*Deformation quantization via graph complexes*

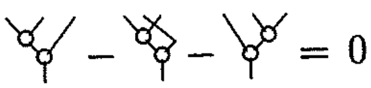
Sergei Merkulov

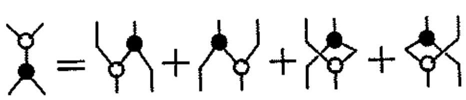
Biological structures are resolutions of their genom.

Some local geometric structures are resolutions of surprisingly small graphical data called their *genom.*

Poisson structure is one of these, and its genom is given by two graded "genes"

with the following engineering rules:





Knowledge of genom can be useful.

For example, one can give rather short proofs of the following two deformation quantization theorems:

THEOREM (Kontsevich 1997)

- Any formal germ, $\nu$, of Poisson structure in $V = \mathbb{R}^n$ can be deformation quantized, i.e. there exists an associated star product, $\star_\hbar$, on $\widehat{\odot} V$.

- If $\nu$ is a linear Poisson structure (i.e. $(V^*, \nu)$ is a Lie algebra), then $\star_\hbar$ is equivalent to the associated universal enveloping algebra.

THEOREM (Me 2005)

- Any strongly homotopy Lie bialgebra structure, $\nu$, in a finite dimensional graded vector space $V$ can be deformation quantized, i.e. there exists an associated strongly homotopy associative bialgebra structure, $\Phi_\hbar$, on $\widehat{\odot} V$.

- If $\nu$ is a Lie bialgebra structure, then $\Phi_\hbar$ is equivalent to the Etingof-Kazhdan quantization of $\nu$.

$$\Phi_\hbar \Leftarrow \begin{cases} \text{Drinfeld associator} \\ \text{diagonal in associahedra} \\ \text{diagonal in permutahedra} \end{cases}$$

Let $V = \bigoplus_{i \in \mathbb{Z}} V^i$ be a graded vector space.

Let $\mathcal{O}_V := \hat{\odot} V$ be the algebra of smooth functions on $V$ viewed as a formal manifold,

$\wedge^\bullet \mathcal{T}_V$ the Lie algebra (with Schouten br.) of formal polyvector fields.

<u>Fact</u> $\mathrm{Hom}(\mathcal{O}_V^{\otimes \bullet}, \mathcal{O}_V)$ has a structure of dg Lie algebra (Hochschild) whose MC elements are in 1-1 correspondence with $A_\infty$ structures on $\mathcal{O}_V$.

<u>Formality Theorem</u> (Kontsevich 1997):

For any finite dimensional $V$

$$\mathrm{Hom}(\mathcal{O}_V^{\otimes \bullet}, \mathcal{O}_V) \quad \text{and} \quad \wedge^\bullet \mathcal{T}_V$$

are quasi-isomorphic dg Lie algebras.

$\mathcal{O}_V = \hat{\odot} V$ is naturally a bialgebra.

**Fact.** $\mathrm{Hom}(\mathcal{O}_V^{\otimes \bullet}, \mathcal{O}_V^{\otimes \bullet})$ has a structure of $L_\infty$ algebra whose MC elements are in 1-1 correspondence with $BiAl_\infty$ structures on $\mathcal{O}_V$

**Observation:** $M := V[1] \times V^*[1]$ viewed as a formal even dimensional manifold has a natural degree 2 symplectic form $\omega$.

Denote by $\{\ ,\ \}$ the associated Poisson bracket (of degree $-2$) on the algebra, $\mathcal{O}_M$, of smooth formal functions on $M$.

"Formality" Theorem (Me 2006):

For any finite dimensional graded vector space $V$

$$\mathrm{Hom}(\mathcal{O}_V^{\otimes \bullet}, \mathcal{O}_V^{\otimes \bullet}) \quad \text{and} \quad (\mathcal{O}_M, \{\ ,\ \})$$

are quasi-isomorphic $L_\infty$ algebras.

**Problem:** Find QFT a la Cattaneo-Felder for this "formality".

Theorem (Me 2005):

For any formal Poisson structure in $\mathbb{R}^n$ there exists deformation quantization, $*_\hbar$, over $\mathbb{Q}$, i.e.

$\forall$ smooth functions $f, g$

$$f *_\hbar g \in \mathbb{Q}[[\partial_{I_1}\nu, \partial_{I_2} f, \partial_{I_3} g, \hbar]]$$

$\nu$ is Poisson structure.

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Fact: $\forall$ smooth manifold $M$,

$\bigoplus_{i\geq 1} \wedge^i T_M$ is a Lie subalgebra of $\underbrace{\bigoplus_{i\geq 0} \wedge^i T_M}$

1) to cover $i=0$ bit one has to introduce a few other genes to our picture

2) in the case of EK quantization these EXTRA genes correspond to <u>COBOUNDARY</u> Lie Bialgebras.

Graph coding: an example.

DEFINITION: An *associative algebra* is the data

(1) graded vector space $V$,

(2) linear map $\circ : V \otimes V \to V$ of degree 0

(3) associativity condition:

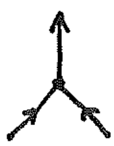
$$(a \circ b) \circ c = a \circ (b \circ c) \quad \forall a, b, c \in V.$$

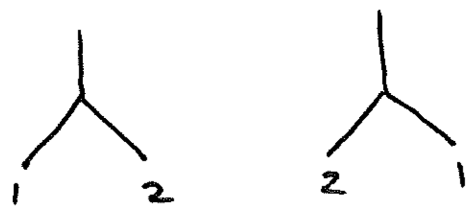
We shall create a GRAPH COMPLEX out of species "ASSOCIATIVE ALGEBRAS" in three steps:

STEP 1:

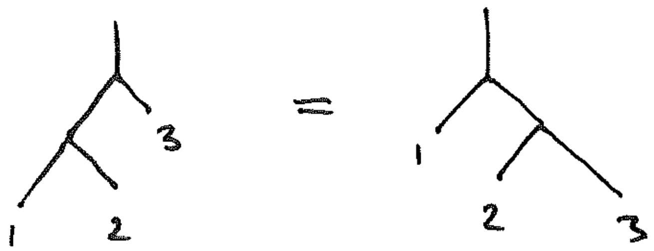
(1) forget the graded vector space $V$,

(2) $\circ : V \otimes V \to V$ gets replaced by its calculational meaning: a "machine" with two inputs and one output represented as a degree 0 graph





