Pure spinors and moment maps I

Based on joint work with

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§1 Motivation

G simply connected Lie group with bi-invariant pseudo-Riemannian metric

(e.g.: G semi-simple)

CLAIM: Every conjugacy class $\mathcal{C} \subset G$ carries a canonical invariant volume form.

Remarks:

- The assumption $\pi_0(G) = \pi_1(G) = \{0\}$ can be weakened.
- Without any assumption on π_0, π_1 one still gets a canonical invariant measure.

Let $\mathcal{B} : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ be the bilinear form on $\mathfrak{g} = \mathrm{Lie}(G)$.

Invariant 2-form on conj. classes $\mathcal{C} \subset G$

$$\omega(\xi^\#, \zeta^\#) = B\left(\frac{\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}}}{2}\,\xi, \zeta\right) \qquad \xi, \zeta \in \mathfrak{g}$$

(Guruprasad - Huebschmann - Jeffrey - Weinstein)

Thm: If $\pi_0(G) = \pi_1(G) = \{0\}$ the formula

$$\psi_g = \det{}^{1/2}\left(\frac{\mathrm{Ad}_g + 1}{2}\right) \exp\left(\frac{1}{2} B\left(\frac{\mathrm{Ad}_g - 1}{\mathrm{Ad}_g + 1}\,\theta^L, \theta^L\right)\right)$$

defines an invariant form $\psi \in \Omega(G)$.

Here $\theta^L \in \Omega^1(G, \mathfrak{g})$ is the left-Maurer Cartan form "$g^{-1}dg$"

Thm: For every conjugacy class $\mathcal{C} \subset G$,

$$\left(\exp(\omega) \wedge \iota_{\mathcal{C}}^* \psi\right)_{[top]} \neq 0$$

§ 2 Pure spinors

V vector space $\mathbb{V} = V \oplus V^*$

inner product on $\mathbb{V}$: $\langle x, x\rangle = 2\langle \alpha, v\rangle$ $(x = v \oplus \alpha)$

$$\boxed{\text{Def: } E \subset \mathbb{V} \text{ is Lagrangian if } E = E^\perp.}$$

Equivalently, $\dim E = \frac{1}{2}\dim \mathbb{V}$ and $\forall x \in E : \langle x, x\rangle = 0$

Examples: V, V^*, $Gr_\omega = \{(v, \omega(v,\cdot))\}$, $Gr_\pi = \{(\pi(\alpha,\cdot), \alpha)\}$

$\omega \in \Lambda^2 V^*$ $\pi \in \Lambda^2 V$

Exercise: Show that

$$\mathrm{Lag}(\mathbb{V}) \cong O(n) \qquad (n = \dim V)$$

(Hint: $\mathbb{V} \cong \mathbb{R}^{n,n}$ so $O(n) \subset O(n,n) \cong O(\mathbb{V})$.)

Consider contravariant spinor representation

$$\varrho : \mathbb{V} \longrightarrow \mathrm{End}(\Lambda V^*)$$

$$\varrho(x).\varphi = \iota(v)\varphi + \alpha \wedge \varphi \qquad (x = v \oplus \alpha)$$

Note $\varrho(x)^2\varphi = \langle \alpha, v\rangle \varphi = \frac{1}{2}\langle x, x\rangle \varphi$. Hence all vectors in

$$N_\varphi = \{ x \in \mathbb{V} \mid \varrho(x).\varphi = 0 \}$$

are isotropic : $\langle x, x\rangle = 0$.

$$\boxed{\text{Def } \varphi \in \Lambda V^* \setminus \{0\} \text{ is a pure spinor } :\Longleftrightarrow N_\varphi \text{ is Lagrangian.}}$$

$\rightsquigarrow$ fiber bundle $\mathbb{R}^\times \longrightarrow \left\{ \begin{matrix}\text{pure}\\ \text{spinors}\end{matrix} \right\} \longrightarrow \mathrm{Lag}(\mathbb{V})$

Examples :

	φ	$E = N_\varphi$	
	1	V	
	$\exp(-\omega)$	Gr_ω	$(\omega \in \Lambda^2 V^*)$
volume form $\longrightarrow$	μ	V^*	
	$\exp(-\iota(\pi))\mu$	Gr_π	$(\pi \in \Lambda^2 V)$

