Deformation of graded Poisson (Batalin-Vilkovisky) structures

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1. Introduction

We are interested in
the graded Poisson structures
Especially
the Batalin-Vilkovisky structures
Because

Mathematically

many geometrical and algebraic structures are described as a Batalin-Vilkovisky structure.

→ Unify "all" geometries
Classify "all" geometries
Find a new geometry

Physically

a Batalin-Vilkovisky structure is the fundamental structure of gauge theories. '81 Batalin-Vilkovisky

→ Unify "all" physics
Classify "
Find a new physics

Our tool is

deformation theory

based on the concept of

Kodaira-Spencer

First we consider

a simple BV-structure

(abelian topological sigma models)

and

construct all deformations.

Plan of my talk

§2 Graded Poisson Structures on the Graded Vector Bundles

§3 Batalin-Vilkovisky Structures of Abelian Topological Sigma Models

§4 Deformation Theory

§5 $n=2$
$n=3$

(§6 Quantum Version)

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2. Graded Poisson Structures on the Graded Vector Bundles (AKSZ formalism)

'95 Alexandrov, Kontsevich, Schwarz, Zaboronsky

M : a smooth manifold in d dimensions

T^*M : a cotangent bundle

<u>Def</u> ΠT^*M : supermanifold

is a cotangent bundle with reversed parity of the fiber.

- local coordinate

$\{\phi^i\}$ on M $\quad i = 1, 2, \cdots, d$

$\{B_i\}$ on $\Pi T^*_\phi M$

We assign a grading the total degree

$\{\phi^i\}$ is 0

$\{B_i\}$ 1

More generally.

We define a <u>graded cotangent bundle</u>

$T^*[p]M$. p : nonnegative integer

the total degree of the fiber is p.

$\{\phi^i\}$ total degree 0

$\{B_i\}$ 〃 p

Note) if p even $\Rightarrow$ B_i : commutative

odd $\Rightarrow$ B_i : anti commutative

∘ We can define

$T[p]M$: a graded tangent bundle

$E[p]$: a graded vector bundle

in a similar manner.

Generally we call <u>a graded manifold</u>

② P-structure

We consider a Poisson manifold N.

↓ shift the grading.

Then We obtain a (super) manifold $\tilde{N}$ and
graded

$$N \longrightarrow \tilde{N}$$

Poisson bracket $\qquad\qquad$ antibracket

$$\{*,*\} \longrightarrow (*,*)$$

It is called the P-structure.

Example 1 $\quad T^*M$

T^*M has a natural Poisson (symplectic) structure:

i.e. $\{F,G\} \equiv F\frac{\overleftarrow{\partial}}{\partial\phi^i}\frac{\overrightarrow{\partial}}{\partial B_i}G - F\frac{\overleftarrow{\partial}}{\partial B_i}\frac{\overrightarrow{\partial}}{\partial\phi^i}G$

F, G : functions on T^*M

ϕ^i, B_i : Darboux coordinate

shift to

antibracket on $T^*[p]M$

$$\boxed{(F,G) \equiv F\frac{\overleftarrow{\partial}}{\partial\phi^i}\frac{\overrightarrow{\partial}}{\partial B_{p,i}}G - F\frac{\overleftarrow{\partial}}{\partial B_{p,i}}\frac{\overrightarrow{\partial}}{\partial\phi^i}G}$$

where $B_{p,i}$: total degree p

This antibracket satisfies

$$(*)\begin{cases} (F,G) = -(-)^{(|F|-p)(|G|-p)}(G,F) \\ (F,GH) = (F,G)H + (-)^{(|F|-p)|G|}G(F,H) \\ (FG,H) = F(G,H) + (-)^{G(|H|-p)}(F,H)G \\ (-1)^{(|F|-p)(|H|-p)}(F,(G,H)) \\ \quad + (\text{cyclic permutation of } F\,G\,H) = 0 \end{cases}$$

where $|F|, |G|, |H|$ are the total degrees of F, G & H.

$(*,*)$ carries the total degree $-p$

<u>Def</u> <u>An antibracket</u> on a graded mfd $\tilde{N}$ with a total degree $-p$ is a bilinear form which satisfies $(*)$.

