Introduction to Deformation Quantization

Giuseppe Dito

Institut de Mathématiques de Bourgogne
UMR CNRS 5584
Université de Bourgogne, Dijon, France

The Fourth Afro-Franco-Brazilian Meeting
on Mathematics and Physics

IMSP, Dangbo – July 3 - 12, 2017

Plan of the mini-course

1. **Moyal quantization**
   - Hamiltonian Mechanics
   - Moyal star product
   - Relation with Quantum Mechanics
   - The exemple of the harmonic oscillator
2. **Basics of Deformation Quantization**
   - Basic concepts
   - Gerstenhaber Theory
   - Brief panorama of results

1 Moyal quantization

1.1 Hamiltonian Mechanics

- Traditionally, Hamilton's equations of motion of a mechanical system are formulated on a flat $2n$-dimensional phase space

- Fix canonical coordinates $(q,p) := (q^1,\ldots,q^n,p_1,\ldots,p_n)$ on phase space, one can identify the phase space with $\mathbb{R}^{2n}$.

- The classical observables are real-valued smooth functions defined on phase space, denoted $C^\infty(\mathbb{R}^{2n})$.

- For two such classical observables f and g, their Poisson bracket takes the form:

$$\{f,g\} = \sum_{i=1}^{n} \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i}\frac{\partial f}{\partial p_i}. \tag{1}$$

Two important algebraic structures defined on the space $C^\infty(\mathbb{R}^{2n})$.

- **Commutative algebra**. $C^\infty(\mathbb{R}^{2n})$ is a vector space endowed with the commutative associative product:

  $$(f,g) \mapsto f \cdot g,$$

  given by $(f \cdot g)(q,p) = f(q,p)g(q,p)$.

- **Lie algebra**. $C^\infty(\mathbb{R}^{2n})$ with the Poisson bracket is an infinite-dimensional Lie algebra.

Since the Poisson bracket involves only first derivatives, the Lie algebra $(C^\infty(\mathbb{R}^{2n}), \{\ ,\ \})$ enjoys a supplementary property, i.e., Leibniz rule applies to each argument of the Poisson bracket:

$$\{fg,h\} = f\{g,h\} + \{f,h\}g.$$

The space $C^\infty(\mathbb{R}^{2n})$ endowed with the pointwise product of functions and the Poisson bracket is called the *algebra of classical observables*.

The dynamics of a mechanical system is governed by its Hamiltonian H and Hamilton's equations of motion read

$$\begin{aligned} \frac{dq^i}{dt} &= \{q^i, H\} \\ \frac{dp_i}{dt} &= \{p_i, H\}. \end{aligned} \tag{2}$$

This is in essence how Classical Mechanics is cast into the Hamiltonian formalism as explained in any standard textbook on Classical Mechanics.

Time evolution
Suppose that at $t = 0$ the system is in the pure state $(q, p) \in \mathbb{R}^{2n}$.

- The system evolves according to Hamilton's equations, and describes a path in phase space

$$t \mapsto (\tilde{q}(t;q,p), \tilde{p}(t;q,p))$$

defined, in general, for t in a neighborhood of $t = 0$, and satisfying the initial conditions:

$$\tilde{q}(0;q,p) = q \quad \text{and} \quad \tilde{p}(0;q,p) = p.$$

- One has the Hamiltonian flow $\phi_t : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ defined by

$$\phi_t(q,p) = (\tilde{q}(t;q,p), \tilde{p}(t;q,p)).$$

.

- The Hamiltonian flow ϕ_t is a canonical transformation of phase space.
- Since Hamilton's equations are first-order differential equations (vector fields) the Hamiltonian flow has the group property: $\phi_t \circ \phi_s = \phi_{t+s}$ (for t and s small enough) and $\phi_{t=0}$ is the identity transformation.

There is a dual description of Hamilton's equations in the algebra of classical observables.

- Solutions of Hamilton's equation induce in a natural way a path $t \mapsto f_t$ starting at f in the algebra of observables $C^\infty(\mathbb{R}^{2n})$ by setting $f_t(q,p) := f(\phi_t(q,p))$.
- The path $t \mapsto f_t$ also satisfies Hamilton's equation:

$$\frac{df_t}{dt} = \{f_t, H\}, \qquad f_{t=0} = f. \tag{3}$$

- There is a one-to-one correspondence between paths in phase space which are solutions of Hamilton's equations and paths in the algebra of classical observables which are solutions of Hamilton's equation.

- The algebraic structure of the algebra of classical observables is compatible with time evolution, i.e. the Hamiltonian flow, in the sense that:

$$\begin{aligned} (fg)_t &= f_t g_t \\ \{f,g\}_t &= \{f_t, g_t\}. \end{aligned} \tag{4}$$

Remark. At the quantum level, this feature no longer survives. A similar property holds for other algebraic structures. Moreover, the one-to-one correspondence between paths in phase space and in $C^\infty(\mathbb{R}^{2n})$ does not hold in general. It is a manifestation of uncertainty principle and the non existence of trajectory in phase space.

1.2 The Moyal star product

The basic idea here is to deform the commutative algebra of classical observables into a noncommutative algebra by introducing a noncommutative associative product, the Moyal star product.

For reasons that will become clear later, one has to extend the algebra of classical observables to a larger space, namely, the vector space of formal power series in the parameter λ with coefficients in $C^\infty(\mathbb{R}^{2n})$.

- This space is denoted $\mathsf{A}_\lambda = C^\infty(\mathbb{R}^{2n})[\![\lambda]\!]$. An element f_λ in A_λ is simply a formal power series of functions:

$$f_\lambda = \sum_{r\geq 0} \lambda^r f_r, \qquad f_r \in C^\infty(\mathbb{R}^{2n}).$$

- The space A_λ inherits the commutative $\mathbb{R}$-algebra structure of $C^\infty(\mathbb{R}^{2n})$ in the sense that for any two elements of A_λ,

$f_\lambda = \sum_{r\geq 0} \lambda^r f_r$ and $g_\lambda = \sum_{r\geq 0} \lambda^r g_r$, addition, product by a scalar and product in A_λ are defined by:

$$f_\lambda + g_\lambda = \sum_{r\geq 0} \lambda^r (f_r + g_r),$$
$$\mu f_\lambda = \sum_{r\geq 0} \lambda^r (\mu f_r), \qquad \mu \in \mathbb{R},$$
$$f_\lambda g_\lambda = \sum_{r\geq 0} \lambda^r (\sum_{i+j=r} f_i g_j).$$

- Extending the multiplication by scalars to elements $\mu_\lambda = \sum_{r\geq 0} \lambda^r \mu_r \in \mathbb{R}[\![\lambda]\!]$ by

$$\mu_\lambda f_\lambda = \sum_{r\geq 0} \lambda^r (\sum_{i+j=r} \mu_i f_j)$$

gives to A_λ the structure of a commutative $\mathbb{R}[\![\lambda]\!]$-algebra.