<u>Lemma</u>: For every pure spinor $\varphi \in \wedge V^*$ there exist unique

$$S \subset V, \quad \omega \in \Lambda^2 S^*, \quad \theta \in \Lambda^{top}(\mathrm{ann}(S)) \setminus \{0\}$$

such that

$$\varphi = e^{-\omega} \wedge \theta$$

Note:

1) $\varphi = e^{-\omega} \wedge \theta$ is pure spinor defining

$$E = \{ v \oplus \alpha \mid v \in S, \ \alpha|_S = \omega(v, \cdot) \}$$

2) For any $E \in \mathrm{Lag}(\mathbb{V})$ get

$S = pr_V(E)$

$\omega \in \Lambda^2 S^*$ where $\omega(v, \cdot) := \alpha|_S \quad (v \oplus \alpha \in E)$

Consequences of $\varphi = e^{-\omega} \wedge \theta$:

- Every pure spinor is even or odd.
- $\varphi^{top} \neq 0 \iff N_\varphi \cap V = 0 \iff N_\varphi = Gr_\pi$
- $A : V \longrightarrow V'$ linear, $\varphi' \in \Lambda(V')^*$ pure $\Rightarrow$ $A^*\varphi' =: \varphi$ is pure (unless $\varphi = 0$)
- $\varphi_1, \varphi_2 \in \Lambda V^*$ pure $\Rightarrow$ $\varphi_1 \wedge \varphi_2 =: \varphi$ is pure (unless $\varphi = 0$).

Let $(\alpha_1 \wedge \ldots \wedge \alpha_r)^T = \alpha_r \wedge \ldots \wedge \alpha_1$ (principal anti-automorphism of ΛV^*)

Thm: (Chevalley) If φ, ψ are pure spinors representing $E, F \in Lag(\mathbb{V})$ then

$$E \cap F = \{0\} \iff (\varphi^T \wedge \psi)^{top} \neq 0$$

Pf : Exercise

Remark: There is a parallel theory using the covariant spinor representation

$$\tilde{\varrho} : \mathbb{V} \longrightarrow \mathrm{End}(\wedge V)$$

$$\tilde{\varrho}(x).\chi = v \wedge \chi + \iota(\alpha)\chi \qquad x = v \oplus \alpha$$

Def: A linear Dirac structure on V is a Lagrangian subspace $E \subset \mathbb{V}$.
A linear map between Dirac spaces

$$A : V \longrightarrow V'$$

is a (strong) Dirac map if it takes covariant spinors defining E to covariant spinors defining E'.

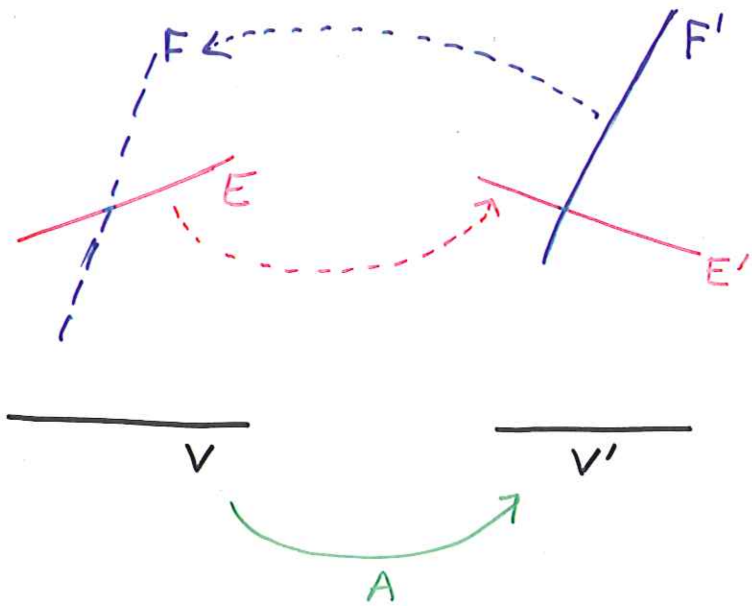
Example: Recall that for $E \in \mathrm{Lag}(\mathbb{V})$, $S = \mathrm{pr}_V(E)$ carries 2-form ω.
Inclusion $S \hookrightarrow V$ is a strong Dirac map w.r.t. Gr_ω, E.

