Deformation quantization and reduction

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Introduction and notations

(M, π) Poisson manifolds: $\pi \in \Gamma(\Lambda^2 TM)$ Poisson bivector field,

$$\{ f , g \} := \pi(\mathrm{d}f, \mathrm{d}g).$$

- $\pi^\sharp : T^*M \to TM$:

$$\pi^\sharp(\alpha)(\beta) = \pi(\alpha, \beta) \qquad \alpha, \beta \in \Omega^1(M).$$

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Deformation quantization. . .

. . . means finding an associative product $\star$ on $C^\infty(M)[[\epsilon]]$ s.t.:
$f \star g = fg + \sum_{n=0}^{\infty} \epsilon^n B_n(f,g)$,

$1 \star f = f \star 1 = f$,

$\{ f \, , \, g \} = B_1(f,g) - B_1(g,f)$.

Theorem 1 (Kontsevich). *Every Poisson manifold has a deformation quantization.*

Remark 1. For other Poisson algebras there is no existence result in general. We will have to consider such a case.

Remark 2. Kontsevich's result is a corollary of the formality theorem. It may also be obtained from the Poisson sigma model.

Reduction

If (M, ω) is a symplectic manifold and C a submanifold:

Define $T^{\perp}C := \bigcup_{x \in C} (T_x M)^{\perp} \subset T_C M$.

$D := T^{\perp}C \cap TC$ a subbundle of $TC \implies$ integrable distribution on C (presymplectic submanifold).

$\underline{C} := C/D$ a manifold $\implies$ it inherits a symplectic structure.

Special case: $T^{\perp}C \subset TC$ (coisotropic).

Again not so special:
Lemma 2. *Given C presymplectic submanifold of M, $\exists C' \subset M$ s.t. 1) C' symplectic submanifold and 2) C coisotropic in C'.*

Now let (M, π) be a Poisson manifold and C a submanifold.

Let N^*C be the conormal bundle of C:

$$0 \to N^*C \to T^*_C M \to T^*C \to 0$$

dual to

$$0 \to TC \to T_C M \to NC \to 0$$

NC normal bundle.

Definition 1. C coisotropic if $D := \pi^\sharp(N^*C) \subset TC$.

Then D is a (singular) integrable distribution on C.

If $\underline{C} := C/D$ is a manifold, then it inherits a Poisson structure.

Remark 3. (M, ω) symplectic, $\pi = \omega^{-1}$: $\pi^\sharp(N^*C) = T^\perp C$.

Other examples

Example 1. M is coisotropic in M for every Poisson structure.

Example 2. $\pi \equiv 0$. Any submanifold is coisotropic.

Example 3. M, N Poisson manifolds, Φ a map $M \to N$.

$\mathrm{Graph}(\Phi)$ coisotropic in $\overline{M} \times N \iff \Phi$ is a Poisson map.

Example 4. $\mathfrak{g}$ a Lie algebra, $M = \mathfrak{g}^*$ with Kirillov–Kostant Poisson structure (i.e., $\{\, X \,,\, Y \,\} := [\, X \,,\, Y \,]$, $X, Y \in \mathfrak{g} \subset C^\infty(M)$).

Let $\mathfrak{h}$ be a subspace of $\mathfrak{g}$ and h^0 its annihilator.

$C = \mathfrak{h}^0$ is coisotropic in $\mathfrak{g}^*$ iff $\mathfrak{h}$ is a Lie subalgebra.

More generally, $C = \mathfrak{h}^0 + \lambda$ is coisotropic for $\mathfrak{h}$ Lie subalgebra and λ a character of $\mathfrak{h}$ (i.e., $\lambda([\, X \,,\, Y \,]) = 0 \ \forall X, Y \in \mathfrak{h}$).

Algebraic description

Let A be a Poisson algebra. An ideal I in A as a commutative algebra is called a coisotrope if it also is a Lie subalgebra. Then A/I is an I-module.