Consider the algebra of classical observables $C^\infty(\mathbb{R}^{2n})$ endowed with the canonical Poisson bracket.

- Write $P(f,g)$ for the Poisson bracket of f and g.
- Label the $2n$ canonical variables (q,p) using a single symbol x by letting $x^i = q^i$ and $x^{n+i} = p_i$, for $1 \leq i \leq n$
- Introduce the following skew-symmetric matrix of order $2n$:

$$\Pi = \begin{pmatrix} \mathbb{O}_n & \mathbb{I}_n \\ -\mathbb{I}_n & \mathbb{O}_n \end{pmatrix}. \tag{5}$$

- The Poisson bracket takes the form

$$P(f,g) = \sum_{1\leq\alpha,\beta\leq 2n} \Pi^{\alpha\beta} \frac{\partial f}{\partial x^\alpha} \frac{\partial g}{\partial x^\beta}. \tag{6}$$

The rth-power of P is defined by the following expression

$$P^r(f,g) := \sum_{1\le\alpha_1,\beta_1,\ldots,\alpha_r,\beta_r\le 2n} \Pi^{\alpha_1\beta_1}\cdots\Pi^{\alpha_r\beta_r}\frac{\partial^r f}{\partial x^{\alpha_1}\cdots\partial x^{\alpha_r}}\frac{\partial^r g}{\partial x^{\beta_1}\cdots\partial x^{\beta_r}}. \tag{7}$$

One conventionally set $P^0(f,g) = fg$. The powers of P are bidifferential operations, i.e., (7) is a differential operator in each of its arguments.
The expansion of (7) gives the explicit form:

$$P^r(f,g) = \sum_{k=0}^{r}(-1)^{r-k}\binom{r}{k}\times\cdots$$
$$\times\sum_{1\le i_1,\ldots,i_r\le n}\frac{\partial^r f}{\partial q^{i_1}\cdots\partial q^{i_k}\partial p_{i_{k+1}}\cdots\partial p_{i_r}}\frac{\partial^r g}{\partial p_{i_1}\cdots\partial p_{i_k}\partial q^{i_{k+1}}\cdots\partial q^{i_r}}.$$

There is a useful equivalent way to define the rth-power of P by applying r times the differential operator in the $4n$ variables (q, p, q', p')

$$\tilde{P} = \sum_{i=1}^{n} \frac{\partial^2}{\partial q^i \partial p'_i} - \frac{\partial^2}{\partial q'^i \partial p_i}. \tag{8}$$

to the function $f(q, p)g(q', p')$ and evaluating on the diagonal $q' = q$ and $p' = p$:

$$P^r(f, g)(q, p) = \Big(\sum_{i=1}^{n} \frac{\partial^2}{\partial q^i \partial p'_i} - \frac{\partial^2}{\partial q'^i \partial p_i}\Big)^r (f(q, p)g(q', p'))\big|_{q'=q, p'=p}. \tag{9}$$

Definition 1. *The Moyal star product is the* $\mathbb{R}[\![\lambda]\!]$*-bilinear map* $\star\colon \mathsf{A}_\lambda \times \mathsf{A}_\lambda \to \mathsf{A}_\lambda$ *defined by*

$$f \star g = fg + \sum_{r\geq 1} \frac{\lambda^r}{r!} P^r(f,g), \quad f,g \in C^\infty(\mathbb{R}^{2n}). \tag{10}$$

For two formal power series $f_\lambda = \sum_{r\geq 0} \lambda^r f_r$ and $g_\lambda = \sum_{r\geq 0} \lambda^r g_r$, one has:

$$f_\lambda \star g_\lambda = \sum_{r\geq 0} \lambda^r \Big(\sum_{\substack{i+j+k=r \\ i,j,k\geq 0}} \frac{1}{k!} P^k(f_i, g_j) \Big). \tag{11}$$

Theorem 1. *The Moyal star product is an associative product on* A_λ *satisfying* $f \star 1 = 1 \star f = f$ *for any function* f.

Proof. The associativity condition is $(f \star g) \star h = f \star (g \star h)$ for arbitrary functions $f, g, h \in C^\infty(\mathbb{R}^{2n})$.
The associativity condition is an equality between formal series of tridifferential operators, hence it is sufficient to check the associativity when f, g and h are exponential functions:

$$f(x) = \exp(a \cdot x), \quad g(x) = \exp(b \cdot x), \quad h(x) = \exp(c \cdot x).$$

where a, b and c are $2n$-tuples. A simple computation leads to

$$(f \star g)(x) = \exp(\lambda \Pi(a,b)) \exp((a+b) \cdot x),$$

where $\Pi(a,b) = \sum_{1 \leq \alpha, \beta \leq 2n} \Pi^{\alpha\beta} a_\alpha b_\beta = \sum_{i=1}^{i=n} (a_i b_{n+i} - b_i a_{n+i})$.
The associativity condition is then equivalent to the following identity valid for any $2n$-tuples a, b and c:

$$\exp(\lambda \Pi(a,b)) \exp(\lambda \Pi(a+b,c)) = \exp(\lambda \Pi(a,b+c)) \exp(\lambda \Pi(b,c)).$$

□

- A_λ has the structure of a commutative $\mathbb{R}[\![\lambda]\!]$-algebra for the natural extension of the pointwise product. We have defined another associative product on A_λ.

- The term of order λ in the Moyal star product is the Poisson bracket, we get a noncommutative algebra $(\mathsf{A}_\lambda, \star)$ with unit 1. This algebra is called the *algebra of quantum observables*.

- Heuristically, as λ tends to zero, one recovers the algebra of classical observables. In that sense the Moyal star product is a noncommutative *associative deformation* of the usual product of functions. The idea to view Quantum Mechanics as a deformation of Classical Mechanics was introduced by Flato and his collaborators in the 70's and led to what is called *deformation quantization*.

- The algebra A_λ possesses in a natural way a Lie algebra structure for the star commutator defined by

$$[f,g]_\star = \frac{1}{2\lambda}(f \star g - g \star f). \quad (12)$$

The rescaling by a factor 2λ is to ensure that the classical term of the star commutator is exactly the Poisson bracket.

-

$$[f,g]_\star = \sum_{r\geq 0} \frac{\lambda^{2r}}{(2r+1)!} P^{2r+1}(f,g) = \{f,g\} + O(\lambda^2). \quad (13)$$

The star commutator is a Lie algebra deformation of the Poisson bracket and was studied by Moyal (1949) in relation with statistical properties of Quantum Mechanics.