Prop: Suppose (V,E) (V',E') Dirac spaces and

$$A : V \longrightarrow V'$$

is strong Dirac map.
Suppose $F' \in Lag(\mathbb{V}')$ with $E' \cap F' = \{0\}$.
defined by $\psi' \in \Lambda(V')^*$. Then $\psi = A^* \psi'$ defines $F \in Lag(\mathbb{V})$ with $E \cap F = \{0\}$.

Pf: Exercise



§3 Almost Dirac structures

Def: An almost Dirac structure on a manifold M is a Lagrangian sub-bundle $E \subset \mathbb{T}M = TM \oplus T^*M$

Def: $\varphi \in \Omega(M)$ is a pure spinor if each $\varphi_x \in \Lambda T_x^* M$ is a pure spinor on $T_x M$.

E.g.: $\omega \in \Omega^2(M) \rightsquigarrow \varphi = \exp(-\omega)$

$\pi \in \mathfrak{X}^2(M) \rightsquigarrow \varphi = \exp(-\iota(\pi))\,\mu$

volume form

Integrability condition: tomorrow

Back to our Lie group G

$\theta^L, \theta^R \in \Omega^1(G, \mathfrak{g})$ Maurer-Cartan form

$\xi^L, \xi^R \in \mathfrak{X}(G)$ left, right invt. vector fields corr. to $\xi \in \mathfrak{g}$

Define sections $e(\xi), f(\xi) \in \Gamma(\mathbb{T}G)$ by

$$e(\xi) = (\xi^L - \xi^R) \oplus B\left(\frac{\theta^L + \theta^R}{2}, \xi\right)$$

$$f(\xi) = \left(\frac{\xi^L + \xi^R}{2}\right) \oplus B\left(\frac{\theta^L - \theta^R}{4}, \xi\right)$$

Prop: The sub-bundles

$$E = \operatorname{span}\{e(\xi),\ \xi \in \mathfrak{g}\}$$

$$F = \operatorname{span}\{f(\xi),\ \xi \in \mathfrak{g}\}$$

are Lagrangian, with $E \cap F = 0$.

E is called the Cartan-Dirac structure

$$E = \mathrm{span}\{e(\xi),\ \xi \in \mathfrak{g}\}$$

$$e(\xi) = (\xi^L - \xi^R) \oplus B\left(\frac{\theta^L + \theta^R}{2}, \xi\right)$$

Note that $pr_{TG}(E) \subset TG$ is the distribution tangent to adjoint action.

$\Rightarrow$ Get 2-forms on conjugacy classes, $\mathcal{C} \subset G$.

These are exactly the GHJW-2 forms!

$\Rightarrow$

Lemma: The inclusions $i_{\mathcal{C}} : \mathcal{C} \longrightarrow G$ are strong Dirac maps.

Hence, if $\psi \in \Omega(G)$ is a pure spinor defining $F = \mathrm{span}\{f(\xi), \xi \in \mathfrak{g}\}$ then

$$\iota_e^* \psi \in \Omega(G)$$

defines a complement F_e to $E_e = Gr_\omega$.

Hence, by Chevalley's theorem

$$\left(\exp(\omega) \wedge \iota_e^* \psi\right)^{top} \neq 0$$

Thm :

$$\psi = \det{}^{1/2}\left(\frac{Ad_g + 1}{2}\right) \exp\left(\frac{1}{2} B\left(\frac{Ad_g - 1}{Ad_g + 1} \theta^L, \theta^L\right)\right)$$

is a pure spinor defining F.

Finally, where do this formulas come from?

If (M, B) is a pseudo-Riemannian mf. there is an isometry

$$\kappa : TM \oplus \overline{TM} \longrightarrow \mathbb{T}M = TM \oplus T^*M$$

$$B \oplus (-B) \qquad\qquad \langle\, , \rangle$$

given by $\qquad O(TM) \hookrightarrow O(\mathbb{T}M)$

$$\kappa(v \oplus w) = (v+w) \oplus B^\flat\left(\tfrac{v-w}{2}\right).$$

Hence, for $C \in \Gamma(O(TM))$ get transverse Lagrangians E, F

$$E = Ad_\kappa \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \cdot T^*M$$

$$F = Ad_\kappa \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix} \cdot TM$$

For $M = G$ take the transformation

$$TG \xleftrightarrow[\text{trivialization}]{\text{left}} G \times \mathfrak{g} \xleftrightarrow[\text{trivialization}]{\text{right}} TG$$

Then $C \in \Gamma(SO(TG))$ takes T^*G, TG to E, F respectively.