Lemma 3. *If I is the vanishing ideal of a closed submanifold C of a Poisson manifold M and $A = C^\infty(M)$, then C is coisotropic iff I is a coisotrope. Moreover, $C^\infty(\underline{C}) = (A/I)^I$. If $\underline{C}$ is not a manifold, take it as a definition.*

In Dirac's terminology: first-class constraints.

Let $N(I) := \{a \in A : \{a, I\} \subset I\}$ (normalizer). Then

$N(I)$ is a Poisson subalgebra. I is a Poisson ideal in $N(I)$. $N(I)/I$ is isomorphic to $(A/I)^I$ as a commutative algebra.

So one may induce a Poisson structure on $(A/I)^I$. Is there a deformation quantization of it?

Coordinate description

Let $\{x^I\}$ be adapted local coordinates of M. That is, locally C is given by $x^I = 0$, $I > k = \dim C$.

Notation: $i, j, \ldots$ for $I \leq k$ and $\mu, \nu, \ldots$ for $I > k$. So C is given by $x^\mu = 0$.

C coisotropic iff $\pi^{\mu\nu}|_C = 0$.

$f \in C^\infty(C)$ descends to $\underline{C}$ iff $\pi^{\mu i}\partial_i f = 0$.

Poisson bracket on $\underline{C}$: $\{\, f \,,\, g \,\}_{\underline{C}} := \pi^{ij}\partial_i f \partial_j g$, $f, g \in \mathrm{C}^\infty(C)$ invariant.

Strongly regular submanifolds

A good generalization of the notion of presymplectic submanifold is a submanifold C of M such that $\mathrm{rank}(TC + \pi^\sharp N^*C)$ is constant (strongly regular submanifold [Calvo–Falceto]).

Remark 4. If M is symplectic, then strongly regular = symplectic.

It turns out that $K := (\pi^\sharp)^{-1}TC \cap N^*C$ is a subbundle (actually a subalgebroid of T^*M) and $\pi^\sharp(K)$ integrable characteristic distribution.

Again, coisotropicity is not so special:

Theorem 4 (Calvo–Falceto, C-Zambon)**.** *C strongly regular in $M \Longrightarrow \exists C' \subset M$ Poisson–Dirac submanifold which contains C as a coisotropic submanifold.*

C' Poisson–Dirac: for every symplectic leaf $\mathcal{O}$ of M, $C' \cap \mathcal{O}$ symplectic in $\mathcal{O}$. C' inherits a Poisson structure with these intersections as its symplectic leaves.

The Poisson sigma model [Schaller–Strobl, Ikeda]. . .

. . . is a 2D topological field theory with target a Poisson manifold (M, π) which, among other things, produces Kontsevich's star product [C–Felder].

$$S(X, \eta) := \int_\Sigma \langle \, \eta \, , \, \mathrm{d}X \, \rangle + \frac{1}{2} \langle \, \eta \, , \, \pi^\sharp(X)\eta \, \rangle \, ,$$

where

Σ is a 2-manifold

X is a map $\Sigma \to M$ and $\eta \in \Gamma(T^*\Sigma \otimes X^*T^*M)$; in other words (X, η) is a bundle map $T\Sigma \to T^*M$.

Regard $\mathrm{d}X$ as a section of $T^*\Sigma \otimes X^*TM$ and $\pi^\sharp(X)$ of $X^* \operatorname{Hom}(T^*M, TM)$.

$\langle \; , \; \rangle$ is pairing of TM and T^*M.

Boundary conditions ("branes")

Variational principle:

$$\delta S = \int_\Sigma \langle\, \delta\eta\,,\, \mathrm{d}X\,\rangle + \langle\, \eta\,,\, \mathrm{d}\delta X\,\rangle + \cdots = \int_\Sigma (\langle\, \delta\eta\,,\, \mathrm{d}X\,\rangle + \langle\, \mathrm{d}\eta\,,\, \delta X\,\rangle) - \int_{\partial\Sigma} \langle\, \eta\,,\, \delta X\,\rangle + \cdots$$

To eliminate the boundary contribution, we have either to fix the variation of X and or set η to 0. More precisely: Fix $C \subset M$ and require

$$X(\partial\Sigma) \subset C, \qquad \iota^*_{\partial\Sigma}\eta \in \Gamma(T^*\partial\Sigma \otimes X^*N^*C).$$

Notation: $\mathcal{M}_C := \{$ pairs (X, η) satisfying these boundary conditions$\}$.