Example 2 $\quad E \oplus E^*$

has a natural pairing on the fiber

$\{A^a\}$ local coordinate on the fiber of E

$\{B_a\}$ local coordinate on the fiber of E^*

We can define a Poisson bracket

$$\{F, G\} \equiv F \frac{\overleftarrow{\partial}}{\partial A^a} \frac{\overrightarrow{\partial}}{\partial B_a} G - F \frac{\overleftarrow{\partial}}{\partial B_a} \frac{\overrightarrow{\partial}}{\partial A^a} G$$

If we consider a graded bundle

$E[p] \oplus E^*[q]$

the Poisson bracket shifts to an antibracket

$$\boxed{(F, G) \equiv F \frac{\overleftarrow{\partial}}{\partial A_p^a} \frac{\overrightarrow{\partial}}{\partial B_{q,a}} G - (-1)^{pq} F \frac{\overleftarrow{\partial}}{\partial B_{q,a}} \frac{\overrightarrow{\partial}}{\partial A_p^a} G}$$

$(*, *)$ total degree $-p-q$

$a = 1, \cdots, \mathrm{rank}\, E$

this antibracket satisfies ($\star$).

② Q-structure

Def S : a functional on a graded bundle is called a Batalin-Vilkovisky (BV) action (an action) if

S satisfies the classical master equation

$$(S, S) = 0$$

It is called Q-structure.

Def a Batalin-Vilkovisky structure is P-structure & Q-structure on a graded manifold $\hat{N}$.

Def $(S, *) = \delta(*)$ is called the BRST transformation which defines coboundary operator δ

s.t. $\boxed{\delta^2 = 0}$ $\because) \ (S, S) = 0$

3 Batalin-Vilkovisky Structures of Abelian Topological Sigma Models

Now we consider a sigma model.
(toplogical)

We introduce two manifolds X and M.

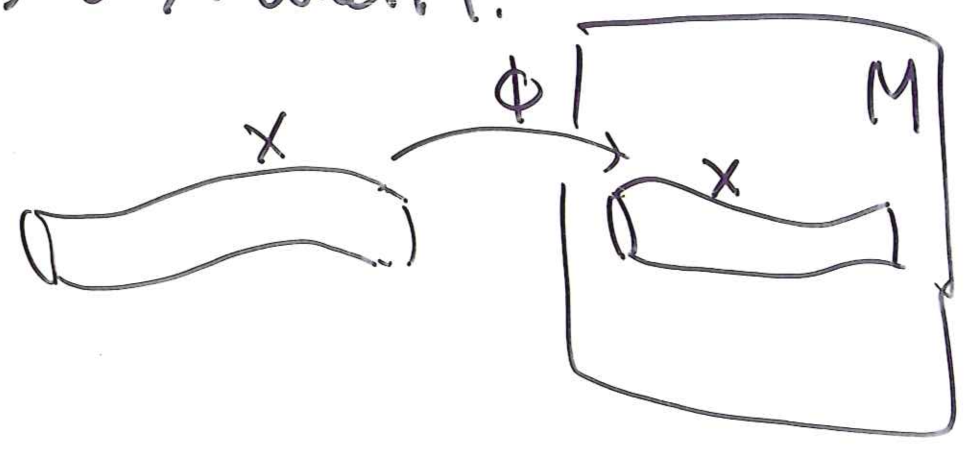
And consider a smooth map $\phi : X \to M$.

○A sigma model

is a classical or quantum field theory on X constructed from a map ϕ (& auxially other maps).

○A topological sigma model

is a sigma model independent of metrics on X and M.



X : n-dim is called world sheet ($n=2$)
world volume ($n \geq 3$)

M : d-dim target space

◎ We extend a map $\phi : X \to M$
to $\boldsymbol{\phi} : T[1]X \to M$

◎ Next, We construct the BV structure
from $\boldsymbol{\phi} : T[1]X \to M$

From now on, we consider
two cases T^*M & $E \oplus E^*$

3.1 P-structure

Example 1: T^*M

The dimension n of X labels a natural grading on $T^*[p]M$ as

$p = n-1$

Antibracket on $T^*[n-1]M$

$$\boxed{(F, G) \equiv F \frac{\overleftarrow{\partial}}{\partial \Phi^i} \frac{\overrightarrow{\partial}}{\partial \mathbb{B}_{n-1,i}} G - F \frac{\overleftarrow{\partial}}{\partial \mathbb{B}_{n-1,i}} \frac{\overrightarrow{\partial}}{\partial \Phi^i} G}$$

$\Phi^i \in \Pi T^*X \otimes M$

$\mathbb{B}_{n-1,i} \in \Gamma\left(\wedge^{n-1}_{\Pi} T^*X \otimes \Phi^*(T^*[n-1]M)\right)$

$(\, , \,)$ has the total degree $-n+1$

Example 2 $E[p] \oplus E^*[q]$

In order to make the total degree of an antibracket $-n+1$, we take

$q = n-p-1$.