$\subseteq \Gamma(C\ell(\mathbb{T}G))$

The lift $\tilde{C} \in \Gamma(\mathrm{Spin}(TG))$ takes defining spinors $\mu, 1$ for T^*G, TG, to defining spinors φ, ψ for E, F.

A calculation yields our formula for $\psi \in \Omega(G)$.

Pure spinors and moment maps II

§1 Review

M manifold

- $\mathbb{T}M = TM \oplus T^*M$; inner product $\langle \cdot , \cdot \rangle$
- $\varrho : \Gamma(\mathbb{T}M) \longrightarrow \mathrm{End}_{C^\infty(M)}(\Omega(M))$; $v \oplus \alpha \longmapsto \iota(v) + \alpha$
- An almost Dirac structure on M is a Lagrangian (= max. isotropic) sub-bundle $E \subset \mathbb{T}M$
- $\varphi \in \Omega(M)$ is a pure spinor defining E
  $\Longleftrightarrow$ $\varrho(x)\varphi = 0 \quad \forall x \in \Gamma(E)$ and $\varphi \neq 0$ everywhere

Examples :

① $\omega \in \Omega^2(M) \rightsquigarrow \varphi = e^{-\omega}$ defines $E = \mathrm{Gr}_\omega$

② $\pi \in \mathfrak{X}^2(M) \rightsquigarrow \varphi = e^{-\iota(\pi)}\mu$ defines $E = \mathrm{Gr}_\pi$

(Here $\mu \in \Omega^{top}(M)$ is any volume form.)

(3) G Lie group

$B : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{R}$ invt. inner product on $\mathfrak{g} = \mathrm{Lie}(G)$

$E = \mathrm{span}\{ e(\xi) \mid \xi \in \mathfrak{g} \}$ where

$$e(\xi) = (\xi^L - \xi^R) \oplus B\left(\frac{\theta^L + \theta^R}{2}, \xi\right)$$

$F = \mathrm{span}\{ f(\xi), \xi \in \mathfrak{g} \}$ where

$$f(\xi) = \left(\frac{\xi^L + \xi^R}{2}\right) \oplus B\left(\frac{\theta^L - \theta^R}{4}, \xi\right)$$

These are obtained from T^*G resp. TG by a certain rotation $A \in \Gamma(O(\mathbb{T}G))$. Applying the lifted rotation $\tilde{A} \in \Gamma(\mathrm{Pin}(\mathbb{T}G))$ to $\mu \in \Omega^{top}(G)$ resp. $1 \in \Omega(G)$ one obtains pure spinors φ, ψ for E, F.

$$\varphi, \psi \in \Omega(G)^G$$

§ 2 Integrability

Fix closed $\eta \in \Omega^3(M)$, $d\eta = 0$.

$\Rightarrow$ $d + \eta$ is a differential.

Def: The Courant bracket $[\![\cdot,\cdot]\!]$ on $\Gamma(\mathbb{T}M)$ is defined by

$$\varrho([\![x,y]\!]) = [\varrho(x), [\varrho(y), d+\eta]]$$

Let

$$\Upsilon(x_1,x_2,x_3) = [\varrho(x_1), [\varrho(x_2), [\varrho(x_3), d+\eta]]] \in C^\infty(M)$$

Lemma: The restriction of Υ to $\Gamma(E)$ defines a tensor $\Upsilon_E \in \Gamma(\Lambda^3 E^*)$ "Courant tensor"

Pf: Let $\varphi \in \Omega(M)$ be a pure spinor defining E. Then $\forall x_i \in \Gamma(E)$

$$\varrho(x_1)\varrho(x_2)\varrho(x_3)(d+\eta)\varphi = [\varrho(x_1), [\varrho(x_2), [\varrho(x_3), d+\eta]]]\varphi$$
$$= \Upsilon_E(x_1,x_2,x_3)\,\varphi$$

Now claim is evident. ∎

$$E = N_\varphi \subseteq \mathbb{T}M$$

$$\underline{\varrho(x_1)\varrho(x_2)\varrho(x_3)(d+\eta)}\varphi = Y_E(x_1, x_2, x_3)\varphi$$

$$x_i \in \Gamma(E)$$

Def: An almost Dirac structure $E \subset \mathbb{T}M$ is a Dirac structure if it is integrable, i.e.

$$Y_E = 0.$$

Equivalently, E is integrable $\Leftrightarrow$ $\Gamma(E)$ preserved under $[\![\cdot,\cdot]\!]$.