- For a Hamiltonian H, the quantum equation of motion are given by the Heisenberg equations:

$$\frac{df_t}{dt} = [f_t, H]_\star = \{f_t, H\} + O(\lambda^2).$$

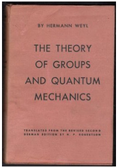
- It defines a trajectory in A_λ, but not on phase space in general as higher derivatives contribute in the equation of motion.

To sum up, the Moyal star product realizes the following correspondence between the algebras of classical and quantum observables:

$$\big(C^\infty(\mathbb{R}^{2n}), \cdot\, , \{\ ,\ \}\big) \rightsquigarrow \big(\mathsf{A}_\lambda, \star, [\ ,\ ]_\star\big).$$

Let us see what is the relation of Moyal quantization with the standard formulation of Quantum Mechanics.

1.3 Relation with Quantum Mechanics



Hermann Weyl (1885-1955)

In its standard framework, the quantization of the classical phase space $\mathbb{R}^{2n}$ is accomplished by representing the canonical coordinates (q, p) by unbounded self-adjoint operators $(\hat{q}, \hat{p})$ in a Hilbert space.

These operators are required to satisfy the canonical commutation relations (CCR) on a certain domain:

$$
\begin{aligned}
[\hat{q}^i, \hat{q}^j] &= 0 \\
[\hat{p}_i, \hat{p}_j] &= 0 \\
[\hat{q}^i, \hat{p}_j] &= \imath\hbar\delta^i_j,
\end{aligned}
\tag{14}
$$

for $i, j = 1, \ldots, n$. When $\hbar$ is seen as a central element, (14) defines the Heisenberg Lie algebra $\mathfrak{h}_{2n}$.

It is usually a delicate matter to manipulate unbounded operators and, as early as 1927, Weyl realized the advantages to deal with an integrated form of the CCR by introducing the n-parameter unitary groups:

$$U(a) = e^{\frac{\imath}{\hbar} a\cdot\hat{q}}, \quad V(b) = e^{\frac{\imath}{\hbar} b\cdot\hat{p}},$$

for any real n-tuples $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$.

These unitary (hence bounded) operators satisfy the Weyl relations:

$$
\begin{aligned}
U(a)U(b) &= U(b)U(a), \\
V(a)V(b) &= V(b)V(a), \\
U(a)V(b) &= e^{-\frac{\imath}{\hbar} a \cdot b} V(b)U(a).
\end{aligned}
$$

From the early days of Quantum Mechanics, a specific representation of the CCR was used by Schrödinger to write down his celebrated equation. It is nowadays known as the Schrödinger representation and it acts on the Hilbert space of square-integrable functions $L^2(\mathbb{R}^n, dx)$, where $dx = dx^1 \cdots dx^n$ is the Lebesgue measure:

$$
\begin{aligned}
(\hat{q}^i \phi)(x) &= x^i \phi(x) \qquad (15) \\
(\hat{p}_j \phi)(x) &= -\imath\hbar \frac{\partial \phi}{\partial x^j}(x)
\end{aligned}
$$

where $\phi \in L^2(\mathbb{R}^n, dx)$ is, e.g., a smooth function with compact support. Each of he operators $\hat{q}^i$ and $\hat{p}_j$ admit a unique self-adjoint extension and the Schrödinger representation (15) can be integrated to unitary operators. One has for any square-integrable function $\phi \in L^2(\mathbb{R}^n, dx)$:

$$
\begin{aligned}
(U(a)\phi)(x) &= e^{\frac{\imath}{\hbar} a \cdot x}\phi(x) \\
(V(b)\phi)(x) &= \phi(x+b).
\end{aligned}
\tag{16}
$$

There is a fundamental result due to von Neumann which essentially assert the uniqueness of the Schrödinger representation. Von Neumann uniqueness theorem (1931) states that two (continuous) n-parameter unitary groups satisfying the Weyl relations are unitarily equivalent, up to multiplicity, to the Schrödinger representation (16).

Die Eindeutigkeit der Schrödingerschen Operatoren.

Von

J. v. Neumann in Berlin.

1. Die sogenannte Vertauschungsrelation

$$PQ - QP = \frac{h}{2\pi i} 1$$

ist in der neuen Quantentheorie von fundamentaler Bedeutung, sie ist es, die den „Koordinaten-Operator“ R und den „Impuls-Operator“ P im wesentlichen definiert[1]). Mathematisch gesprochen, liegt darin die folgende Annahme: Seien P, Q zwei Hermitesche Funktionaloperatoren des Hilbertschen Raumes, dann werden sie durch die Vertauschungsrelation bis auf eine Drehung des Hilbertschen Raumes, d. i. eine unitäre Transformation U, eindeutig festgelegt[2]). Es liegt im Wesen der Sache, daß noch der Zusatz gemacht werden muß: vorausgesetzt, daß P, Q ein irreduzibles System bilden (vgl. weiter unten Anm. [6])). Wird nun, wie es sich durch die Schrödingersche Fassung der Quantentheorie als besonders günstig erwies, der Hilbertsche Raum als Funktionenraum interpretiert — der Einfachheit halber etwa als Raum aller komplexen Funktionen $f(q)$ $(-\infty < q < +\infty)$ mit endlichem $\int_{-\infty}^{+\infty} |f(q)|^2 dq$ —, so gibt es nach Schrödinger ein besonders einfaches Lösungssystem der Vertauschungsrelation

$$Q:\ f(q) \to q f(q), \qquad P:\ f(q) \to \frac{h}{2\pi i}\frac{d}{dq} f(q)$$ [3]).