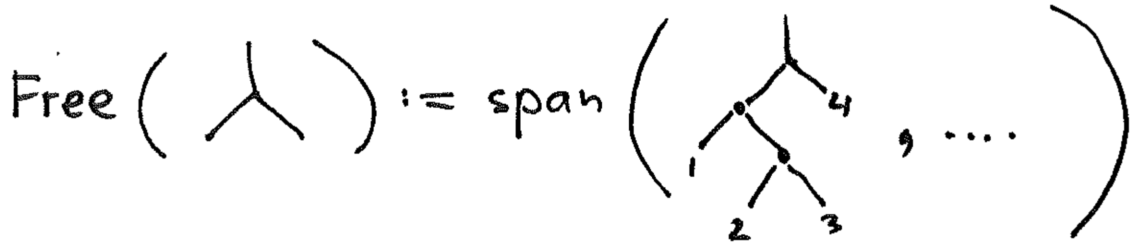

(3) associativity condition:



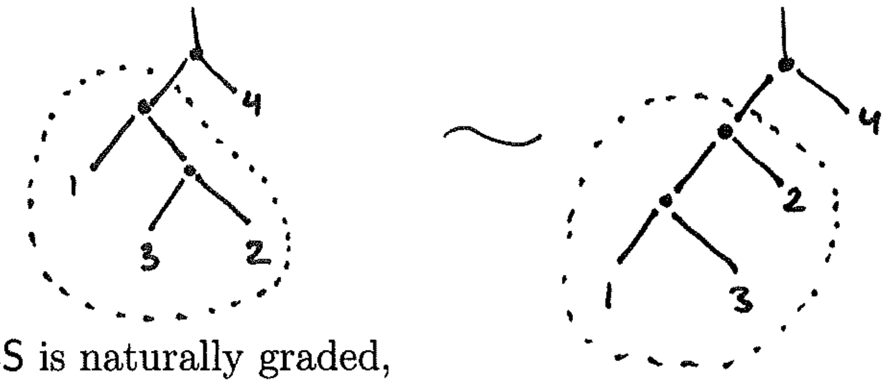

+ $\mathbb{S}_3$-permutations

STEP 2:

- Create the vector space of all possible calculational schemes,



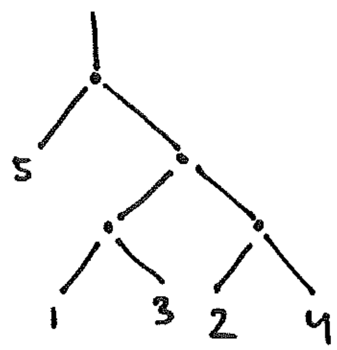

- Take the quotient of this vector space by the subspace of graphs related to each other via "associativity condition",




ASS is naturally graded,

$$\mathsf{ASS} = \bigoplus_{n\geq 2} \mathsf{ASS}(n) \quad := \frac{\mathrm{Free}(\curlywedge)}{\sim},$$

where $\mathsf{ASS}(n)$ is a right $\mathbb{S}_n$-module spanned by graphs (trees) with $n$ input legs,



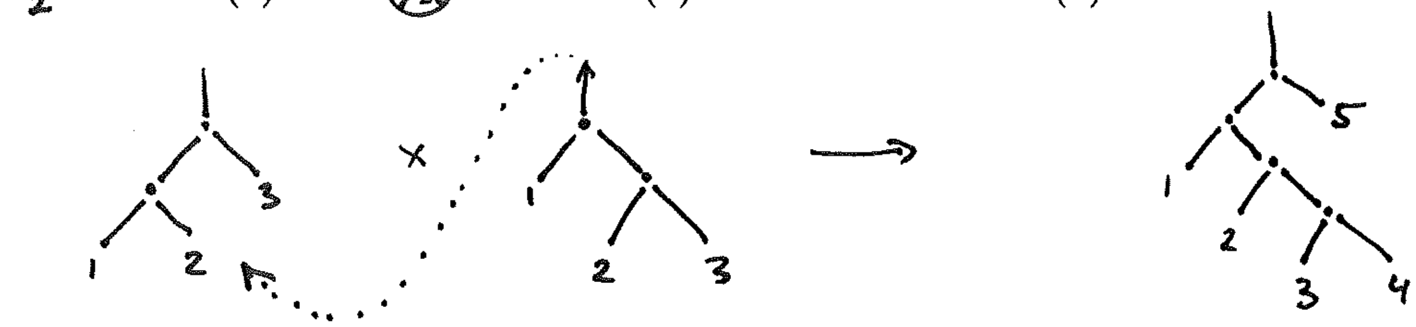

$$\in \mathrm{Ass}(5).$$

For any $i \in \{1, 2, \ldots, n\}$ there is an obvious composition — gluing root leg into $i$-th leg:

$$\circ_i : \quad \mathsf{ASS}(n) \otimes \mathsf{ASS}(m) \longrightarrow \mathsf{ASS}(n+m-1)$$

For example

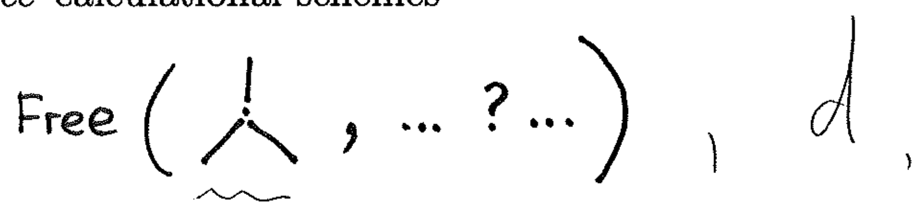
$$\circ_2 : \quad \mathsf{ASS}(3) \quad \otimes \quad \mathsf{ASS}(3) \quad \longrightarrow \quad \mathsf{ASS}(5)$$