We consider

$$E[p] \oplus E^*[n-p-1]$$

the antibracket is

$$\boxed{(F,G) \equiv F \frac{\overleftarrow{\partial}}{\partial A_p^{a_p}} \frac{\overrightarrow{\partial}}{\partial B_{n-p-1\, a_p}} G - (-)^{np} F \frac{\overleftarrow{\partial}}{\partial B_{n-p-1, a_p}} \frac{\overrightarrow{\partial}}{\partial A_p^{a_p}} G}$$

where $A_p^{a_p} \in T(\wedge^p T T^* X \otimes \phi^*(E[p]))$

$B_{n-p-1, a_p} \in T(\wedge^{n-p-1} T T^* X \otimes \phi^*(E^*[n-p-1]))$

$(\ , \ )$ has the total degree $-n+1$.

We want to consider $T^*[n-1]M$ & $E[p] \oplus E^*[n-p-1]$ simultaneously. $(p=1,2,\cdots)$

We identify

$$E^*[n-p-1] \oplus (E^*)^*[p] \triangleq E[p] \oplus E^*[n-p-1].$$

as a dual space.

$\lfloor x \rfloor$: the floor fn which gives the largest integer less than or equal to x.

Let E_p be $\lfloor \frac{n-1}{2} \rfloor$ series of vector bundles, $1 \leq p \leq \lfloor \frac{n-1}{2} \rfloor$

We consider a vector bundle

$$\boxed{\left(\sum_{p=1}^{\lfloor \frac{n-1}{2} \rfloor} E_p[p] \oplus E_p^*[n-p-1] \right) \oplus T^*[n-1]M}$$

the combination of Ex. 1 & 2

The antibracket is the combination of Ex. 1 & 2

$$\boxed{(F,G) \equiv \sum_{p=0}^{\lfloor \frac{n-1}{2} \rfloor} F \frac{\overleftarrow{\partial}}{\partial A_p^{a_p}} \frac{\overrightarrow{\partial}}{\partial B_{n-p-1,a_p}} G - (-)^{np} F \frac{\overleftarrow{\partial}}{\partial B_{n-p-1,a_p}} \frac{\overrightarrow{\partial}}{\partial A_p^{a_p}} G}$$

where $p=0$ part is the antibracket on

$T^*[n-1]M$,

$A_0{}^{a_0} = \phi^i$

$(\, , \,)$ total degree $-n+1$

<u>Def</u> the form degree is a grading on $\Pi T^* X$

the form degree 0 on the base mfd X

" 1 on the fiber $\Pi T^*_\sigma X$

$\deg F$: form degree of F

<u>Def</u> the ghost number $gh\, F$ is defined $\boxed{gh F = |F| - \deg F}$

3-2 Q-structure

A simplest natural Q-structure is

(Q)
$$\boxed{S_0 = \sum_{p=0}^{\left[\frac{n-1}{2}\right]} (-1)^{n-p} \int_X \mathbb{B}_{n-p-1, a_p} d\mathbb{A}_p{}^{a_p}}$$

where d: exterior differential on X

$\int_X$: integration over the n-form part

analogy of $\underline{\theta = p_i dq^i}$ a fundamental form for a symplectic form

$\omega = dq^i \wedge dp_i = -d\theta$.

It is called <u>an abelian BF theory in n dim</u>

We can conform

$(S_0, S_0) = 0$

i.e. S_0 is Q-structure.

$|S_0| = n$ total degree

4. Deformation Theory

'99 Izawa
'00~ N. I.

<u>Def</u> <u>Deformation of the BV structures</u>

Fix a <u>P-structure</u> $(*,*)$ on a graded bundle.