Pf:
$$\begin{aligned} Y(x_1, x_2, x_3) &= [\varrho(x_1), [\varrho(x_2), [\varrho(x_3), d+\eta]]] \\ &= [\varrho(x_1), \varrho([\![x_2, x_3]\!])] \\ &= \langle \cancel{\varrho}(x_1), \cancel{\varrho}([\![x_2, x_3]\!]) \rangle \end{aligned}$$

Claim follows since E is maximal isotropic.

Prop (Alekseev-Xu, Gualtieri) $E \subset \mathbb{T}M$ is a Dirac structure if and only if

$$(d+\eta)\varphi = \varrho(s)\varphi$$

for some $s \in \Gamma(\mathbb{T}M)$.

Given Dirac structure $E \subset \mathbb{T}M$ let

$$[x,y]_E := [\![x,y]\!] \quad (x,y \in \Gamma(E))$$

$$\rho([x,y]_E) = [\rho(x),[\rho(y),d+\eta]]$$

$$p = pr_{TM} : E \longrightarrow TM$$

$\boxed{\text{Prop: } (E, [\,,]_E, p) \text{ is a Lie algebroid. That is } [\,,]_E \text{ is a Lie bracket with } [x, fy]_E = f[x,y]_E + (p(x)f)\, y}$

$\Rightarrow$ $p(E) \subset TM$ defines a foliation of M

Examples:

1) $\varphi = e^{-\omega}$ $\quad E = Gr_\omega$

integrable $\Leftrightarrow$ $d\omega = \eta$ (trivial foliation $p(E) = TM$)

2) $\varphi = e^{-\iota(\pi)}\mu$ $\quad E = Gr_\pi$

integrable $\Leftrightarrow$ $\frac{1}{2}[\pi,\pi] = \pi^\#(\eta)$

$p(E) = \pi^\#(T^*M) \subset TM$

3) G Lie group $B: \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$

$$E = \text{span}\{e(\xi),\ \xi \in \mathfrak{g}\}$$
$$e(\xi) = (\xi^L - \xi^R) \oplus B\left(\frac{\theta^L + \theta^R}{2}, \xi\right)$$

Let

$$\eta = \frac{1}{12} B(\theta^L, [\theta^L, \theta^L]) \qquad \text{Cartan 3-form}$$
$$\in \Omega^3(G)$$

<u>Thm</u>: The defining spinor $\varphi \in \Omega(G)$ for E satisfies
$$(d + \eta)\varphi = 0$$
In particular E is a Dirac structure.
"Cartan – Dirac structure"

Pf.: For $x \in \Gamma(E)$

$$\begin{aligned} \varrho(x)(d+\eta)\varphi &= [\varrho(x), d+\eta]\varphi \\ &= \left[\iota(\xi^L - \xi^R) + B\left(\frac{\theta^L + \theta^R}{2}, \xi\right), d+\eta\right]\varphi \\ &= L(\xi^L - \xi^R)\varphi = 0 \end{aligned}$$

$\Rightarrow (d+\eta)\varphi = f\varphi$ some f. By parity $f = 0$. ∎

$$E = \mathrm{span}\{e(\xi),\ \xi \in \mathfrak{g}\}$$ Cartan Dirac structure

$$e(\xi) = (\xi^L - \xi^R) \oplus B\left(\frac{\theta^L + \theta^R}{2}, \xi\right)$$

Defined by pure spinor φ with $(d+\eta)\varphi = 0$.

Foliation defined by $\mathrm{pr}_{TG}(E) = \mathrm{span}\{\xi^L - \xi^R,\ \xi \in \mathfrak{g}\}$ is given by conjugacy classes $\mathcal{C} \subset G$

Remarks

1) If $E \subset \mathbb{T}M$ is a Dirac structure w.r.t. a closed $\eta \in \Omega^3(M)$ then the 2-forms on the leaves $S \subset M$ satisfy $d\omega = \iota^*\eta$.