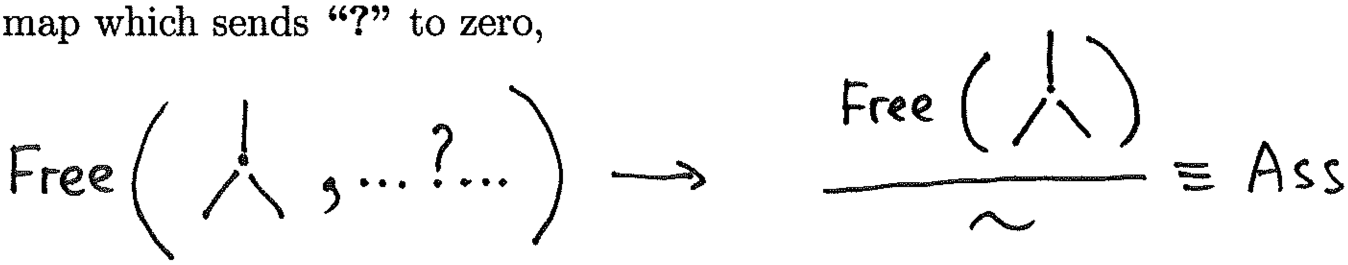

DEFINITION: An (pseudo)operad, $P$, is a graded vector space, $P = \oplus_{n\geq 2} P(n)$, together with compositions $P(n) \circ_i P(m) \to P(n+m-1)$ satisfying the axioms which just mimics the obvious properties of the compositions $\circ_i$ in $\mathsf{ASS}$.

STEP 3:

Get rid of relations in $\mathsf{ASS}$, i.e. find a graded vector space spanned by *free* calculational schemes



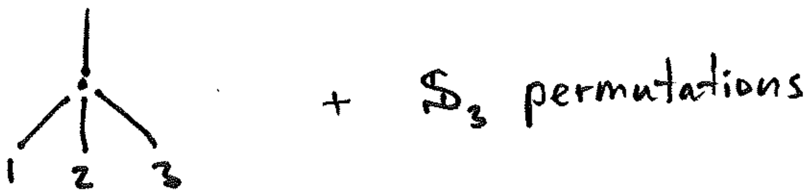

(probably with more basic operations "?") together with a differential, $d : \mathsf{ASS}_\infty \to \mathsf{ASS}_\infty$, such that the natural morphism map which sends "?" to zero,



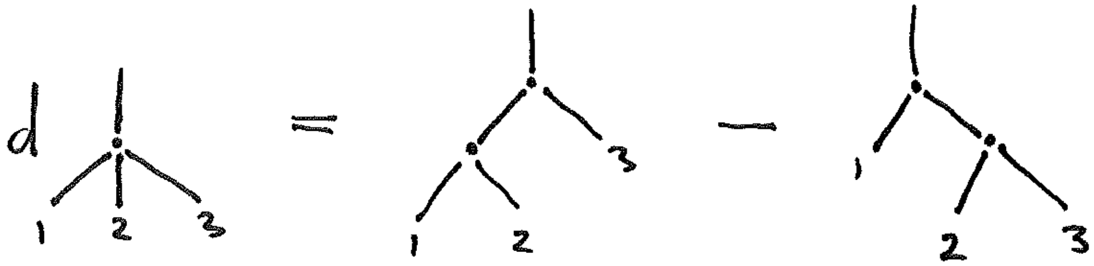

q. iso

Obviously the original binary corolla must be a cycle,

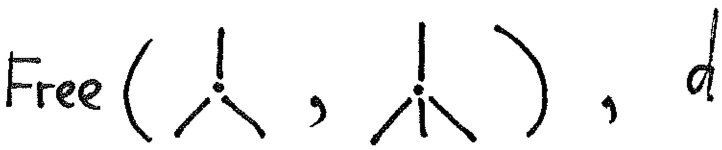

The associativity relations must be coboundaries in $(\mathsf{ASS}_\infty, d)$ we are forced to introduce a new basic corolla,