Consider infinitesimal defomations of <u>Q-structure</u> where $S = \int_X F(A_p, B_{n-p-1})$

$$S = S_0 + g S_1 + g^2 S_2 + \cdots$$

Such that $(S, S) = 0$

up to $S' = S + \underbrace{\delta_0 (*)}$ ← δ_0 exact

$\Rightarrow S' \underset{\text{equivalent}}{\sim} S$

where g: constant (deformation parameter)

$$\delta_0 (*) \equiv (S_0, *)$$

<u>Note)</u> $\delta_0^2 = 0$; We look for δ_0-cohomology class.

$$S = S_0 + \underbrace{g S_1 + g^2 S_2 + \cdots}_{g S_1'}$$

We set

We consider the total bundle

$$\boxed{\left(\sum_{p=1}^{\lfloor \frac{n-1}{2} \rfloor} E_p[p] \oplus E_p^*[n-p-1]\right) \oplus T^*[n-1]M}$$

P-structure

$$\boxed{(F, G) = \sum_{p=0}^{\lfloor \frac{n-1}{2} \rfloor} F \frac{\overleftarrow{\partial}}{\partial A_p{}^{a_p}} \frac{\overrightarrow{\partial}}{\partial B_{n-p-1, a_p}} G - (-)^{n-p} F \frac{\overleftarrow{\partial}}{\partial B_{n-p-1, a_p}} \frac{\overrightarrow{\partial}}{\partial A_p{}^{a_p}} G}$$

Q-structure

$$\boxed{S_0 = \sum_{p=0}^{\lfloor \frac{n-1}{2} \rfloor} (-1)^{n-p} \int_X B_{n-p-1, a_p} \, dA_p{}^{a_p}}$$

Now we assume

1. For an arbitrary function G

$$\int_X dG(\Phi, d\Phi) = 0$$

where $\Phi = A_p{}^{a_p}$ or $B_{n-p-1,\, a_p}$

no obstruction

$\Leftrightarrow H^2(\delta_0) = 0$ no D-brane

2. total degree

$$|S| = |S_1| = n$$

is physically required but mathematically can be relaxed.

Procedure

$$S = S_0 + g S_1'$$

Since

$$(S,S)=0 \Rightarrow (S_0+gS_1', S_0+gS_1')=0$$

We obtain

(i) g^0 $\quad (S_0,S_0)=0$ $\qquad$ ok

(ii) g^1 $\quad (S_0,S_1')=0$

Since

$$(S_0, \mathbb{A}_p{}^{a_p}) = d\mathbb{A}_p{}^{a_p}$$

$$(S_0, \mathbb{B}_{n-p-1,a_p}) = d\mathbb{B}_{n-p-1,a_p}$$

The general solution is

$$S_1' = \int_X F(\Phi, d\Phi)$$

F: an arbitrary function

$(S_0,S_1')=0$ $\qquad$ assumption 1

Theorem

$\boxed{\text{If a monomial } F(\Phi, d\Phi) \text{ includes at least one } d\Phi \text{ and } (S_0, \int_X F)=0, \text{ then } \int_X F(\Phi, d\Phi) \text{ is } \delta_0\text{-exact.}}$

∵) F includes at least one d ⟹

$$\int_X F(\Phi, d\Phi) = \sum_{p=0}^{n-1} \int_X F_{n-p-1}\, dG_p$$

F_{n-p-1} : form degree $n-p-1$

G_p : 〃 p

expansion by the form degrees

$(S_0, \int_X F)=0$

$$\Rightarrow (*) \begin{cases} \delta_0 F_0 = 0 \\ \delta_0 F_{n-p-1} = dF_{n-p-2} & -1 \le p \le n-2 \\ dF_n = 0, \quad \delta_0 G_0 = 0 \\ \delta_0 G_p = dG_{p-1} & 1 \le p \le n \\ dG_n = 0 \end{cases}$$

for even p adjacent 2 terms

$$F_{n-p-1} dG_p + F_{n-p-2}\, dG_{p+1}$$

$$\overset{(*)}{=} (-)^{n-p-1} \delta_0 (F_{n-p-1} G_{p+1}) - (-)^{n-p-1} d (F_{n-p-2} G_{p+1})$$