---

[1]) Vgl. Born-Heisenberg-Jordan, Zeitschr. f. Phys. **34** (1925), S. 858—888, ferner Dirac, Proc. Roy. Soc. **109** (1925) u. f. Besonders in der letztgenannten Darstellung ist die Rolle dieser Relation fundamental. Einen interessanten Versuch zur Begründung

A full quantization procedure however requires more than a representation of the CCR, the latter specifies only the quantization of canonical coordinates. One should also prescribe how a "large" class of classical observables are quantized. This is the so-called ordering problem. For example, consider a system with one degree of freedom with canonical coordinates q and p and corresponding operators $\hat{q}$ and $\hat{p}$ given by (15). Polynomials in the variables (q,p) can be quantized by assigning a differential operator T_{mn} in $L^2(\mathbb{R},dx)$ to monomials $q^m p^n$ in the following way. Introduce the notation $\hat{a}_1 = \cdots = \hat{a}_m = \hat{q}$ and $\hat{a}_{m+1} = \cdots = \hat{a}_{m+n} = \hat{p}$. The operator T_{mn} is given by the complete symmetrization of the product $\hat{a}_1 \cdots \hat{a}_{m+n}$:

$$T_{mn} = \mathrm{Sym}(\hat{a}_1 \cdots \hat{a}_{m+n}) := \frac{1}{(m+n)!} \sum_{\sigma \in \mathfrak{S}_{m+n}} \hat{a}_{\sigma(1)} \cdots \hat{a}_{\sigma(m+n)}, \tag{17}$$

where $\mathfrak{S}_{m+n}$ is the symmetric group of degree $m+n$. Applying this rule e.g. to the monomial $q^2 p$ gives the correspondence:

$$q^2 p \mapsto \frac{1}{3}(\hat{q}\hat{q}\hat{p} + \hat{q}\hat{p}\hat{q} + \hat{p}\hat{q}\hat{q}).$$

Clearly this rule can be extended by linearity to polynomial functions on phase space $\mathbb{R}^{2n}$. Hence we can associate an operator $W(f)$ to any polynomial f: For $f(q,p) = \sum_{I,J\in\mathbb{N}^n} f_{IJ} q^I p^J$, where the f_{IJ} are complex numbers and only finitely many of them are nonzero, one has

$$W(f) = \sum_{I,J\in\mathbb{N}^n} f_{IJ} W(q^I p^J) = \sum_{I,J\in\mathbb{N}^n} f_{IJ} T_{IJ},$$

where the T_{IJ} are direct generalization to the n degrees of freedom case of the T_{mn} introduced in (17). For any polynomial f, the operator $W(f)$ can be considered as defined on the dense subspace of $L^2(\mathbb{R}^n, dx)$ consisting of smooth functions with compact support.

The rule $f \mapsto W(f)$ is called the Weyl transform, the corresponding ordering is known as the Weyl ordering and is implicitly assumed in many quantization problems. Other ordering rules such as the standard ordering or the normal ordering are also of physical interest. We now proceed to explain the link between the Weyl ordering and the Moyal star product.
Let us denote by $\mathbb{C}[q,p]$ the algebra of complex-valued polynomials in the $2n$ variables $(q,p) = (q^1,\ldots,q^n,p_1,\ldots,p_n)$. The Moyal star product of polynomial functions on phase space is polynomial in λ and we have a noncommutative product on $\mathbb{C}[q,p][\lambda]$, the space of polynomials in λ with coefficients in $\mathbb{C}[q,p]$:

$$\star\colon \mathbb{C}[q,p][\lambda] \times \mathbb{C}[q,p][\lambda] \to \mathbb{C}[q,p][\lambda].$$

As far as the algebra of observables is concerned, the following theorem gives in essence the relation between Moyal quantization and standard Quantum Mechanics.

Theorem 2. *The Moyal star product with deformation parameter* $\lambda = \frac{\imath\hbar}{2}$ *satisfies*

$$W(f \star g) = W(f)W(g)$$

for any polynomials f, g *in* $\mathbb{C}[q,p][\hbar]$.

Through the Weyl transform, the star commutator (12) is sent to the (scaled) commutator:

$$W([f,g]_\star) = \frac{1}{\imath\hbar}[W(f), W(g)]. \tag{18}$$

This should be put in relation with the Groenewold-van Hove no-go theorem that basically says that there is no consistent quantization of all polynomial functions on phase space implementing Dirac's idea that the Poisson bracket $\{\ ,\ \}$ should be mapped to the (scaled) commutator $\frac{1}{\imath\hbar}[\ ,\ ]$.

The Weyl transform was introduced by H. Weyl in 1927 as an attempt to understand the ambiguity inherent to the quantization process. It was presented in the following form.
Let f be an integrable function defined on $\mathbb{R}^{2n}$. The Fourier transform of f is defined by

$$\tilde{f}(\xi,\eta) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} f(q,p) e^{-\frac{\imath}{\hbar}(\xi\cdot q+\eta\cdot p)}\, dqdp. \qquad (19)$$

Weyl proposed the following correspondence between functions on phase space and operators:

$$W(f) = \frac{1}{(2\pi\hbar)^n} \int_{\mathbb{R}^{2n}} \tilde{f}(\xi,\eta) e^{\frac{\imath}{\hbar}(\xi\cdot\hat{q}+\eta\cdot\hat{p})}\, d\xi d\eta, \qquad (20)$$

where the operators $\hat{q}^i$ and $\hat{p}_i$ are given by (15). One can make sense of (20) when f is a polynomial in the distribution (generalized functions) sense. A few years after Weyl, E. Wigner introduced a kind of trace formula that associates a function to an operator: the Wigner transform (nowadays we will say the symbol of an operator) and inaugurated the phase space approach to Quantum Mechanics. The Wigner transform appears to be the inverse of the Weyl transform.

The Weyl transform has triggered several works on the foundation of quantization. Von Neumann has exploited this notion in proving his uniqueness theorem for the Schrödinger representation.

Incidentally, the paper by von Neumann seems to be the first reference in which the Moyal product appeared in an integral form.

The work of Groenewold (1946) is also based on the Weyl transform and led him to the (re)discovery of the Moyal star product by computing $W^{-1}(W(f)W(g))$.

$$e^{\frac{1}{2} i(\alpha v - \beta u)} \boldsymbol{a}(\alpha - u, \beta - v) \quad \text{bzw.} \quad e^{-\frac{1}{2} i(\alpha v - \beta u)} \boldsymbol{a}(\alpha - u, \beta - v).$$

Haben A, B die bzw. Kerne $\boldsymbol{a}(\alpha, \beta)$, $\boldsymbol{b}(\alpha, \beta)$, so hat $A + B$ offenbar $\boldsymbol{a}(\alpha, \beta) + \boldsymbol{b}(\alpha, \beta)$, bei AB dagegen ist eine kleine Rechnung notwendig:

$$\begin{aligned}(ABf, g) &= (Bf, A^* g) = \iint \boldsymbol{b}(\alpha, \beta)\,(S(\alpha, \beta) f, A^* g)\, d\alpha\, d\beta \\ &= \iint \boldsymbol{b}(\alpha, \beta)\,(A S(\alpha, \beta) f, g)\, d\alpha\, d\beta \\ &= \iiiint \boldsymbol{b}(\alpha, \beta)\, e^{\frac{1}{2} i(\gamma\beta - \delta\alpha)} \boldsymbol{a}(\gamma - \alpha, \delta - \beta)\,(S(\gamma, \delta) f, g)\, d\alpha\, d\beta\, d\gamma\, d\delta \\ &= \iint \Big[\iint e^{\frac{1}{2} i(\gamma\beta - \delta\alpha)} \boldsymbol{a}(\gamma - \alpha, \delta - \beta)\, \boldsymbol{b}(\alpha, \beta)\, d\alpha\, d\beta\Big] (S(\gamma, \delta) f, g)\, d\gamma\, d\delta.\end{aligned}$$

Der Kern von AB ist also (statt γ, δ schreiben wir wieder α, β, statt α, β ξ, η) $\iint e^{\frac{1}{2} i(\alpha\eta - \beta\xi)} \boldsymbol{a}(\alpha - \xi, \beta - \eta)\, \boldsymbol{b}(\xi, \eta)\, d\xi\, d\eta$. (Die absolute Integrierbarkeit folgt aus der Deduktion.)