2) The almost Dirac structure $F \subset \mathbb{T}G$,

$$F = \mathrm{span}\{f(\xi),\ \xi \in \mathfrak{g}\}$$

$$f(\xi) = \left(\frac{\xi^L + \xi^R}{2}\right) \oplus B\left(\frac{\theta^L - \theta^R}{4}, \xi\right)$$

is not integrable. Instead the defining pure spinor $\psi \in \Omega(G)$ satisfies

$$(d+\eta)\psi = \frac{1}{4}\,\varrho(e(\Xi)).\psi \qquad e : \mathfrak{g} \to \Gamma(E)$$

where $\Xi \in \Lambda^3\mathfrak{g}$ "structure constants tensor".

$$e(\Xi) = \Gamma(\Lambda^3 E) \subset \Gamma(\mathcal{Cl}(\mathbb{T}M))$$

§ 3 Group-valued moment maps

Def (Alekseev–Malkin–M) A $\mathfrak{g}$-Hamiltonian $\mathfrak{g}$-space is a manifold M with 2-form $\omega \in \Omega^2(M)$ and moment map $\Phi : M \longrightarrow G$ s.th.

(1) $d\omega = \Phi^* \eta$

(2) M is a $\mathfrak{g}$-manifold, and Φ is $\mathfrak{g}$-equivariant

(3) $\iota(\xi^\#)\,\omega = \Phi^* B\left(\frac{\theta^L + \theta^R}{2}, \xi\right)$

(4) $\ker(\omega_m) = \{ \xi^\#_m \mid \mathrm{Ad}_{\Phi(m)} \xi = -\xi \}$

Remark (Xu, Burstyn-Crainic) Condition (4) can be replaced by equivalent condition

(4′) $\ker(\omega_m) \cap \ker((d\Phi)_m) = \{0\}$

Example:

Conjugacy classes $M = \mathcal{C}$, $\Phi = \iota_{\mathcal{C}}$,

$$\omega(\xi^\#, \zeta^\#)_g = \tfrac{1}{2} B((\mathrm{Ad}_g - \mathrm{Ad}_{g^{-1}})\xi, \zeta)$$

$\boxed{\begin{array}{l}\textbf{Thm} \text{ (Bursztyn – Crainic) Conditions (2)–(4) above may be replaced by:} \\ \quad (*) \; \Phi \text{ is a strong Dirac map from } (M, Gr_\omega) \text{ to } (G, E_G)\end{array}}$

In particular, the $\mathfrak{g}$ - action comes for free!

Remark: More generally, a Dirac manifold (M, E_M) together with a Dirac map Φ to (G, E_G) s.th. $\Phi^* \mathcal{Y}_G = \mathcal{Y}_M$ is equivalent to a Hamiltonian $\mathfrak{g}$-Poisson manifold as defined by Alekseev – Kosmann-Schwarzbach – M.

The new viewpoint is extremely useful! E.g.

$\boxed{\begin{array}{l}\text{Thm: For any } q\text{-Hamiltonian } G\text{-space } (M,\omega,\Phi) \\ \exists \text{ an invariant volume form} \\ \qquad (\exp(\omega)\,\Phi^*\psi)^{top} \neq 0 \\ \text{where} \\ \qquad \psi_g = \det{}^{1/2}\left(\frac{Ad_g+1}{2}\right)\ \exp\left(\frac{1}{2}B\left(\frac{Ad_g-1}{Ad_g+1}\theta^L,\theta^L\right)\right)\end{array}}$

Immediate from our theory: volume form $\hat{=}$ pairing of pure spinors

<u>Remark</u>: Get bi-vector field $\pi \in \mathfrak{X}^2(M)$ with

$$\exp(\omega)\,\Phi^*\psi = e^{-\iota(\pi)}\left(\exp(\omega)\,\Phi^*\psi\right)^{top}$$

From equation for $(d+y)\psi$ one deduces

$$\frac{1}{2}[\pi,\pi] = \frac{1}{4}\Xi^{\#} \in \mathfrak{X}^3(M)$$

(tri-vector field defined by $\Xi \in \Lambda^3\mathfrak{g}$)