Integration of the above is δ_0-exact

If n even $\Rightarrow$ $\displaystyle\sum_{p=0}^{n-1} F_{n-p-1}\, dG_p$

even numbers of terms

$\Rightarrow$ Integration is δ_0-exact

If n odd $\Rightarrow$ $\displaystyle\sum_{p=0}^{n-1} F_{n-p-1}\, dG_p$

odd number of terms

but the last term

$$F_0\, dG_{n-1} \overset{(*)}{=} \delta_0 (F_0 G_n)$$

$\Rightarrow$ Integration is δ_0-exact //

From the theorem the general solution

$$\boxed{S_1' = \int_X F(\Phi)}$$

F : an arbitrary function

$$S_1' = \sum \int_X F_{a_{p(1)} \cdots a_{p(k)}}{}^{b_{q(1)} \cdots b_{q(l)}} (A_0{}^{a_p})$$

$$\times A_{p_1}{}^{a_{p_1}} \cdots A_{p_k}{}^{a_{p_k}} \mathbb{B}_{q_1 b_{q(1)}} \cdots \mathbb{B}_{q_l b_{q(l)}}$$

the total degree n terms

(assumption 2)

arbitrary fn F for $A_0{}^{a_p} = \phi^{a_p}$

(iii) g^2] $(S_1', S_1') = 0$

$\Rightarrow$ We obtain the conditions for the functions $F_{a\cdots a}{}^{b\cdots b}(\mathbb{A}_0{}^{a_p})$, which determine the algebraic and geometric structure.

The solution

$$S = S_0 + g S_1'$$

$$= \sum_{p=0}^{\lfloor \frac{n-1}{2} \rfloor} (-)^{n-p} \int_X \mathbb{B}_{n-p-1, a_p} d\mathbb{A}_p{}^{a_p}$$

$$+ g \int_X F(\Phi)$$

&

$$(S_1', S_1') = 0$$

5 Structures in lower n

5-1 $n=2$

the total bundle

$$T^*[1]M = \Pi T^*M$$

$\boxed{P}$

$$(F,G) = F \frac{\overleftarrow{\partial}}{\partial \phi^i}\frac{\overrightarrow{\partial}}{\partial B_{1,i}} G - F \frac{\overleftarrow{\partial}}{\partial B_{1,i}}\frac{\overrightarrow{\partial}}{\partial \phi^i} G$$

$\boxed{Q}$

$$S = S + gS_1'$$

$$= \int_X B_{1,i}\, d\phi^i + g\int_X \frac{1}{2} f^{ij}(\phi)\, B_{1i} B_{1j}$$

$$(S_1', S_1') = 0$$

$$\Longleftrightarrow f^{kl}\frac{\overrightarrow{\partial}}{\partial\phi^l} f^{ij} + f^{il}\frac{\overrightarrow{\partial}}{\partial\phi^l} f^{jk} + f^{jl}\frac{\overrightarrow{\partial}}{\partial\phi^l} f^{ki} = 0$$

$\Longleftrightarrow -f^{ij}\frac{\partial}{\partial\phi^i}\wedge\frac{\partial}{\partial\phi^j}$ is a Poisson bivector

the Poisson sigma model

$$\{F,G\} = -f^{ij}(\phi)\frac{\partial F}{\partial\phi^i}\frac{\partial G}{\partial\phi^j} = ((S,F),G)\big|$$

I: Lie algebroid and observables

Def Lie algebroid

- vector bundle $\mathcal{E} \to M$
- for section $e_1, e_2 \in \Gamma(\mathcal{E})$
  the bracket $[e_1, e_2]$ is defined
  with a Lie algebra str.
- bundle map (anchor)
  $\rho : \mathcal{E} \to TM$

s.t.

(1) for $\forall e_1, e_2 \in \Gamma(\mathcal{E})$

$$[\rho(e_1), \rho(e_2)] = \rho([e_1, e_2])$$

(2) for $\forall e_1, e_2 \in \Gamma(\mathcal{E})$, $F \in C^\infty(M)$

$$[e_1, F e_2] = F[e_1, e_2] + (\rho(e_1)F)e_2$$

If we define

$$[e_1, e_2] = ((S, e_1), e_2)$$

$$\rho(e)F(\phi) = (e, (S, F(\phi)))$$

where S : Poisson sigma model

$\mathcal{E}$ is Lie algebroid
$\Leftrightarrow (S, S) = 0$

where $\mathcal{M} = \{\phi^i : \Sigma \to M\}$
$\mathcal{E} = T^*\mathcal{M}$

Batalin-Vilkovisky structure of total degree 2 topological σ-model
$\cong$ Lie algebroid on $T^*\mathcal{M}$