What kind of C may be chosen?

Hamiltonian approach

Let $\Sigma = I \times \mathbb{R}$. Write $\eta = \lambda \mathrm{d}t + \zeta$, with $I = [0,1]$ and $t \in \mathbb{R}$ with λ a path in $\Gamma(I, X^*T^*M)$ and ζ a path in $\Gamma(I, T^*I \otimes X^*T^*M)$.

Let $PM = \mathrm{Map}(I, M) \ni X$. We think of $\Gamma(I, T^*I \otimes X^*T^*M)$ as its cotangent fiber.

So $(X, \zeta) \in T^*PM$ with canonical symplectic structure.

λ is a Lagrange multiplier imposing: $\mathrm{d}X + \pi^\sharp(X)\zeta = 0$.

Let $\mathcal{C}$ be the space of solutions. And $\mathcal{C}_C := \{(X,\zeta) \in \mathcal{C} : X(0) \in C\}$. Then

1. $\mathcal{C}_C$ coisotropic in T^*PM iff C coisotropic in M [C–Felder].
2. $\mathcal{C}_C$ presymplectic in T^*PM iff C strongly regular in M [Calvo–Falceto].

Lagrangian approach

Thus, compatibility with symmetries $\Longrightarrow C$ strongly regular [Calvo–Falceto].
Remark 5. Rescale π to $\pi_\epsilon := \epsilon\pi$. Then $\forall\epsilon$, C still satisfies the strong regularity condition. Actually, $K_\epsilon := (\pi_\epsilon^\sharp)^{-1}TC \cap N^*C$ has the same rank $\forall\epsilon \neq 0$.

For $\epsilon = 0$, C is even coisotropic and $K_0 = N^*C$.

If we want to study the PSM perturbatively, we want the symmetries to vary smoothly with $\epsilon \to 0$. One may see that this is the case iff C is coisotropic [C–Felder].

Considering coisotropic submanifolds is anyway no loss of generality:

[Calvo–Falceto]: Let C' be Poisson–Dirac in M containing C as a coisotropic submanifold. Then PSM on M with boundary conditions on C is equivalent to that on C' with the same boundary conditions.

Perturbative expansion of the PSM

Let Σ be the disk. Let C be a coisotropic submanifold of M.

Given $f \in C^\infty(C)$ and $u \in \partial\Sigma$, let $\mathcal{O}_{f,u} := f(X(u))$.

Fix ∞ in $\partial\Sigma$. For $u < v \in \partial\Sigma \setminus \{\infty\}$ and $x \in C$, define

$$(f \star g)(x) := \int_{(X,\eta)\in\mathcal{M}_C : X(\infty)=x} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\,\mathcal{O}_{f,u}\,\mathcal{O}_{g,v}.$$

where the functional integral is defined in perturbation theory around the critical point $X \equiv x$, $\eta \equiv 0$.

For $C = M$:

The result does not depend on u and v.

It defines a star product for the given Poisson structure with $\epsilon = \mathrm{i}\hbar$.

This is actually Kontsevich's star product.

For $C \neq M$, one expects to get a star product for the given Poisson structure on $C^\infty(\underline{C})$ with $\epsilon = \mathrm{i}\hbar$. This actually not the case in general.

Problem 1: There exists a potential anomaly (in $H^2_\delta(N^*C)$ — Lie algebroid cohomology) which prevents this from working.