Fusion

Let $\mathrm{Mult}: G \times G \longrightarrow G$ group multiplication. Then

$$\mathrm{Mult}^* \eta = \mathrm{pr}_1^* \eta + \mathrm{pr}_2^* \eta + d\tau \;,$$
$$\tau = \tfrac{1}{2} B(\mathrm{pr}_1^* \theta^L, \mathrm{pr}_2^* \theta^R) \in \Omega^2(G \times G)$$

Put

$$\tilde{\varphi} = e^{-\tau}(\mathrm{pr}_1^* \varphi \wedge \mathrm{pr}_2^* \varphi) \in \Omega(G \times G)$$

<u>Thm</u> (Alekseev-Burszfyn-M) $\mathrm{Mult}: G \times G \to G$ is a strong Dirac map relative to Dirac structures defined by $\tilde{\varphi}, \varphi$.

(Cf: Add : $\mathfrak{g}^* \times \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ is a Poisson map.)

Thm: Suppose (M_i, ω_i, Φ_i) are q-Hamiltonian $\mathfrak{g}$-spaces. Let

$M = M_1 \times M_2$ diagonal $\mathfrak{g}$-action

$\omega = \omega_1 + \omega_2 + (\Phi_1, \Phi_2)^* \tau$

$\Phi = \Phi_1 \cdot \Phi_2$ (pointwise product)

Then (M, ω, Φ) is a q-Hamiltonian $\mathfrak{g}$-space.

Pf: $\Phi_1 \times \Phi_2 : (M_1 \times M_2 ; e^{-\omega_1} e^{-\omega_2}) \longrightarrow (G \times G, \varphi^1 \wedge \varphi^2)$

is strongly Dirac

Mult : $(G \times G, e^{-\tau}(\varphi^1 \wedge \varphi^2)) \longrightarrow (G, \varphi)$

is strongly Dirac

$\Downarrow$

$\mathrm{Mult} \circ (\Phi_1 \times \Phi_2) : (M_1 \times M_2 , e^{-(\omega_1 + \omega_2 + (\Phi_1, \Phi_2)^* \tau)})$

$\rightarrow (G, \varphi)$

is strongly Dirac.

Q.E.D.

Examples

Suppose $\sigma: G \to G$ is an involution preserving B.
Let $H = G^\sigma$ its fixed point set.
Then $M = G/H$ with $\omega = 0$ is a $\mathfrak{g}$-Hamiltonian $\mathbb{Z}_2 \ltimes G$ - space.
(It's the conjugacy class of (σ, e).)

Hence $M \times M$ with diagonal action is a $\mathfrak{g}$-Hamiltonian G - space.

Special case:

$G = (G \times G)^\sigma$ $\qquad \sigma(g_1, g_2) = (g_2, g_1)$

$M = G \times G / G \cong G$

so $M^2 = G^2$ is a $\mathfrak{g}$-Hamiltonian $G \times G$ -space.
This is called the double. One computes

$$\Phi(a,b) = (ab, a^{-1}b^{-1})$$

$$\omega = \tfrac{1}{2} B(a^*\theta^L, b^*\theta^R) + \tfrac{1}{2}(a^*\theta^R, b^*\theta^L)$$

Fuse one more time, get

$D(G) := G^2$, $\Phi(a,b) = aba^{-1}b^{-1} = [a,b]$

Fuse several copies of $D(G)$:

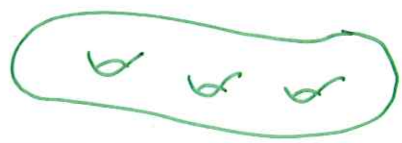
$M = \underbrace{D(G) \times \ldots \times D(G)}_{r} = G^{2r}$

$\Phi(a_1, b_1, \ldots, a_r, b_r) = \prod_{i=1}^{r} [a_i, b_i]$

$\omega = \ldots$

Thm (AMM) The 2-form ω descends to the standard (Atiyah-Bott) symplectic form on

$$M/\!/G = \Phi^{-1}(e)/G = \mathrm{Hom}(\pi_1(\Sigma), G)/G$$

$\Sigma =$ 

$\mathcal{A}(\Sigma) = \Omega^1(\Sigma, \mathfrak{g})$

$\cup$

Note: Construction of ω doesn't require compactness of G and also works in complex category.

$\mathcal{A}_{flat}(\Sigma)$

$M/\!/G \cong \mathcal{A}_{flat}(\Sigma)/\mathrm{Map}(\Sigma, G)$