Levin, Olshanetsky

5-2, $n=3$

<u>the total bundle</u>

$$(E[1]\oplus E^*[1])\oplus T^*[2]M$$

$\boxed{P}$ $$(F,G)=F\frac{\overleftarrow{\partial}}{\partial\phi^i}\frac{\overrightarrow{\partial}}{\partial B_{2i}}G-F\frac{\overleftarrow{\partial}}{\partial B_{2i}}\frac{\overrightarrow{\partial}}{\partial\phi^i}G$$
$$+F\frac{\overleftarrow{\partial}}{\partial A_1^i}\frac{\overrightarrow{\partial}}{\partial B_{1i}}G+F\frac{\overleftarrow{\partial}}{\partial B_{1i}}\frac{\overrightarrow{\partial}}{\partial A_1^i}G$$

$\boxed{Q}$ $$S=S_0+gS_1'$$

$$S_0=\int_X -B_{2i}d\phi^i+B_{1a}dA_1^a$$

$$S_1'=\int f_{1a}{}^i(\phi)A_1^aB_{2i}+f_2^{ib}(\phi)B_{2i}B_{1b}$$
$$+\frac{1}{3!}f_{3abc}(\phi)A_1^aA_1^bA_1^c+\frac{1}{2}f_{4ab}{}^c(\phi)A_1^aA_1^bB_{1c}$$
$$+\frac{1}{2}f_{5a}{}^{bc}(\phi)A_1^aB_{1b}B_{1c}+\frac{1}{3!}f_6^{abc}(\phi)B_{1a}B_{1b}B_{1c}$$

$(S_1', S_1') = 0 \Leftrightarrow$

① $f_{1e}{}^{i} f_2{}^{\gamma e} + f_2{}^{ie} f_{1e}{}^{\gamma} = 0$

② $-\frac{\partial f_{1c}{}^{i}}{\partial \phi^{\gamma}} f_{1b}{}^{\gamma} + \frac{\partial f_{1b}{}^{i}}{\partial \phi^{\gamma}} f_{1c}{}^{\gamma} + f_{1e}{}^{i} f_{4bc}{}^{e} + f_2{}^{ie} f_{3ebc} = 0$

③ $f_{1b}{}^{\gamma} \frac{\partial f_2{}^{ic}}{\partial \phi^{\gamma}} - f_2{}^{\gamma c} \frac{\partial f_{1b}{}^{i}}{\partial \phi^{\gamma}} + f_{1e}{}^{i} f_{5b}{}^{ec} - f_2{}^{ie} f_{4eb}{}^{c} = 0$

④ $-f_2{}^{\gamma b} \frac{\partial f_2{}^{ic}}{\partial \phi^{\gamma}} + f_2{}^{\gamma c} \frac{\partial f_2{}^{ib}}{\partial \phi^{\gamma}} + f_{1e}{}^{i} f_6{}^{ebc} + f_2{}^{ie} f_{5e}{}^{bc} = 0$

⑤ $-f_{1[a}{}^{\gamma} \frac{\partial f_{4bc]}{}^{d}}{\partial \phi^{\gamma}} + f_2{}^{\gamma d} \frac{\partial f_{3abc}}{\partial \phi^{\gamma}} + f_{4e[a}{}^{d} f_{4bc]}{}^{e}$
$+ f_{3e[ab} f_{5c]}{}^{de} = 0$

⑥ $-f_{1[a}{}^{\gamma} \frac{\partial f_{5b]}{}^{cd}}{\partial \phi^{\gamma}} + f_2{}^{\gamma[c} \frac{\partial f_{4ab}{}^{d]}}{\partial \phi^{\gamma}} + f_{3eab} f_6{}^{ecd}$
$+ f_{4e[a}{}^{[c} f_{5b]}{}^{d]e} + f_{4ab}{}^{e} f_{5e}{}^{cd} = 0$

⑦ $-f_{1a}{}^{\gamma} \frac{\partial f_6{}^{bcd}}{\partial \phi^{\gamma}} + f_2{}^{\gamma[b} \frac{\partial f_{5a}{}^{cd]}}{\partial \phi^{\gamma}} + f_{4ea}{}^{[b} f_6{}^{cd]e}$
$+ f_{5e}{}^{[bc} f_{5a}{}^{d]e} = 0$

⑧ $-f_2{}^{\gamma[a} \frac{\partial f_6{}^{bcd]}}{\partial \phi^{\gamma}} + f_6{}^{e[ab} f_{5e}{}^{cd]} = 0$