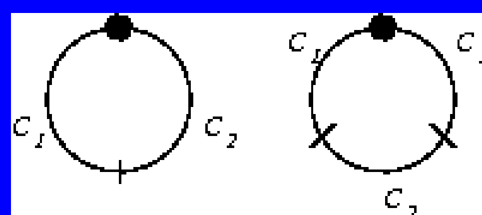
Problem 2: In the absence of anomaly, one gets an associative algebra on quantum invariant elements of $C^\infty(C)[[\epsilon]]$. The condition of quantum invariance is a deformation of the classical invariance: $\pi^{\mu i}\partial_i f + \cdots = 0$. If $H^1_\delta = 0$, then $C^\infty(C)[[\epsilon]]^{\mathrm{inv}} \simeq (C^\infty(C)^{\mathrm{inv}})[[\epsilon]]$.

Many branes

It is possible to cut the boundary of the disk Σ into two pieces and associate to each of them a different coisotropic submanifold C_1, C_2.

If the anomalies vanish, this produces a bimodule structure on $C^\infty(\underline{C_1 \cap C_2})[[\epsilon]]$ for the deformation quantizations of $\underline{C_1}$ and $\underline{C_2}$.

Putting three different coisotropic submanifolds on three different components of the boundary yields morphisms of bimodules.



What happens with more than three boundary conditions? Not clear. Actually, the topology changes and some propagators can no longer be closed forms.

Comments

Quantization of coisotropic submanifolds can also be achieved by the BVF method. The present method seems to be more systematic.

The construction of bimodule structure is something new. In particular if applied to graphs, it yields a quantization of Poisson maps.

The construction of morphisms of bimodules may also be interesting. A dream is to use this procedure to get quantum groups by deformation quantization of Poisson–Lie groups.

The method has also interesting applications to Lie theory (work in progress with C. Torossian).

Graded manifold interpretation

The case with one brane can nicely be reinterpreted using graded manifolds.

The PSM with target a graded manifold has a duality corresponding roughly speaking to the exchange of X and η. This yields equivalent PSMs with "dual" targets $M_1 \leftrightarrow M_2$.

With boundary: $(M_1, C_1) \leftrightarrow (M_2, C_2)$. It is possible to get $C_2 = M_2$! (Technical tool: Legendre mapping by Tulczyjew, Mackenzie–Xu, Roytenberg.)

The result is actually more general:

C coisotropic implies N^*C Lie algebroid. On $\Gamma(\Lambda NC)$ there is an algebroid differential but also a flat L_∞-structure [Oh–Park]. Quantization yields a (possibly non flat) A_∞-structure on $\Gamma(\Lambda NC)[[\epsilon]]$.

With many branes: A_∞-bimodules, A_∞-morphisms of bimodules

The correct definition of the Poisson sigma model (including ghosts and antighosts) is:

$$S(\tilde{X}, \tilde{\eta}) := \int_\Sigma \left\langle \tilde{\eta} \, , \, \mathrm{d}\tilde{X} \right\rangle + \frac{1}{2} \left\langle \tilde{\eta} \, , \, \pi^\sharp(\tilde{X})\tilde{\eta} \right\rangle ,$$

with $(\tilde{X}, \tilde{\eta}) \in \mathrm{Map}(T[1]\Sigma, T^*[1]M)$. This fiber over $\mathrm{Map}(T[1]\Sigma, M) \ni \tilde{X}$. We denote by $\tilde{\eta}$ the element of the fiber.

The quadratice term $= \int_\Sigma \left\langle \tilde{\eta} \, , \, \mathrm{d}\tilde{X} \right\rangle$ is invariant under exchanging $\tilde{X} \leftrightarrow \tilde{\eta}$. Actually, it is invariant also under exchanging only some components.

Consequence: The PSM with target M is equivalent to the PSM for a different target M' corresponding to the above exchange. If C labels boundary conditions in M, we will get some C' labelling the corresponding boundary conditions in M'. It is is possible to choose M' so that $C' = M'$!

More on the quantization of a reduced space

Deformation quantization of Poisson manifolds is actually a corollary of Kontsevich's formality theorem.