Schließlich zeigen wir: wenn A verschwindet, so ist auch sein Kern (bis auf eine Lebesguesche Nullmenge) gleich 0. Aus $A = 0$ folgt nämlich $S(-u, -v) A S(u, v) = 0$, also, da dieses den Kern $e^{i(\alpha v - \beta u)} \boldsymbol{a}(\alpha, \beta)$ hat,

$$\iint e^{i(\alpha v - \beta u)} \boldsymbol{a}(\alpha, \beta)\,(S(\alpha, \beta) f, g)\, d\alpha\, d\beta = 0.$$

Somit ist jedenfalls

$$\iint P(\alpha, \beta)\, \boldsymbol{a}(\alpha, \beta)\,(S(\alpha, \beta) f, g)\, d\alpha\, d\beta = 0,$$

wenn $P(\alpha, \beta)$ ein Linearaggregat von endlich vielen $e^{i(k\alpha + l\beta)}$ ist, also für jedes trigonometrische Polynom mit einer Periode $p > 0$ in α, β. Da der zweite Faktor absolut integrierbar ist, und der dritte beschränkt, können wir mit dem ersten ($P(\alpha, \beta)$) Grenzübergänge ausführen, falls dieser dabei gleichmäßig beschränkt bleibt. So können wir die Klasse der $P(\alpha, \beta)$ sukzessiv erweitern: 1. zu allen stetigen Funktionen mit einer Periode $p > 0$ in α, β, 2. zu allen beschränkten stetigen Funktionen, 3. zu allen beschränkten Funktionen der ersten Baireschen Klasse. Wenn also $\mathfrak{R}$ ein beliebiges (endliches) Rechteck in der α, β-Ebene ist, so können wir $P(\alpha, \beta)$ in $\mathfrak{R}$ gleich 1 und außerhalb $= 0$ setzen, es wird:

$$\iint \boldsymbol{a}(\alpha, \beta)\,(S(\alpha, \beta) f, g)\, d\alpha\, d\beta = 0$$

1.4 The example of the harmonic oscillator

Let us explore further the relation of Moyal quantization with Quantum Mechanics. At this stage one may wonder how from Moyal quantization one can get the spectrum of an observable. We will briefly illustrate this point for the simple case of the harmonic oscillator.

Consider the Hamiltonian of the (normalized) harmonic oscillator in one degree of of freedom:

$$H_0(q,p) = \frac{1}{2}(p^2+q^2), \quad (q,p) \in \mathbb{R}^2. \tag{21}$$

In Quantum Mechanics, the spectral analysis of the corresponding Schrödinger equation

$$\imath\hbar\frac{\partial}{\partial t}\psi(t,q) = \frac{1}{2}(-\hbar^2\frac{\partial^2}{\partial q^2}+q^2)\psi(t,q)$$

leads to the spectrum of the quantum Hamiltonian: $E_n = \hbar(n + \frac{1}{2})$, $n = 0, 1, 2, \dots$
The tool that allows to determine the spectrum is the star exponential. For the example at hand, let $\star$ be the Moyal product on $\mathbb{R}^2$ with deformation parameter $\lambda = \frac{\imath\hbar}{2}$, one defines the star exponential of a classical Hamiltonian H formally by the series

$$\mathrm{Exp}_\star\left(\frac{tH}{\imath\hbar}\right) := \sum_{n\geq 0} \frac{1}{n!}\left(\frac{t}{\imath\hbar}\right)^n \underbrace{H \star \cdots \star H}_{n \text{ factors}}. \tag{22}$$

One should be aware that are several issues that should be addressed about this series and its convergence. First, the series (22) involves negative powers of $\hbar$ and hence is not a formal series in $\hbar$. Thus it does not belong to the algebra of quantum observables A_λ with $\lambda = \frac{\imath\hbar}{2}$. The series is at best an element of $C^\infty(\mathbb{R}^2)[[\hbar^{-1}, \hbar]]$, i.e., formal in both $\hbar^{-1}$ and $\hbar$. Moreover, in physical applications, one would like to consider $\hbar$ not as a formal parameter but as the

physical Planck's constant divided by 2π. Therefore (22) should be seen as a series of functions defined on $\mathbb{R}^2$.
An alternative definition of the star exponential is through an evolution linear partial differential equation for $\phi(t) := \mathrm{Exp}_\star\left(\frac{tH}{\imath\hbar}\right)$. On can define the star exponential as the (unique) solution of the evolution equation

$$\imath\hbar\frac{\partial}{\partial t}\phi(t) = H \star \phi(t), \tag{23}$$

with the initial data $\phi(0) = 1$. Still one has to properly deal with the right-hand side of (23) when H is a general function. Indeed it might not be a polynomial in $\hbar$ and hence involves infinitely many derivatives with respect to q and p: it will not be a differential operator, but can be a (formal) pseudo-differential operator.
Without entering into mathematical considerations regarding the existence of the solutions of equation (23), let us mention that in a more general geometrical setting these analytical issues have been

addressed by interpreting the star exponential as the symbol of an operator living in a bigger algebra than A_λ.
For the Hamiltonian of the harmonic oscillator (21) the details have been worked out in the Ann. Phys paper by Flato et al.
The situation here is greatly simplified by the fact that the star powers of H_0 appearing in (22) are polynomials in H_0. More explicitly, for a smooth function F of H_0, one has

$$H_0 \star F(H_0) = H_0\, F(H_0) - \frac{\hbar^2}{4}\big(F'(H_0) + H_0 F''(H_0)\big),$$

hence the equation (23) takes the form

$$\imath\hbar \frac{\partial}{\partial t}\phi(t,H_0) = H_0\,\phi(t,H_0) - \frac{\hbar^2}{4}\big(\frac{\partial}{\partial H_0}\phi(t,H_0) + H_0\frac{\partial^2}{\partial H_0^2}\phi(t,H_0)\big)$$

whose unique solution satisfying $\phi(0,H_0) = 1$ exists for $|t| < \pi$

and is given by

$$\phi(t,H_0) = \mathrm{Exp}_\star(\frac{tH_0}{\imath\hbar}) = \frac{1}{\cos(t/2)}\exp\big(\frac{2H_0}{\imath\hbar}\tan(t/2)\big). \tag{24}$$