⑨ $-f_{1[a}{}^{\gamma} \frac{\partial f_{3bcd]}}{\partial \phi^{\gamma}} + f_{4[ab}{}^{e} f_{3cd]e} = 0$

Theorem

$$(S_1', S_1') = 0 \Leftrightarrow ① \sim ⑨$$

$\Leftrightarrow$ a Courant algebroid on $E[1] \oplus E^*[1]$

$\Leftrightarrow$ $n=3$ Batalin-Vilkovisky structure

Courant algebroid

'90 Courant
'97 Liu, Weinstein, Xu
'01 Roytenberg

vector bundle $\mathcal{E} \to \mathcal{M}$

with $\langle \cdot, \cdot \rangle$: (graded) symmetric bilinear form

$\circ$: bilinear form (Dorfman bracket)

ρ : $\mathcal{E} \to T\mathcal{M}$ anchor

s.t.

1. $e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3)$
2. $\rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)]$
3. $e_1 \circ F e_2 = F(e_1 \circ e_2) + (\rho(e_1) F) e_2$
4. $e_1 \circ e_2 = \frac{1}{2} \mathcal{D} \langle e_1, e_2 \rangle$
5. $\rho(e_1) \langle e_2, e_3 \rangle = \langle e_1 \circ e_2, e_3 \rangle + \langle e_2, e_1 \circ e_3 \rangle$

where $e_1, e_2, e_3 \in \Gamma(\mathcal{E})$

$F \in C^\infty(\mathcal{M})$

$\mathcal{D} : \mathcal{M} \to \Gamma(\mathcal{E})$ s.t. $\langle \mathcal{D} F, e \rangle = \rho(e) F$

We take

$\mathcal{M} = \{\phi : X \to M\}$

fiber of $\mathcal{E}$: $E[1] \oplus E^*[1]$ bases $\{A^a, B_a\}$

If we set

$$\begin{cases} \langle e_1, e_2 \rangle \equiv (e_1, e_2) & \text{antibracket} \\ e_1 \circ e_2 \equiv ((S, e_1), e_2) & \\ \rho(e) F(\phi) \equiv (e, (S, F(\phi))) & \\ \mathcal{D}(*) \equiv (S, *) & \end{cases}$$

defined as derived bracket

then

$$\boxed{\begin{array}{l} \langle\ ,\ \rangle, \circ, \rho, \mathcal{D} \text{ is Courant algebroid} \\ \Longleftrightarrow (S, S) = 0 \quad \text{master eq} \end{array}}$$

'99 Roytenberg

S is $|S| = 3$

Therefore We call

S : <u>Courant sigma model</u>

The form degree n deformation defines an algebroid on the bundle defined from the master eq. on M

$$\boxed{(S,S)=0}$$

We name BV-n-algebroid

n-algebroid	classical	quantum
2 Lie algebroid	Poisson	$*$-product on Poisson
3 Courant	Courant algebroid	?
n n-algebroid	BV-n-algebroid	?

◎ Substructures of BV structures

$\underline{n=2}$

∘ Symplectic (Kähler)

A-model

$$S=\int_{X_2} \mathbb{B}_{1i} d\Phi^i + \int_{X_2} \frac{1}{2} f^{ij} \mathbb{B}_{1i} \mathbb{B}_{1j}$$

∘ complex

B-model $\quad i=(a,\bar{a})$

$$S=\int_{X_2} \mathbb{B}_{1a} d\Phi^a + \mathbb{B}_{1\bar{a}} d\Phi^{\bar{a}} - \mathbb{B}_{0\bar{a}} d\mathbb{A}_1^{\bar{a}} - \mathbb{B}_{1\bar{a}} \mathbb{A}_1^{\bar{a}}$$

∘ twisted Poisson

$$S=\int_{X_2} \mathbb{B}_{1i} d\Phi^i + \frac{g}{2} f^{ij} \mathbb{B}_{1i} \mathbb{B}_{1j} + \int_{X_3} \frac{1}{2} H_{ijk} d\Phi^i d\Phi^j d\Phi^k$$

$\underline{n=3}$

generalized complex structure

$$S=\int_{X_3} -B_{2i}\, d\phi^i + B_{1i}\, dA_1^i$$

$$+\int_{X_3} -J^i{}_j\, B_{2i} A^j - P^{ij} B_{2i} B_{1j}$$

$$+\frac{1}{2}\left(H_{ijk}+\frac{\partial Q_{jk}}{\partial \phi^i}\right) A_1^i A_1^j A_1^k$$

$$+\frac{1}{2}\left(-\frac{\partial J^k{}_j}{\partial \phi^i}+\frac{\partial J^k{}_i}{\partial \phi^j}\right) A^i A^j B_{1k}$$

$$+\frac{1}{2}\frac{\partial P^{jk}}{\partial \phi^i} A^i B_{1j} B_{1k}$$

where $\mathcal{J}=\begin{pmatrix} J & P \\ Q & -J^* \end{pmatrix}$ GCS

etc.