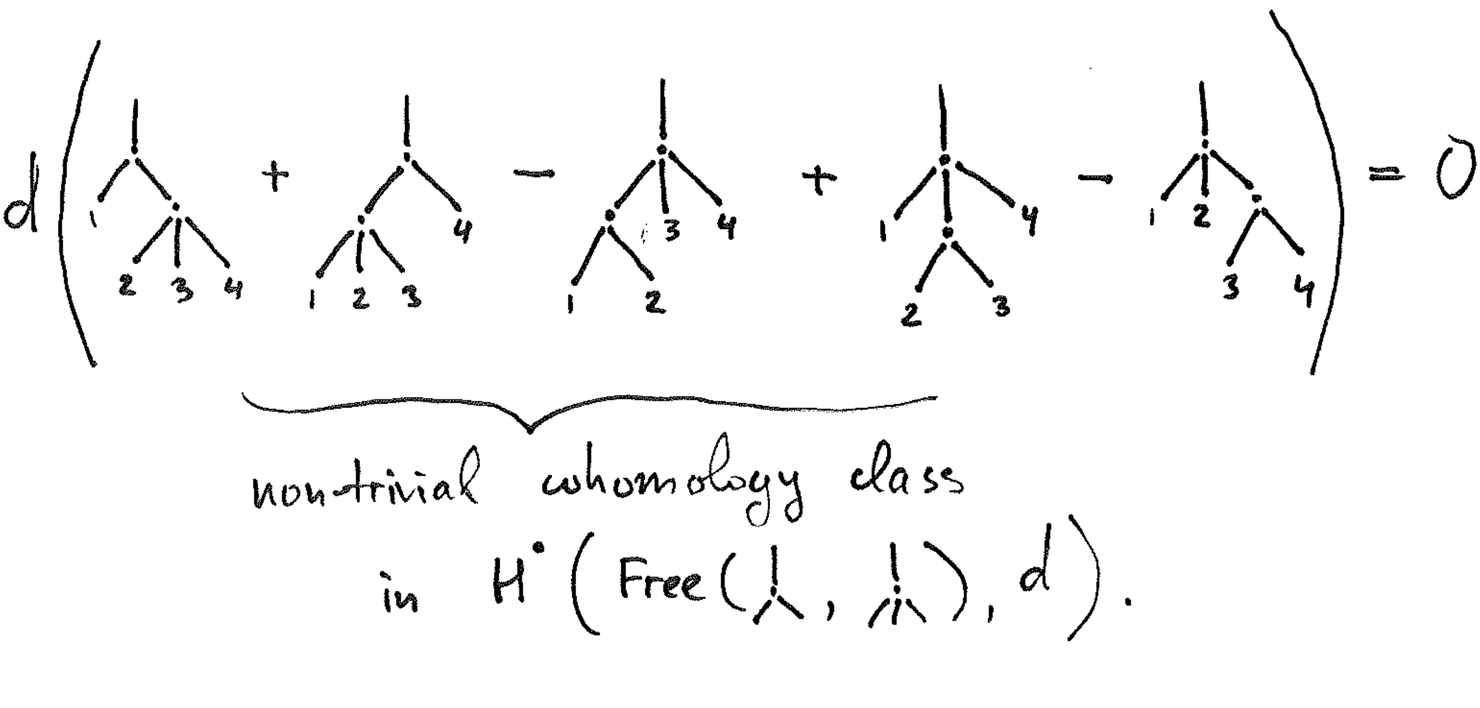

of degree -1 and set



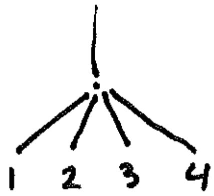

However the cohomology of the graph complex

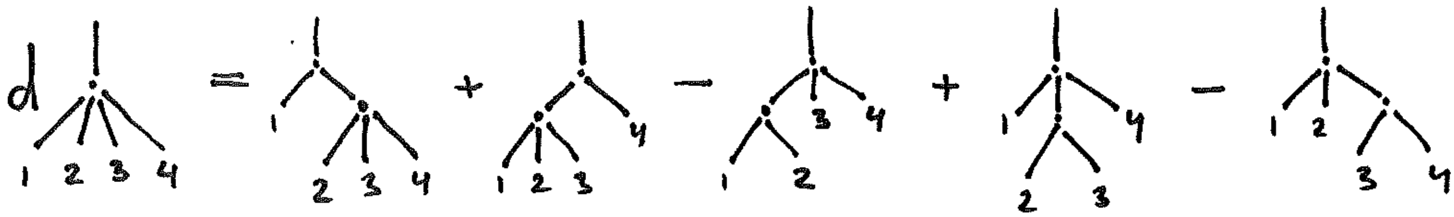



is too large:

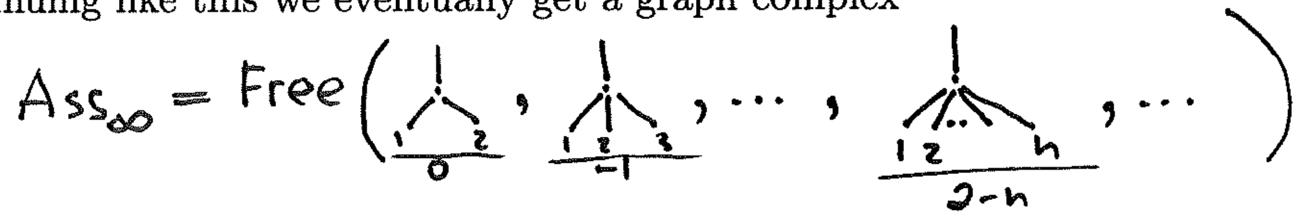


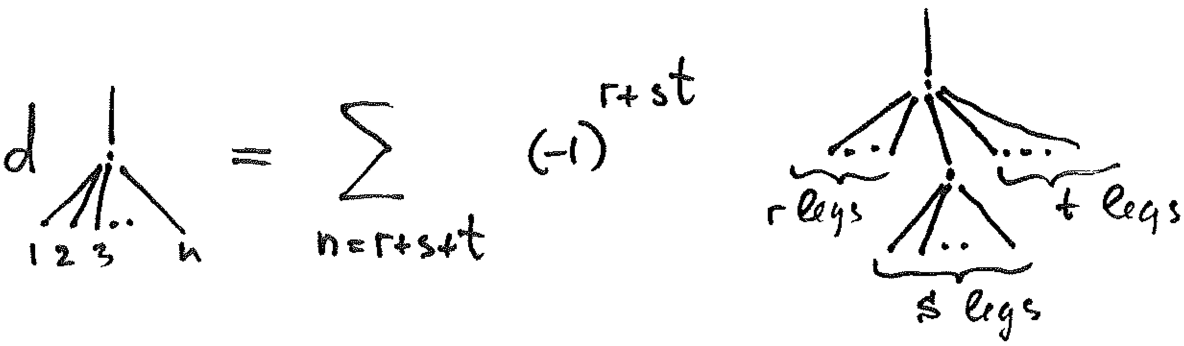
This fact forces us to introduce a new corolla, of degree -2, and set the differential to be



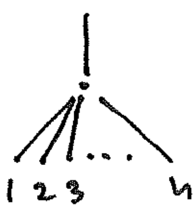


Continuing like this we eventually get a graph complex




Its representation in a dg vector space $V$ is an association



$\longrightarrow$ element in $\mathrm{Hom}_{2-n}(V^{\otimes n}, V)$

$\forall n \geq 2$

DEFINITION: An *Lie n-bialgebra* is the data

(1) graded vector space $V$,

(2) linear map $\Delta : V \to V \wedge V$ of degree 0 making $V$ into Lie coalgebra

(3) linear map $[\ \bullet\ ] : \odot^2 V \to V$ of degree 1 making $V[n]$ into a Lie algebra

(4) co-Jacobi identities for $\Delta$ so that $(V, \delta)$ is a Lie coalgebra

(5) compatibility condition

$$\Delta[a \bullet b] = \sum a_1 \otimes [a_2 \bullet b] + [a \bullet b_1] \otimes b_2 + (-1)^{|a||b|+|a|+|b|}([b \bullet a_1] \otimes a_2 + b_1 \otimes [b_2 \bullet a])$$

where $\Delta a = \sum a_1 \otimes a_2$, $\Delta b = \sum b_1 \otimes b_2$

Case $n = 0$ gives us the ordinary definition of Lie bilagebra.