To a coisotropic submanifold one may associate a graded manifold and get deformation quantization from the formality theorem of graded manifolds.

Summary:

1. L_∞ algebras and L_∞ morphisms

2. MC elements

3. Formality theorem for graded manifolds

L_∞

Let V be a graded vector space and SV its symmetric algebra, but regarded as a coalgebra. An L_∞-structure on $V[-1]$ consists of a coderivation D of degree 1 on SV s.t. $[\,D\,,\,D\,] = 0.$

$\mathrm{Coder}(SV) \simeq \mathrm{Hom}(SV, V)$, so D actually consists of multilinear operations $L_n \colon S^nV \to V$ satisfying quadratic relations. The L_∞-algebra is called flat if $L_0 = 0$. In this case L_1 is a differential for L_2 and H_{L_1} is a GLA.
Example 5. If $\mathfrak{g}$ is a DGLA, then it is a flat L_∞-algebra with $L_2 =$bracket and $L_1 =$differential, $L_n = 0\ n \neq 1, 2.$

An L_∞-morphism between two L_∞-algebras $V[-1]$ and $W[-1]$ is a coalgebra map $SV \to SW$ compatible with the coderivations.

Observe that such a map is determined by its projections $U_n \colon S^nV \to W$. We write $U \colon V[-1] \rightsquigarrow W[-1]$. If they are flat and $U_0 = 0$, then U_1 is a chain map.

MC

A MC element in an L_∞-algebra $\mathfrak{g}$ is an element $A \in \mathfrak{g}^1$ satisfying

$$\sum_n \frac{1}{n!} L_n(A, \ldots, A) = 0.$$

Example 6. If $\mathfrak{g}$ is a DGLA, then MC simply means $\mathrm{d}A + \frac{1}{2}[\,A\,,\,A\,] = 0$.

If $U \colon \mathfrak{g} \rightsquigarrow \mathfrak{h}$ is an L_∞-morphism and A is MC in $\mathfrak{g}$ then

$$B := \sum_n \frac{1}{n!} U_n(A, \ldots, A)$$

is MC in $\mathfrak{h}$. But the series should converge.

Use formal power series!

An L_∞-structure on $\mathfrak{g}$ extends to an L_∞-structure on $\epsilon\mathfrak{g}[[\epsilon]]$.

An L_∞-morphism $U\colon \mathfrak{g} \rightsquigarrow \mathfrak{h}$ determines an L_∞-morphism $U\colon \epsilon\mathfrak{g}[[\epsilon]] \rightsquigarrow \epsilon\mathfrak{h}[[\epsilon]]$.

If A is MC in $\mathfrak{g}$ and $\mathfrak{g}$ is a GLA, then ϵA is MC in $\epsilon\mathfrak{g}[[\epsilon]]$.

Then

$$B := \sum_n \frac{\epsilon^n}{n!} U_n(A, \ldots, A)$$

is MC in $\epsilon\mathfrak{h}[[\epsilon]]$.

Kontsevich's formality theorem

Let M be a smooth manifold. Let $\mathcal{V}(M) := \Gamma(\Lambda TM)$ be the GLA of multivector fields (with Schouten–Nijenhuis bracket). Let $\mathcal{D}(M)$ be the DGLA of multidifferential operators on M:

$$\begin{aligned} \phi \circ \psi &:= \sum_i \pm \phi(a_1, \ldots, a_i, \psi(a_{i+1}, \ldots, a_{i+l}), a_{i+l+1}, \ldots), \\ [\,\phi\,,\,\psi\,] &:= \phi \circ \psi \pm \psi \circ \phi, \\ \mathrm{d}\phi &:= [\,\mu\,,\,\phi\,], \qquad \mu \text{ multiplication.} \end{aligned}$$

Theorem 5 (Kontsevich). *There is an L_∞-q.i. $U\colon \mathcal{V}(M) \rightsquigarrow \mathcal{D}(M)$.*

π MC in $\mathcal{V}(M) \Leftrightarrow \pi$ Poisson structure.