6 Quantum Deformation Theory

<u>Def</u> BV Laplacian

$$\Delta F \equiv \sum_{p=0}^{\left[\frac{n-1}{2}\right]} \frac{\overrightarrow{\partial}}{\partial A_p^{a_p}} \frac{\overrightarrow{\partial}}{\partial B_{n-p-1,a_p}} F$$

<u>property</u>

$$\Delta(FG) = (\Delta F)G + (-)^{(n+1)|F|}(F, G) + (-)^{|F|} F \Delta G$$

We add the <u>gauge fixing term</u> S_{GF}

$$S_q = S + S_{GF}$$

to restrict the theory on the "Lagrangian" submanifold of the total bundle.

S_q satisfies the quantum master equation

$$(S_q, S_q) - 2i\hbar \Delta S_q = 0$$

$$\hbar = g$$

<u>Def</u> <u>observable $\mathcal{O}$</u> : fn on the total bundle

$$(S_g, \mathcal{O}) - i\hbar \Delta \mathcal{O} = 0$$

the "correlation function"

$\langle \mathcal{O}_1 \cdots \mathcal{O}_k \rangle$ — path integral

$$= \int \prod_{p=0}^{\left[\frac{n-1}{2}\right]} \mathcal{D}A_p \mathcal{D}B_{n-p-1} \, \mathcal{O}_1 \cdots \mathcal{O}_k \, e^{\frac{i}{\hbar} S_g}$$

define the geometric or algebraic str.

@ $n=2$ known

quantization of the Poisson sigma model on X = disc

$\Longleftrightarrow$ deformation quantization on a Poisson manifold

'97 Kontsevich

'99 Cattaneo-Felder

In $n=2$

$$S_0$$
$$\downarrow \text{ deformation}$$
$$S = S_0 + g S_1'$$

provides a quantum deformation

$$F(x)\,G(x) = \int_{\phi(\infty)=x} \mathcal{D}\phi\,\mathcal{D}B_1\,F(\phi(1))\,G(\phi(0)) \times e^{\frac{i}{\hbar}(S_0 + S_{GF})}$$

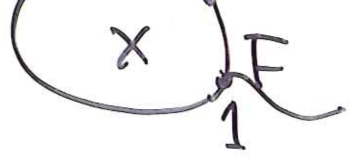
$$\downarrow \text{ star deformation}$$

$$F \star G(x) = \int_{\phi(\infty)=x} \mathcal{D}\phi\,\mathcal{D}B_1\,F(\phi(1))\,G(\phi(0)) \times e^{\frac{i}{\hbar}(S + S_{GF})}$$

$\phi(\infty) = x$

F, G : fn on M Observables

$\hbar = g$



8 Other Results

- Dimensional reduction $n=3 \to n=2$

  Courant algebroid is formulated as a $n=2$ topological sigma model

- complex structure (B-model)

  Kähler structure (A-model)

  generalized geometry (Kapstin, Zucchini, N.I.)

  twisted Poisson str. (WZ-Poisson)

  Contact (Jacobi) str. (Jacobi σ-model)

  are formulated as a sub structure of our model

- Many geometric structures can be formulated as a BV structure

$\Rightarrow$ unification of geometries?
(theory of every geometries?)

n	classical	quantum
2	Poisson	$*$-product
	complex B	B-model
	Kähler A	A-model
	contact (Jacobi)	
	symplectic	
	......	
3	Courant	corresponding quantum geometry
	Dirac	
	generalized geometry	
	twisted Poisson	
	topological M	
	G_2	
⋮		
n	topological membrane	

The list includes many geometries but have not completed yet.

Problems

- Characterization of <u>n-algebroid</u>
  classicaly $(n \geq 4)$
  quantum $(n \geq 3)$

- Extension to the other graded bundles.

Formulate a given geometry as a (graded) Poisson geometry.