It is worth noticing that the convergence of the series (22) for H_0 should be understood in the space of distributions $\mathscr{D}'(\mathbb{R}^2)$ when $|t| < \pi$ is fixed. An important feature of the solution (24) is its periodicity in the variable t and can be Fourier expanded:

$$\frac{1}{\cos(t/2)}\exp\big(\frac{(q^2+p^2)}{\imath\hbar}\tan(t/2)\big) = \sum_{n\geq 0}\exp(-\imath(n+\frac{1}{2})t)\ \pi_n(q,p)$$

where

$$\pi_n(q,p) = 2(-1)^n\exp(-\frac{(p^2+q^2)}{\hbar})L_n(\frac{2(p^2+q^2)}{\hbar}), \tag{25}$$

and where the L_n's are the Laguerre polynomials.

The Fourier expansion (25) is the key ingredient allowing the identification of the quantized energy level of the harmonic oscillator and its corresponding eigenstates. From the "spectral decomposition" (25), one sees that the energy levels are $E_n = \hbar(n + 1/2)$, $n = 0, 1, \ldots$, and the corresponding eigenstates are given by the functions π_n.

The analogy with traditional Quantum Mechanics can be further emphasized.

$$\hat{H}_0|n\rangle = E_n|n\rangle, \qquad |n\rangle\langle n| \circ |n'\rangle\langle n'| = \delta_{nn'}|n\rangle\langle n|,$$
$$\sum_{n\geq 0} |n\rangle\langle n| = 1, \qquad \sum_{n\geq 0} E_n|n\rangle\langle n| = \hat{H}_0.$$

The counterpart of these relations in deformation quantization are:

$$H \star \pi_n = E_n\pi_n, \qquad \pi_n \star \pi_{n'} = \delta_{nn'}\pi_n$$
$$\sum_{n\geq 0} \pi_n = 1, \qquad \sum_{n\geq 0} E_n\pi_n = H_0,$$

which can be established by simple computations with the Moyal star product.

2 Deformation Quantization

2.1 Basic notions

Here we will see how one can generalize the Moyal quantization to more general spaces.

Definition 2 (Poisson Manifold)**.** *Smooth manifold X with a Poisson bracket, i.e., bilinear map:*

$$\{\ ,\ \}\colon C^\infty(X)\times C^\infty(X)\to C^\infty(X)$$

satisfying antisymmetry, Leibniz rule, and Jacobi identity.

- Equivalently, $\mathfrak{p}\in\Gamma(X,\wedge^2 TX)$ with $[\mathfrak{p},\mathfrak{p}]_{SN}=0$. $\{f,g\}=\langle\mathfrak{p}, df\wedge dg\rangle$.
- Locally,

$$\{f,g\}(x)=\sum_{i,j}\mathfrak{p}^{ij}(x)\ \partial_i f(x)\ \partial_j g(x),$$

  where $\mathfrak{p}^{ij}$ are local smooth functions satisfying

  - $\mathfrak{p}^{ij}=-\mathfrak{p}^{ji}$
  - $\sum_a(\mathfrak{p}^{ia}\partial_a\mathfrak{p}^{jk}+\mathfrak{p}^{ja}\partial_a\mathfrak{p}^{ki}+\mathfrak{p}^{ka}\partial_a\mathfrak{p}^{ij})=0,\quad\forall i,j,k.$

Consider a Poisson manifold $(M, \mathfrak{p})$.

Definition 3 (Star-product)**.** *[Flato et al. from mid 70's]*

- $\mathsf{A}_\hbar := C^\infty(X)[[\hbar]]$, *formal series in* $\hbar$ *with coefficients in* $C^\infty(X)$. *Elements:* $f_0 + \hbar f_1 + \hbar^2 f_2 + \cdots$

- $\star_\hbar \colon \mathsf{A}_\hbar \times \mathsf{A}_\hbar \to \mathsf{A}_\hbar$; $\quad f \star_\hbar g = fg + \sum_{r \geq 1} \hbar^r C_r(f, g)$

- C_r *are bidifferential operators null on constants: (*$1 \star_\hbar f = f \star_\hbar 1 = f$*);*

- $\star_\hbar$ *is associative;*

- $C_1(f, g) - C_1(g, f) = 2\{f, g\}$, *so that*

$$[f, g]_\hbar \equiv (f \star_\hbar g - g \star_\hbar f)/(2\hbar) = \{f, g\} + O(\hbar)$$

defines a Lie algebra deformation.

Example Take two commuting vector fields D_1 and D_2 on X. The wedge product $D_1 \wedge D_2$ defines a Poisson tensor $\mathfrak{p}$ on X. The following defines a globally defined star-product on X:

$$f \star g = \sum_{k \geq 0} \frac{\hbar^k}{k!} D_1^k(f) D_2^k(g).$$

(Quantum plane: take $D_1 = x\frac{\partial}{\partial x}$, $D_2 = y\frac{\partial}{\partial y}$.)

Definition 4 (Equivalence)**.** *Two star products $\star_1$ and $\star_2$ are said equivalent if there exists a formal series of differential operators*

$$T(f) = f + \sum_{r\geq 1} \hbar^r T_r(f)$$

such that

$$T(f \star_1 g) = T(f) \star_2 T(g).$$

Example Normal product. $z = \frac{1}{\sqrt{2}}(p - \imath q)$.

$$f(\bar{z}, z) \star' g(\bar{z}, z) = \sum_{k\geq 0} \frac{\hbar^k}{k!} \frac{\partial^k f}{\partial z^k} \frac{\partial^k g}{\partial \bar{z}^k}$$

Normal and Moyal star products are equivalent: $T = \exp(-\frac{\hbar}{4} \frac{\partial^2}{\partial z \partial \bar{z}})$.

- It is a highly non-trivial fact that star-products exist on any Poisson manifold. But one can easily find the 2 first terms:

$$f \star g = fg + \hbar\{f,g\} + \hbar^2 C_2(f,g)$$

$$\begin{aligned} C_2(f,g) \;&= \sum \Big(\frac{1}{2}\, \mathfrak{p}^{i_1 j_1}\, \mathfrak{p}^{i_2 j_2}\, \partial_{i_1 i_2} f\, \partial_{j_1 j_2} g \\ &\quad + \frac{1}{3}\, \mathfrak{p}^{i_1 j_1}\, \partial_{i_1} \mathfrak{p}^{i_2 j_2} \left(\partial_{j_1 j_2} f\, \partial_{i_2} g + \partial_{i_2} f\, \partial_{j_1 j_2} g \right) \Big) \end{aligned}$$

- Associativity is equivalent to solve infinitely many non-linear equations (Maurer-Cartan equations).