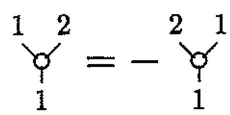
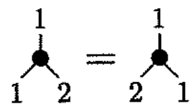
Case $n = 1$ is relevant to Poisson geometry.

GRAPH COMPLEX out of species "Lie 1-bialgebras" via the same three steps:

STEP 1:

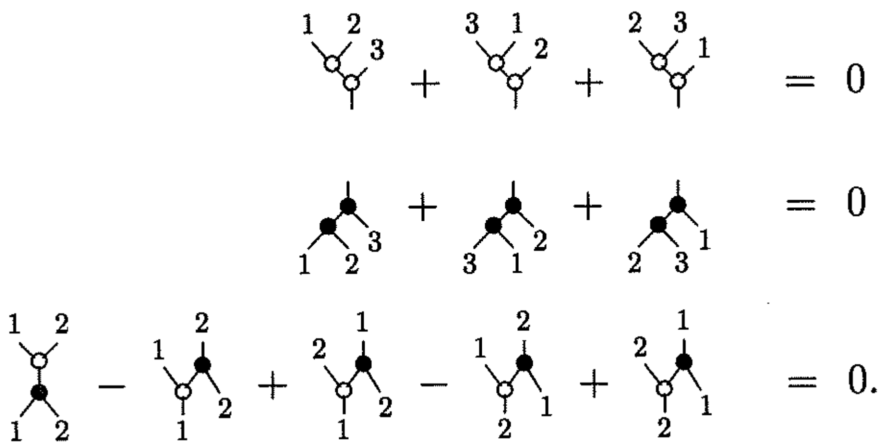
(1) forget $V$

(2) code two basic operations into corollas, "genes",

$\Delta \leftrightarrow$  , $[\bullet] \Leftrightarrow$ 

0 1

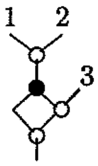
(3) code relations between basic operations into graph relations:



STEP 2: Define the vector space,

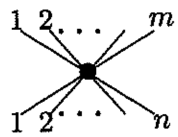
$$\mathsf{Lie^1Bi} = \bigoplus_{m+n\geq 3} \mathsf{Lie^1Bi(m,n)} := \frac{Free\langle \quad , \quad \rangle}{Relations}$$

spanned by all possible graphs,

*e.g.*  $\in \mathsf{Lie^1Bi}(3,1)$

generated by our two genes and modulo the equivalence relation shown above.

STEP 3: Resolve $\mathsf{Lie^1Bi}$ into a *complex* generated by *graphs* exactly as in the case of $\mathsf{ASS}$,

$\mathsf{Lie^1Bi}_\infty := Free\Big\langle$  $\Big\rangle$

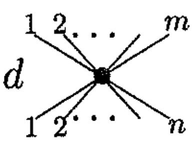
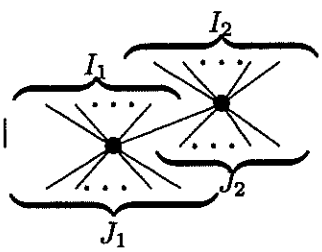
$m, n \geq 1$

$m+n \geq 3$

$2-m$

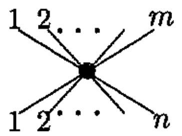
$n$

with the differential given by

$d$  $= \sum_{\substack{I_1\sqcup I_2=(1,\dots,m)\\ J_1\sqcup J_2=(1,\dots,n)\\ |I_1|\geq 0, |I_2|\geq 1\\ |J_1|\geq 1, |J_2|\geq 0}} (-1)^{\sigma(I_1\sqcup I_2)+|I_1||I_2|}$ 

PROPOSITION: There is a 1-1 correspondence between representations,

$$\mathsf{Lie^1Bi}_\infty \dashrightarrow \mathrm{End}\langle V\rangle$$



$$\dashrightarrow \quad \mu_{m,n} \in \mathrm{Hom}_{2-m}(\odot^n V, \wedge^m V)$$

of the dg PROP $(\mathsf{Lie^1Bi}_\infty, d)$ in a dg vector space $V$ and degree 1 polyvector fields,

$$\Gamma = \sum_{m\geq 1} \Gamma^{a_1\ldots a_m}(x) \frac{\partial}{\partial x^{a_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{a_m}} \in \wedge^\bullet \mathcal{T}_V$$

on $V$ (which is viewed as formal graded manifold with distinguished point 0) satisfying the equation,

$$[\Gamma, \Gamma]_S = 0,$$

and the condition $\Gamma|_{x=0} = 0$.

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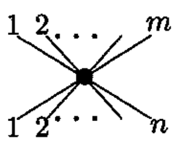
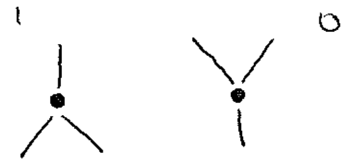
Key to the above correspondence:

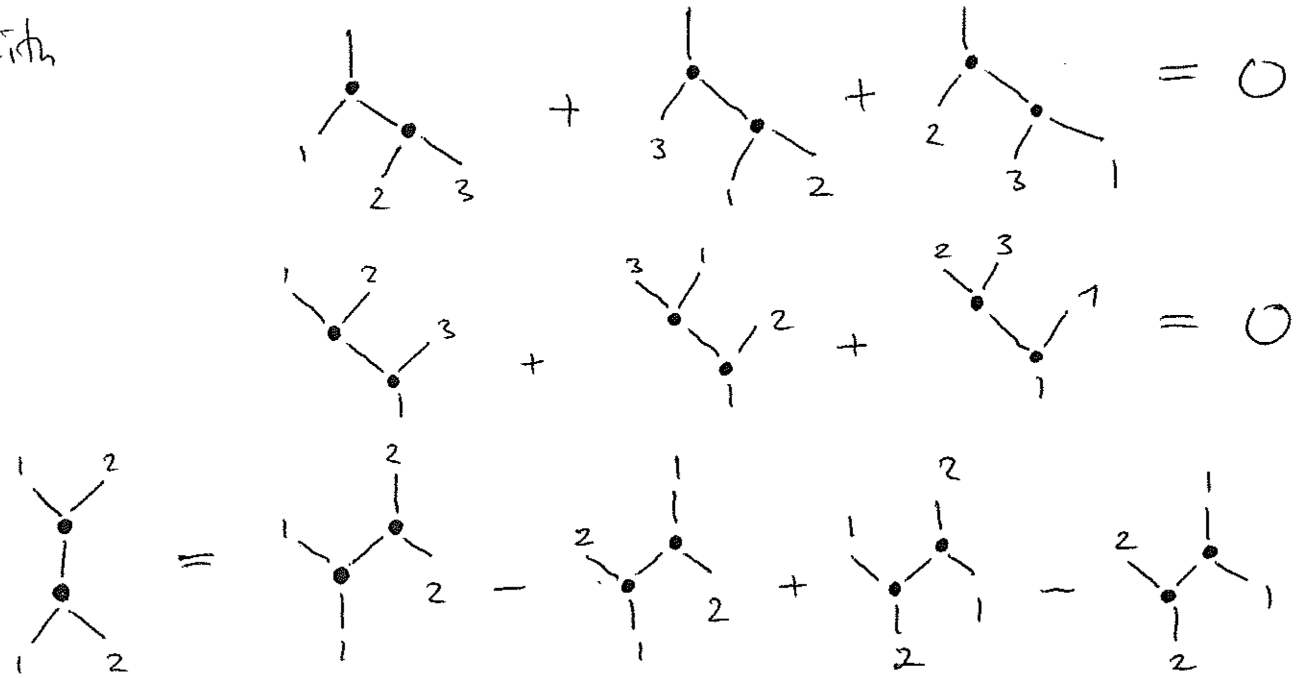


$$\dashrightarrow \quad \frac{\partial^n \Gamma^{a_1\ldots a_m}(x)}{\partial x^{a_1}\cdots \partial x^{a_n}}\Big|_{x=0}.$$

## Genom of Poisson geometry:



with



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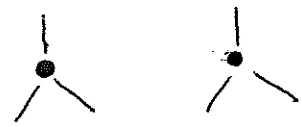
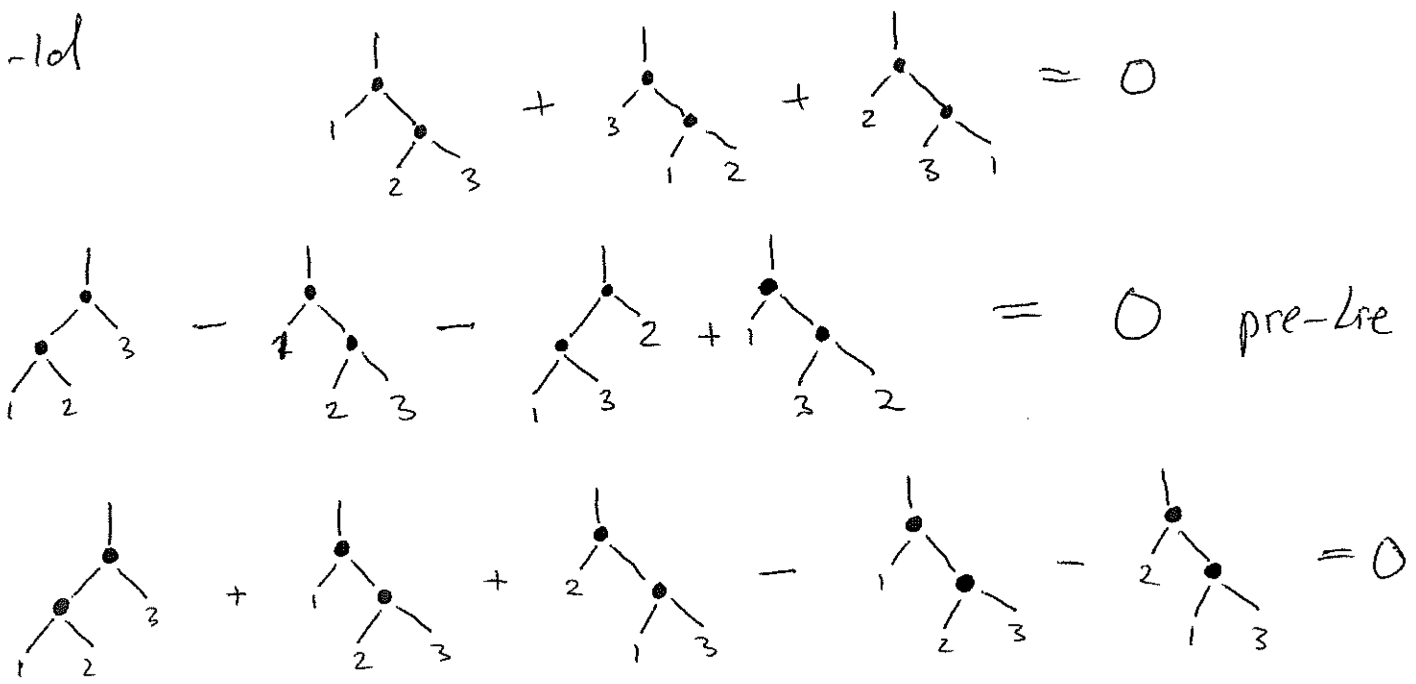
## Genon of Nijenhuis geometry:



$J \in \mathrm{End}(T_M)$ s.t. $N(J) = 0$

$J^2 = -\mathrm{Id}$

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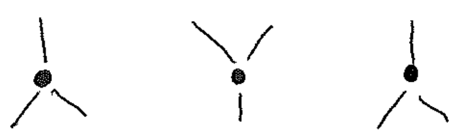
## Genetic engineering (Henrik Strohmayer):