B MC in $\epsilon\mathcal{D}(M)[[\epsilon]] \Leftrightarrow \mu + B$ star product.

Kontsevich's formality theorem extends to graded manifolds (i.e., supermanifolds with $\mathbb{Z}$-grading on coordinates. For us, this grading matches with parity.) [C-Felder]

Because of the grading a MC element π in $\mathcal{V}(M)$ is not necessarily a bivector field. Actually

$$\pi = \pi_0 + \pi_1 + \pi_2 + \cdots ,$$

π_0 a function of degree 2, π_1 a vector field of degree 1, π_2 a bivector field of degree zero, ...

Let $\lambda_0 := \pi_0$, $\lambda_1(f) := [\,\pi_1\,,\,f\,]$, $\lambda_2(f,g) = [\,[\,\pi_2\,,\,f\,]\,,\,g\,]$,. . . .

By a result of T. Voronov these derived brackets define an L_∞-structure on $C^\infty(M)$ iff π is MC.

B is MC in $\epsilon\mathcal{D}(M)[[\epsilon]] \Leftrightarrow \mu + B$ is an A_∞-structure on $C^\infty(M)[[\epsilon]]$. [C–Felder, Lyakhovic–Sharapov]

A_∞ is "like" L_∞ without graded commutativity.

In particular, if the degree-0 operation vanishes, the degree-one operation is a differential and the cohomology is an associative algebra.

Problem: $\pi_0 = 0$ does not imply $\mu + B$ flat.

One may see that $H^2_{\lambda_1} = \{0\}$ is a sufficient condition for $\mu + B$ to be equivalent to a flat one.

If this is the case, $H^0_{B_1}$ is an associative algebra "quantizing" a Poisson subalgebra of $H^0_{\lambda_1}$.

Legendre mapping

Let $E \longrightarrow M$ be a graded vector bundle. We consider it as a graded manifold with $C^\infty(E) := \Gamma(\hat{S}E^*)$.

Theorem 6 (Roytenberg, after Mackenzie-Xu, Tulczyjew). *$T^*[l]E$ and $T^*[l]E^*[l]$ are antisymplectomorphic $\forall l$.*

For a graded vector space: $T^*[l]V = V \oplus V^*[l]$ while $T^*[l]V^*[l] = V^*[l] \oplus V$.

Notation. $\mathcal{V}(E) =$ GLA of multivector fields.

$\mathcal{V}(E) = C^\infty(T^*[1]E)$. So $\mathcal{V}(E)$ and $\mathcal{V}(E^*[1])$ are antiisomorphic.

Application

Let $C \subset M$. Let NC be the normal bundle. Let $N[0]C$ be the graded manifold with functions $\Gamma(\hat{S}N^*C)$. Upon choosing an embedding of NC into M, we may regard $N[0]C$ as a formal neighborhood of C.

The Poisson bivector field on M then yields a Poisson bivector field π' on $N[0]C$.

The Legendre mapping maps π' to a MC element in $N^*[1]C$. This defines an L_∞-structure on $\Gamma(\Lambda NC)$. ▮

Flat iff C coisotropic. In this case, λ_1 is the Lie algebroid differential [Oh–Park].

Finally, U yields an A_∞-structure on $\Gamma(\Lambda NC)[[\epsilon]]$.

If $H^2_{\lambda_1} = \{0\}$, we may make it flat. It turns out that $H^0_{B_1} = A[[\epsilon]]$ where A is a Poisson subalgebra of $H^0_{\lambda_1} = C^\infty(\underline{C})$.

Studying the long exact sequence associated to

$$0 \to \Gamma(\Lambda NC)[[\epsilon]] \xrightarrow{\epsilon} \Gamma(\Lambda NC)[[\epsilon]] \to \Gamma(\Lambda NC) \to 0,$$

one may easily see that $H^1_{\lambda_1} = \{0\}$ implies $A = C^\infty(\underline{C})$.