2.2 Gerstenhaber bracket and Hochschild cohomology

The normalized differential Hochschild cochain complex of the associative algebra $A := C^\infty(X)$, with product $m_0(f,g) = fg$, with values in itself

$$\mathscr{C}^\bullet(A,A) = \oplus_{m\geq 0}\mathscr{C}^m(A,A)$$

consists of polydifferential operators on X that are vanishing on constants.

Locally, an m-cochain $C \in \mathscr{C}^m(A,A)$ has the form

$$C(f_1,\ldots,f_m) = \sum C_{\alpha_1\cdots\alpha_m}\partial^{\alpha_1} f_1\cdots\partial^{\alpha_m} f_m, \qquad f_1,\ldots,f_m \in A, \tag{26}$$

where the sum is finite and runs over multi-indices $\alpha_i \in \mathbb{N}^d$ such that $|\alpha_i| \geq 1$, and the $C_{\alpha_1 \cdots \alpha_m}$ are locally defined smooth functions on X.

Usually one considers the Hochschild complex as a $\mathbb{Z}$-graded vector space with a shift in the degree: $D^{\bullet}_{\text{poly}}(X) = \mathscr{C}^{\bullet}(A,A)[1]$. Hence

$$D^k_{\text{poly}}(X) = \begin{cases} \mathscr{C}^{k+1}(A,A) & \text{for } k \geq -1, \\ \{0\} & \text{otherwise.} \end{cases}$$

The Gerstenhaber bracket $[\cdot,\cdot]_G$ on $D^{\bullet}_{\text{poly}}(X)$ is defined on homogeneous elements $D_i \in D^{k_i}_{\text{poly}}(X)$ by:

$$[D_1, D_2]_G = D_1 \circ D_2 - (-1)^{k_1 k_2} D_2 \circ D_1,$$

where $\circ\colon D^{k_1}_{\text{poly}}(X) \times D^{k_2}_{\text{poly}}(X) \to D^{k_1+k_2}_{\text{poly}}(X)$ is a composition law for polydifferential operators:

$$
\begin{aligned}
&(D_1 \circ D_2)(f_0, \ldots, f_{k_1+k_2}) \\
&\qquad\qquad = \sum_{0 \le j \le k_1} (-1)^{jk_2} D_1(f_0, \ldots, f_{j-1}, D_2(f_j, \ldots, f_{j+k_2}), f_{j+k_2+1}, \ldots f_{k_1+k_2}).
\end{aligned}
$$

$$
[m_0, m_0]_G(f,g,h) = 2\big(m_0(m_0(f,g),h) - m_0(f, m_0(g,h))\big) = 0
$$

The Hochschild differential is given by $\delta = [m_0, \,\cdot\,]_G$. It is the standard Hochschild differential d up to a sign: for $D \in D^k_{\mathrm{poly}}(\mathbb{R}^d)$, we have $\delta D = (-1)^k dD$.

Theorem 3 (HKR, Vey)**.** *The cohomology $H^\bullet(A,A)$ of the complex $(\mathscr{C}^\bullet(A,A), \delta)$ is the space of polyvectors $\Gamma(\wedge^\bullet TX)$.*

- $(D^\bullet_{\mathrm{poly}}(X), [\cdot,\cdot]_G, \delta)$ is a differential graded Lie algebra (DGLA).
- In terms of the Gerstenhaber bracket, the associativity of the star-product $\star_\hbar = m_0 + \sum_{r\geq 1} \hbar^r c_r$ is equivalent to the Maurer-Cartan equations:

$$\delta c_k + \frac{1}{2} \sum_{\substack{a+b=k\\ a,b\geq 1}} [c_a, c_b]_G = 0, \qquad \text{for all } k \geq 1. \tag{27}$$

- If $\star^{(k)} = m_0 + \sum_{1\leq r\leq k} \hbar^r c_r$ is associative up to order $\hbar^k$, then

$$\sum_{\substack{a+b=k+1\\ a,b\geq 1}} [c_a, c_b]_G \tag{28}$$

is a Hochschild cocycle.

- One can extend $\star^{(k)}$ to a product $\star^{(k+1)}$, associative up to order $\hbar^{k+1}$, if (28) is a coboundary. Hence the obstructions to associativity belongs to the 3rd Hochschild cohomology space.

- In a similar fashion, the obstructions to equivalence belongs to the 2nd Hochschild cohomology space.

2.3 A panorama of results

Existence

Symplectic manifolds: DeWilde-Lecomte [1982], Glue local Moyal products. Fedosov [1994], Weyl bundle, flat connexion.

Poisson manifolds: Kontsevich [1997], Local formula using graph techniques. Consequence of the Formality Theorem.

Classification

How unique is a star-product?

Symplectic case: Equivalence classes of star-products are parametrized by the 2nd De Rham cohomology space $H^2_{dR}(X)$.

More precisely: $\{\star_\hbar\}/\sim \quad = \quad H^2_{dR}(X)[[\hbar]]$

Poisson case: $\{\star_\hbar\}/\sim \quad =$ equivalence classes of formal Poisson structures.

Denote by $T^{\bullet}_{\text{poly}}(X)$ the graded vector space $T^{\bullet}_{\text{poly}}(X) = \oplus_{k\in\mathbb{Z}} T^{k}_{\text{poly}}(X)$, where

$$T^{k}_{\text{poly}}(X) = \begin{cases} \Gamma(\wedge^{k+1}TX) & \text{for } k \geq -1, \\ \{0\} & \text{otherwise.} \end{cases}$$

$T^{\bullet}_{\text{poly}}(X)$ is endowed with the Schouten-Nijenhuis bracket $[\cdot,\cdot]_{SN}$.

Recall that the Jacobi identity for a bivector $\mathfrak{p} \in \Gamma(\wedge^2 TX)$ is equivalent to the condition $[\mathfrak{p},\mathfrak{p}]_{SN} = 0$.

Kontsevich Quantization Formula

Two algebraic stuctures on a manifold X.

- $(D^{\bullet}_{\text{poly}}(X), [\cdot,\cdot]_G, \delta)$ is a differential graded Lie algebra (DGLA).
- $(T^{\bullet}_{\text{poly}}(X), [\cdot,\cdot]_{SN}, 0)$ is a differential graded Lie algebra.