$\rightsquigarrow$ Poisson-Nijenhuis geometry and generalized complex mnflds

COROLLARY: Formal germs, $\nu$, of Poisson structure in $V = \mathbb{R}^n$ are in 1-1 correspondence with *morphisms* of dg PROPS,

$$(d,\ \mathsf{Lie}^1\mathsf{Bi}_\infty) \xrightarrow{\nu} (\mathrm{End}_{\mathbb{R}^n},\ d) \quad \mathrm{Hom}(V^{\otimes m}, V^{\otimes n})$$

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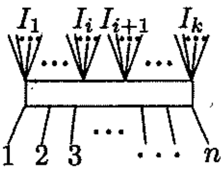
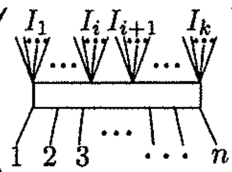
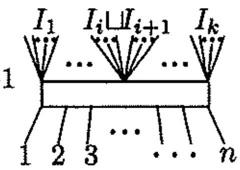
FACT. There exists a graph complex (dg PROP)

$$\mathsf{DefQ} = \mathit{Free}\left\langle \right.$$  $$\left.\right\rangle$$

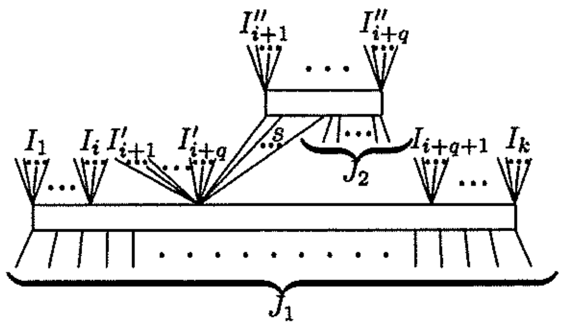


with the differential given by

$$\delta\left(\right.$$  $$\left.\right) = \sum_{i=1}^{k} (-1)^{i+1}$$ 



$$+ \sum_{\substack{p+q=k+1\\ p\geq 1, q\geq 0}} \sum_{i=0}^{p-1} \sum_{\substack{I_{i+1}=I'_{i+1}\sqcup I''_{i+1}\\ \cdots\cdots\cdots \\ I_{i+q}=I'_{i+q}\sqcup I''_{i+q}}} \sum_{[n]=J_1\sqcup J_2} \sum_{s\geq 0} (-1)^{(p+1)q+i(q-1)}$$

$$\frac{1}{s!}$$ 

whose representations,

$$\mathsf{DefQ} \longrightarrow \mathrm{End}_V$$

in a dg vector space $V$ are in 1-1 correspondence with Maurer-Cartan elements,

$$d_H\Gamma + \frac{1}{2}[\Gamma, \Gamma] = 0$$

in the dg Hochschild Lie algebra $\bigoplus_{m\geq 0} Hom(\mathcal{O}_V^{\otimes m}, \mathcal{O}_V)$ of poly-differential operators on the ring, $\mathcal{O}_V$, of formal smooth functions on $V$.

COROLLARY: Star products, $*$, in $\mathcal{O}_{\mathbb{R}^n}$ are in 1-1 correspondence with *morphisms* of dg PROPS,

$$* : \mathsf{DefQ} \longrightarrow \mathrm{End}_{\mathbb{R}^n}$$

---

NEW VIEWPOINT on universal deformation quantization: this is a *morphisms* of graph complexes (dg PROPs),

$$\mathsf{DefQ} \xrightarrow{Q} \mathsf{Lie}^1\mathsf{Bi}_\infty.$$

---

Then quantization of a particular formal Poisson structure $\nu$ on $\mathbb{R}^n$,

$$\mathsf{Lie}^1\mathsf{Bi}_\infty \xrightarrow{\nu} \mathrm{End}_{\mathbb{R}^n}$$

is just the composition

$$\begin{tikzcd} * : \mathsf{DefQ} \arrow[r, "Q"] \arrow[rr, bend right] & \mathsf{Lie}^1\mathsf{Bi}_\infty \arrow[r, "\nu"] & \mathrm{End}_{\mathbb{R}^n}. \end{tikzcd}$$

whose representations,

$$\mathsf{DefQ}^{\hbar} \longrightarrow \mathrm{End}_V$$

in a dg vector space $V$ are in 1-1 correspondence with Maurer-Cartan elements,

$$d_H\Gamma + \frac{1}{2}[\Gamma, \Gamma] = 0$$

in the dg Hochschild Lie algebra $\bigoplus_{m\geq 0} Hom(\mathcal{O}_V^{\otimes m}[[\hbar]], \mathcal{O}_V)$ of polydifferential operators on the ring, $\mathcal{O}_V$, of formal smooth functions on $V$ such that $\Gamma|_{\hbar=0} = 0$.

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PROPOSITION: The dg PROP $(\mathsf{DefQ}^{\hbar}, d)$ is cofibrant (i.e. has structure of CW complex type).

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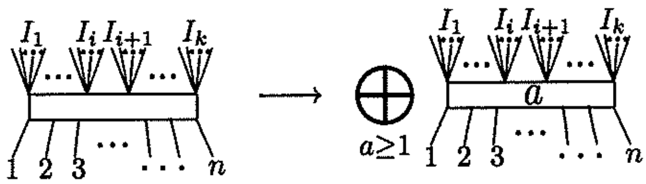
FACT: There exists a canonical morphism of dg PROPs,

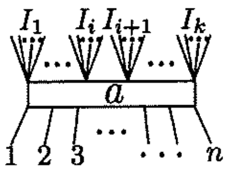
$$\mathsf{DefQ} \xrightarrow{\ i\ } \mathsf{DefQ}^{\hbar}$$




REMARK: We construct a rather narrow family of deformation quantizations ($Lie_\infty$-morphisms) which factor through $i$. This family is defined over $\mathbb{Q}$.

