry to the case of length metric spaces was done by Cohn-Vossen.

Hopf-Rinow-Cohn-Vossen: For a locally compact length space (X, d), the following assertions are equivalent:

(i) X is complete,

(ii) every closed metric ball in X is compact.

If one of the above assertions is true then for any two points $x, y \in X$ there exist a shortest path connecting them. In other words, (X, d) is a geodesic metric space.

If we endow the Riemannian manifold with its length metric we obtain:

Every complete, connected, Riemannian manifold is a geodesic metric space.

Metric Convexity.

A subset C in a metric space (X, d) is said to be **convex** if the restriction of d to C is a finite length metric.

(X, d) geodesic metric space. Then a subset $C \in X$ is convex if and only if for any two points $x, y \in C$ there exists a rectifiable shortest path γ connecting x and y which is entirely contained in C.

(X, d) geodesic metric space. A subset $C \in X$ is **weakly convex** if for any two points $x, y \in C$ there exists a geodesic connecting x and y which is entirely contained in C.

Note that weak convexity does not require that the geodesic be the shortest one.

Maps with Local Convexity Data.

Let X be a connected Hausdorff space and (Y, d) a length space. A continuous mapping $f : X \to Y$ is said to have **local convexity data** if for each $x \in X$ and every sufficiently small neighborhood U_x of x the set $f(U_x)$ is a convex subset of Y.

The main technical tool in the proof of the convexity theorem is the space X_f.

On X_f we define the function $\tilde{d} : X_f \times X_f \to [0, \infty]$ in the following way: for $[x]$ and $[y]$ in X_f let $\tilde{d}([x], [y])$ be the infimum of all the lengths $l_d(\tilde{f} \circ \gamma)$ where γ is a continuous curve in X_f that connects $[x]$ and $[y]$. Recall $\tilde{f} : X_f \to Y$ is the quotient map of $f : X \to Y$. The length l_d is computed with respect to the distance d on Y. From the definition it follows that $d(\tilde{f}([x]), \tilde{f}([y])) \leq \tilde{d}([x], [y])$.

Standing hypotheses: X connected, normal, first countable topological space. $f : X \to Y$ continuous, closed map, locally open onto its image, and locally fiber connected.

• (Y, d) metric space. Then $\tilde{d} : X_f \times X_f \to [0, \infty]$ is a metric on X_f.

• If, in addition, f has local convexity data, then $(X_f, \tilde{d})$ is a length space and the topology induced by $\tilde{d}$ coincides with the quotient topology of X_f.

• If, in addition, (Y, d) is a complete, locally compact, length space (a geodesic metric space), then $(X_f, \tilde{d})$ is a complete, locally compact length space (a geodesic metric space).

To prove this one uses Vaĭnšteĭn's theorem.