The Formality theorem

Theorem 4 (Kontsevich)**.** *There is an L_∞ quasi-isomorphism between the differential graded Lie algebras $(T^{\bullet}_{\text{poly}}(X), [\cdot,\cdot]_{SN}, \mathbf{0})$ and $(D^{\bullet}_{\text{poly}}(X), [\cdot,\cdot]_G, \delta)$.*

- A remarkable consequence of the Formality theorem is the existence of deformation quantization of any smooth Poisson manifold.

- Kontsevich gives an explicit description of the L_∞ quasi-isomorphism for $X = \mathbb{R}^d$ in terms of graphs and weights.

- Consider $\mathbb{R}^n$ endowed with a Poisson structure $\mathfrak{p} \in \Gamma(\mathbb{R}^n, \wedge^2 T\mathbb{R}^n)$.

- The Kontsevich quantization formula defines a deformation quantization of $(\mathbb{R}^n, \mathfrak{p})$:

$$f \star_\hbar^{\mathsf{K}} g = fg + \sum_{r \geq 1} \hbar^r k_r(f, g), \quad f, g \in C^\infty(\mathbb{R}^n).$$

- The k_r's are bidifferential operators defined by weights and bidifferential operators associated to graphs.

Definition 5 (Graphs)**.** *The set $\mathscr{K}$ consists of all the simple directed graphs Γ satisfying:*

1. *The set of vertices V_Γ is finite and is a disjoint union of nonempty sets: $V_\Gamma = V_\Gamma^1 \sqcup V_\Gamma^2$. Vertices belonging to V_Γ^1 (resp. V_Γ^2) are said of type 1 (resp. 2);*

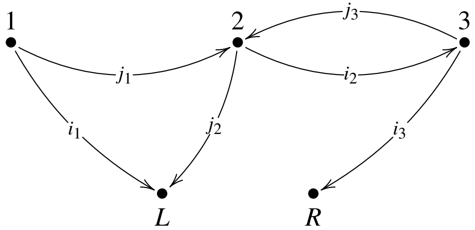
2. *Each vertex of type 1 is of outdegree 2;*

3. *Each vertex of type 2 is of indegree at least 1 and of outdegree 0;*

4. *Vertices and edges are labeled.*

We denote by $\mathscr{K}_{n,m}$ for $m, n \geq 1$ the subset of $\mathscr{K}$ consisting of graphs having n vertices of type 1 and m vertices of type 2. Thus a graph in $\mathscr{K}_{n,m}$ has $2n$ edges.

Example 1. *A graph* Γ *in* $\mathscr{K}_{3,2}$.



- Bidifferential operator B_Γ associated to Γ:

$$\mathscr{B}_\Gamma(f,g) = \sum_{0 \le i_*, j_* \le n} \mathfrak{p}^{i_1 j_1}\, \partial_{j_1 j_3} \mathfrak{p}^{i_2 j_2}\, \partial_{i_2} \mathfrak{p}^{i_3 j_3}\, \partial_{i_1 j_2} f\, \partial_{i_3} g.$$

- To each graph $\Gamma \in \mathscr{K}_{r,2}$ is associated a real number $w(\Gamma)$.
- $w(\Gamma)$ is expressed as an integral over compactification of configuration space of r points:

- $\mathsf{H} = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > 0\}$ be the upper half-plane. H_n will denote the configuration space $\{z_1, \ldots, z_n \in \mathsf{H} \,|\, z_i \neq z_j \text{ for } i \neq j\}$.

- Let $\phi : \mathsf{H}_2 \to \mathbb{R}/2\pi\mathbb{Z}$ be the function:

$$\phi(z_1, z_2) = \frac{1}{2\sqrt{-1}} \mathrm{Log}\Big(\frac{(z_2 - z_1)(\bar{z}_2 - z_1)}{(z_2 - \bar{z}_1)(\bar{z}_2 - \bar{z}_1)}\Big). \tag{29}$$

  $\phi(z_1, z_2)$ is extended by continuity for $z_1, z_2 \in \mathbb{R}$, $z_1 \neq z_2$.

- For a graph $\Gamma \in \mathscr{K}_{r,2}$, the vertex k, $1 \leq k \leq r$, is associated with the variable $z_k \in \mathsf{H}$, the vertex L with $0 \in \mathbb{R}$, and the vertex R with $1 \in \mathbb{R}$.

- The weight $w(\Gamma)$ is defined by integrating an $2r$-form over (a compactification of) H_r:

$$w(\Gamma) = \frac{1}{r!(2\pi)^{2r}} \int_{\mathsf{H}_r} \bigwedge_{1\leq k\leq r} \Big(d\phi(z_k, I_k) \wedge d\phi(z_k, J_k) \Big), \quad (30)$$

  where I_k (resp. J_k) denotes the variable or real number associated with the ending vertex of the arrow i_k (resp. j_k).

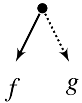
- The weights are universal in the sense that they do not depend on the Poisson structure or the dimension n.

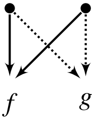
Theorem 5 (Kontsevich). *Let* $k_r(f,g) = \sum_{\Gamma \in \mathscr{K}_{r,2}} w(\Gamma) B_\Gamma(f,g)$, *then*

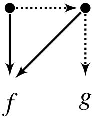
$$f \star_\hbar^\kappa g = fg + \sum_{r \geq 1} \hbar^r k_r(f,g), \quad f,g \in C^\infty(\mathbb{R}^n).$$

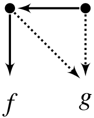
defines an associative product in $C^\infty(\mathbb{R}^n)[\![\hbar]\!]$.

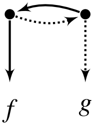
$$k_1(f,g) = \{f,g\} =$$



$k_2(f,g) = \frac{1}{2}$ 

$+\frac{1}{3}$ 

$+\frac{1}{3}$ 

$-\frac{1}{6}$ 

$$\begin{aligned} k_2(f,g) \;\; = \;\; & \frac{1}{2}\,\mathfrak{p}^{i_1 j_1}\,\mathfrak{p}^{i_2 j_2}\,\partial_{i_1 i_2} f\,\partial_{j_1 j_2} g \\ & \quad + \frac{1}{3}\,\mathfrak{p}^{i_1 j_1}\,\partial_{i_1}\mathfrak{p}^{i_2 j_2}\,(\partial_{j_1 j_2} f\,\partial_{i_2} g + \partial_{i_2} f\,\partial_{j_1 j_2} g) \\ & \quad - \frac{1}{6}\,\partial_{j_2}\mathfrak{p}^{i_1 j_1}\,\partial_{j_1}\mathfrak{p}^{i_2 j_2}\,\partial_{i_1} f\,\partial_{i_2} g, \end{aligned}$$