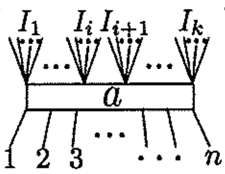
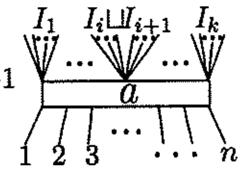
The dg PROP **DefQ** is *not* cofibrant. There exists, however, a cofibrant ("Planck constant") version of it:

FACT. There exists a graph complex (dg PROP)

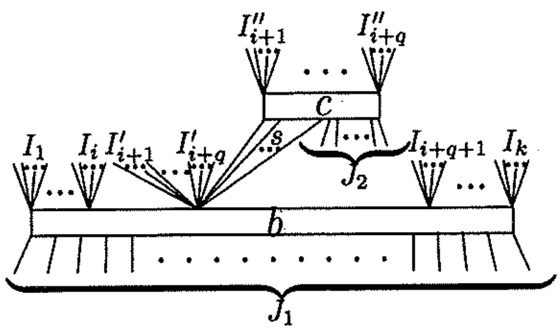
$$\mathsf{DefQ}^{\hbar} = Free \left\langle \right. $$  $$\left. \right\rangle$$

$\hbar^{a}$

with differential given by

$$\delta \left( \right.$$  $$\left. \right) = \sum_{i=1}^{k} (-1)^{i+1}$$ 

$$+ \sum_{\substack{b+c=a \\ b,c\geq 1}} \ \sum_{\substack{p+q=k+1 \\ p\geq 1, q\geq 0}} \ \sum_{i=0}^{p-1} \ \sum_{\substack{I_{i+1}=I'_{i+1}\sqcup I''_{i+1} \\ \cdots \\ I_{i+q}=I'_{i+q}\sqcup I''_{i+q}}} \ \sum_{[n]=J_1\sqcup J_2} \ \sum_{s\geq 0} (-1)^{(p+1)q+i(q-1)}$$

$$\frac{1}{s!}$$ 

SUMMARY: we want to construct morphism $Q$

$$\begin{tikzcd}
\mathsf{DefQ} \arrow[r, dotted, "Q"] \arrow[d, "i"'] & \mathsf{Lie^1Bi}_\infty \arrow[d, "qis"] \\
\mathsf{DefQ}^\hbar & \mathsf{Lie^1Bi}
\end{tikzcd}$$

IDEA: If there exists morphism $q$, then there exists its lift $\hat{q}$ (as $\mathsf{DefQ}^\hbar$ is cofibrant), and hence there exists $Q$ making the diagram commutative:

$$\begin{tikzcd}
\mathsf{DefQ} \arrow[r, dotted, "Q"] \arrow[d, "i"'] & \mathsf{Lie^1Bi}_\infty^{\circlearrowright} \arrow[d, "qis"] \\
\mathsf{DefQ}^\hbar \arrow[r, "q"'] \arrow[ur, dotted, "\hat{q}"] & \mathsf{Lie^1Bi}^{\circlearrowright}
\end{tikzcd}$$

Q

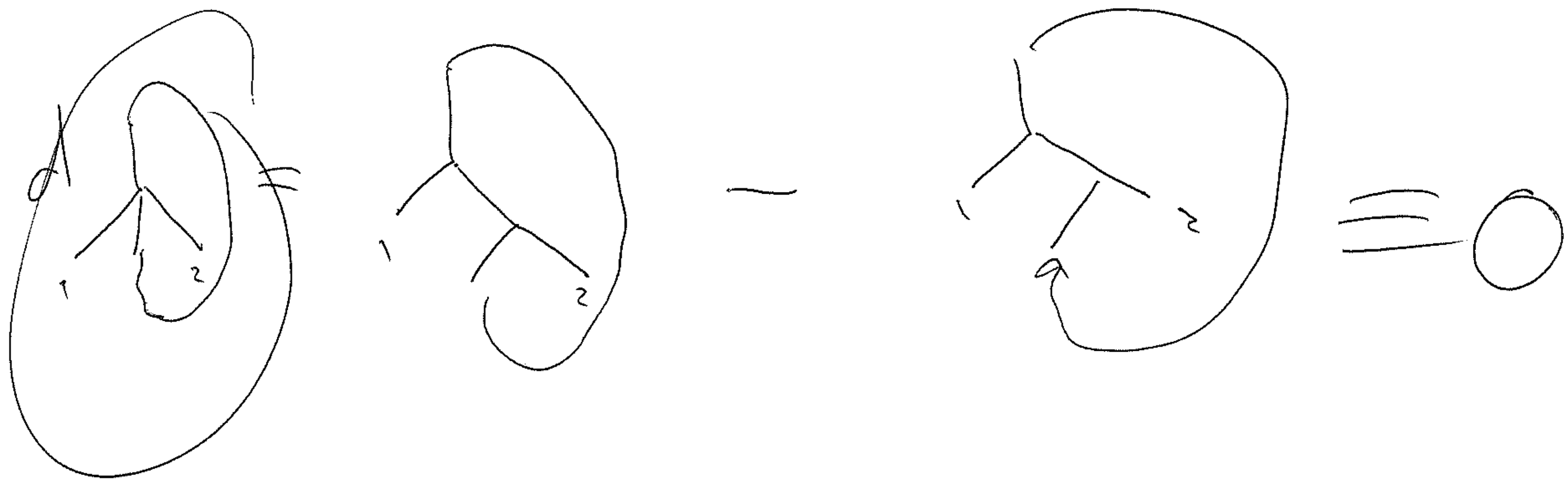
PROPOSITION: $q$ does not exists.

THEOREM: There exists $q : \mathsf{DefQ}^\hbar \to \mathsf{LieBi}^{\circlearrowright}$

THEOREM: The natural map $\mathsf{Lie^1Bi}_\infty^{\circlearrowright} \longrightarrow \mathsf{Lie^1Bi}^{\circlearrowright}$ is a quasi-isomorphism.

COROLLARY: There exists a morphism $Q$.

COROLLARY: Any formal germ of Poisson structure in $\mathbb{R}^n$ can be deformation quantized over $\mathbb{Q}$.




cycle